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DECOMPOSITION OF THE INVARIANT LAPLACIAN IN THE COMPLEX BALL

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Abstract. We, in this note, decompose the invariant Laplacian of the unit complex ball of $\mathbb{C}^n$ by the radial part and tangential part as

$$\widetilde{\Delta} = \widetilde{\Delta}_{rad} + \widetilde{\Delta}_{tan}.$$

We give several properties and interpretations involved with this decomposition.

1. Introduction

Function theory on a subset of $\mathbb{R}^{2n}$ often uses the Lapalcian:

$$\Delta = \sum_{j=1}^{n} \left(\frac{\partial^2}{\partial^2 x_j} + \frac{\partial^2}{\partial^2 y_j} \right),$$

because it is one of the most convenient operator in measuring the local change of a function. Our interest in this note is on the decomposition of the Laplacian. Sometimes a decomposition of Δ gives some benefits in measuring functional behaviors.

Throughout this note, terminologies are used as in [3,4]. Let $B = B_n$ be the open unit ball of the complex n-dimensional space $\mathbb{C}^n$ topologically identified with $\mathbb{R}^{2n}$. Then Δ can be expressed as

$$\Delta = 4 \sum_{j=1}^{n} D_j \bar{D}_j,$$

where $D_j = \frac{\partial}{\partial z_j}$ and $\bar{D}_j = \frac{\partial}{\partial \bar{z}_j}$, $j = 1, 2, \ldots, n$. On B, Δ may be decomposed into the complex tangential Laplacian and the complex radial Laplacian as

$$\Delta = \Delta_{tan} + \Delta_{rad},$$

where Δ_{rad} is defined, for $f \in C^2(B)$ and $z = r\zeta \in B - \{0\}$, to be the Laplacian of the function $\lambda \to f(z + \lambda\zeta)$ at the origin of $\mathbb{C}$ (see [3, 17.3.2]). And lots of

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properties of a function on B can be observed in terms of Δ_{tan} and Δ_{rad}. See for example [1].

We, in this note, decompose another more geometric Laplacian of B. Let $\mathcal{M}$ denote the group of all automorphisms, that is, one to one biholomorphic onto map, of B. $\mathcal{M}$ consists of all maps of the form $U\varphi_a$, where U is a unitary operator of $\mathbb{C}^n$ and φ_a is defined by

$$\varphi_a(z) = \begin{cases} \frac{a - P_a z - \sqrt{1-|a|^2} Q_a z}{1 - \langle z, a\rangle}, & \text{if } a \neq 0, \\ 0, & \text{if } a = 0. \end{cases} \tag{1.1}$$

Here $\langle , \rangle$ is the Hermitian inner product of $\mathbb{C}^n$: $\langle z, w\rangle = \sum_{j=1}^n z_j \bar{w}_j$, z, $w \in \mathbb{C}^n$, $P_a z$ is the projection of $\mathbb{C}^n$ onto the subspace generated by B:

$$P_a z = \frac{\langle z, a\rangle}{\langle a, a\rangle} a, \quad \text{if } a \neq 0 \quad \text{and} \quad P_0 z = 0,$$

and $Q_a(z) = z - P_a z$. Then the ($\mathcal{M}$-)invariant Laplacian $\widetilde{\Delta}$ is defined, for $f \in C^2(B)$, as

$$\widetilde{\Delta} f(a) = \Delta(f \circ \varphi_a)(0), \quad a \in B.$$

It is $\mathcal{M}$-invariant in the sense that

$$\left(\widetilde{\Delta} f\right)(\psi(a)) = \widetilde{\Delta}(f \circ \psi)(a), \quad a \in B,$$

for all $\psi \in \mathcal{M}$, and a straightforward calculation shows that

$$\widetilde{\Delta} f(a) = 4(1 - |a|^2) \sum_{i,j=1}^n (\delta_{i,j} - \bar{a}_i a_j)(\bar{D}_i D_j f)(a), \quad a \in B. \tag{1.2}$$

See [3, 4.1.3]. Now, in this paper, we define:

Definition 1.1. Define $\widetilde{\Delta}_{rad}$, for $f \in C^2(B)$ and $z = r\zeta \in B - \{0\}$, as the (invariant) Laplacian of the function $\lambda \to f \circ \varphi_z(\lambda\zeta)$ at the origin of $\mathbb{C}$. Define

$$\widetilde{\Delta}_{tan} = \widetilde{\Delta} - \widetilde{\Delta}_{rad}.$$

Note that the round bracket in Definition 1.1 comes from the fact that $\widetilde{\Delta} f(0) = \Delta f(0)$ for any $f \in C^2(B)$.

We give several properties involved with this decomposition.

Theorem 1.2. *Let $f \in C^2(B)$ and $a = r\zeta$, $0 < r < 1$, $\zeta \in S$. Then we have the following.*

(a) $$\begin{aligned} \widetilde{\Delta}_{rad} f(a) &= 4(1 - |a|^2)^2 \sum_{j,k=1}^n \zeta_k \bar{\zeta}_j D_k \bar{D}_j f(a) \\ &= (1 - |a|^2)^2 \Delta_{rad} f(a) \\ &= 4(1 - |a|^2)^2 \lim_{\rho \to 0} \frac{1}{\rho^2} \int_0^{2\pi} \left\{ f(a + \rho e^{i\theta}\zeta) - f(a) \right\} \frac{d\theta}{2\pi}. \end{aligned}$$

(b) $\widetilde{\Delta}_{tan} f(a) = 4(1 - |a|^2) \sum_{j,k=1}^n (\delta_{k,j} - \zeta_k \bar{\zeta}_j) D_k \bar{D}_j f(a) = (1 - |a|^2) \Delta_{tan} f(a)$

(c) *If f is radial, then*

$$\widetilde{\Delta}_{rad} f = (1-r^2)^2 \frac{\partial f^2}{\partial^2 r} + (1-r^2)\frac{1}{r}\frac{\partial f}{\partial r}; \quad \widetilde{\Delta}_{tan} f = 2(n-1)\frac{(1-r^2)^2}{r}\frac{\partial f}{\partial r}.$$

(d) $\Delta, \Delta_{rad}, \Delta_{tan}, \widetilde{\Delta}, \widetilde{\Delta}_{rad}, \widetilde{\Delta}_{tan}$ *all commutes with the action of the unitary group.*

(e) $\widetilde{\Delta} = \widetilde{\Delta}_{tan} + \widetilde{\Delta}_{rad} = (1-r^2)\Delta_{tan} + (1-r^2)^2\Delta_{rad}$ *and*

$$\Delta = \frac{\widetilde{\Delta}}{1-r^2} + r^2\Delta_{rad} = \frac{\widetilde{\Delta}}{1-r^2} + r^2(1-r^2)^2\widetilde{\Delta}_{rad}.$$

We next consider several consequences of Theorem 1.2. If we let f_a be defined by $f_a(\lambda) = f(\lambda a)$, $\lambda \in B_1$, and Δ_1 denote the 1-variable Laplacian:

$$\Delta_1 = 4\frac{\partial^2}{\partial\lambda\partial\bar{\lambda}},$$

then

$$\Delta_1 f_a(1) = 4\sum_{i,j=1}^{n} \bar{a}_i a_j (\bar{D}_i D_j f)(a),$$

so that (1.2) can be expressed as

$$\widetilde{\Delta} f(a) = (1-|a|^2)\left\{\Delta f(a) - \Delta_1 f_a(1)\right\} \tag{1.3}$$

(See [3, 4.1.3]). About the formula (1.3), in [3, 19.3.16], W. Rudin asked "Is there a more intuitive way of seeing why $\widetilde{\Delta}$ is a difference of two ordinary Laplacian?" By Theorem 1.1, we can give an interpretation of (1.3) as follows.

Letting $a = r\zeta$, $0 < r < 1$, $\zeta \in S$, S being the boundary of B, we see by a direct computation that

$$\Delta_1 f_a(1) = 4\sum_{i,j=1}^{n} \bar{a}_i a_j (\bar{D}_i D_j f)(a) = 4r^2 \sum_{i,j=1}^{n} \bar{\zeta}_i \zeta_j (\bar{D}_i D_j f)(a),$$

so that $\Delta_1 f_a(1) = r^2\Delta_{rad} f(a)$. Thus, (1.3) means

$$\widetilde{\Delta} = (1-r^2)\left(\Delta - r^2\Delta_{rad}\right). \tag{1.4}$$

We have two observations of (1.4). First, since $r^2\Delta_{rad}$ represents radial directional change only, $\widetilde{\Delta} - (1-r^2)\Delta$ has tangential component canceling property. That is, considering the quantities which equal $-(1-r^2)\Delta_1 f_a(1)$, we have a kind of the tangential component canceling identity as in

Corollary 1.3.

$$\widetilde{\Delta} - (1-r^2)\Delta = \widetilde{\Delta}_{rad} - (1-r^2)\Delta_{rad} = -r^2(1-r^2)\Delta_{rad}.$$

That is, for $f \in C^2(B)$ and $a = r\zeta$, $0 < r < 1$, $\zeta \in S$, the quantity $-(1-r^2)\Delta_1 f_a(1)$ equals

$$\left[\widetilde{\Delta} - (1-r^2)\Delta\right] f(a) = \left[\widetilde{\Delta}_{rad} - (1-r^2)\Delta_{rad}\right] f(a) = -r^2(1-r^2)\Delta_{rad} f(a).$$

Secondly, noting that $\Delta = \Delta_{tan} + \Delta_{rad}$ and dividing tangential components and radial components, (1.4) matches the decomposition

$$\begin{aligned}\widetilde{\Delta} &= (1-r^2)\left(\Delta_{tan} + \Delta_{rad} - r^2\Delta_{rad}\right)\\ &= (1-r^2)\Delta_{tan} + (1-r^2)^2\Delta_{rad} = \widetilde{\Delta}_{tan} + \widetilde{\Delta}_{rad}\end{aligned}$$

by Theorem 1.2(a) and (b), which explicitly represents the tangential components and the radial components. And it is a natural and geometrical expression of $\widetilde{\Delta}$ in the sense that $\widetilde{\Delta}$ can be decomposed into two (orthogonal) directional Laplacians as $\widetilde{\Delta}_{tan} + \widetilde{\Delta}_{rad}$ with the growth properly controlled, namely, twice better controlled in the radial direction (by powers of $1 - r^2$) as

$$\widetilde{\Delta}_{tan} = (1-r^2)\Delta_{tan}; \quad \widetilde{\Delta}_{rad} = (1-r^2)^2\Delta_{rad}.$$

The following gives an illustration of such a phenomenon (See [1] Theorem 3.1 for a proof).

Theorem 1.4. *If $f \in C^2(B)$ satisfies $\widetilde{\Delta} f \equiv 0 \equiv \widetilde{\Delta} f$, then*

$$\sup \widetilde{\Delta}|f|^2 < \infty \iff \sup \widetilde{\Delta}_{tan}|f|^2 < \infty \iff \sup \widetilde{\Delta}_{rad}|f|^2 < \infty.$$

2. Proof of Theorem 1.2

Theorem 1.2 follows from the following two lemmas.

Lemma 2.1. *Let $f \in C^2(B)$ and $a = r\zeta$, $0 \le r < 1$, $\zeta \in S$. Then we have the following.*

(a) $$\begin{aligned}\Delta_{rad} f(a) &= 4\sum_{j,k=1}^{n} \zeta_k \bar{\zeta}_j D_k \bar{D}_j f(a)\\ &= \lim_{\rho\to 0} \frac{4}{\rho^2}\int_0^{2\pi} \left\{f(a+\rho e^{i\theta}\zeta) - f(a)\right\}\frac{d\theta}{2\pi}.\end{aligned}$$

(b) $\Delta_{tan} f(a) = 4\sum_{j,k=1}^{n}(\delta_{k,j} - \zeta_k\bar{\zeta}_j) D_k \bar{D}_j f(a)$.

(c) *If f is radial, then $\Delta_{rad} f = \frac{\partial f^2}{\partial^2 r} + \frac{1}{r}\frac{\partial f}{\partial r}$ and $\Delta_{tan} f = \frac{2(n-1)}{r}\frac{\partial f}{\partial r}$.*

(d) *$\Delta, \Delta_{rad}, \Delta_{tan}, \widetilde{\Delta}$ all commutes with the action of the unitary group.*

(e) *$\widetilde{\Delta} = (1-r^2)\Delta_{tan} + (1-r^2)^2\Delta_{rad}$ and $\Delta = \frac{\widetilde{\Delta}}{1-r^2} + r^2\Delta_{rad}$.*

Lemma 2.2. *Let $f \in C^2(B)$ and $a = r\zeta$, $0 < r < 1$, $\zeta \in S$. Then we have*

(a) $\widetilde{\Delta}_{rad} f(a) = 4(1-|a|^2)^2\sum_{j,k=1}^{n}\zeta_k\bar{\zeta}_j D_k \bar{D}_j f(a)$;

(b) $\widetilde{\Delta}_{tan} f(a) = 4(1-|a|^2)\sum_{j,k=1}^{n}(\delta_{k,j} - \zeta_k\bar{\zeta}_j) D_k \bar{D}_j f(a)$

Proof of Lemma 2.1. (a) Let $F(\lambda) = f(a + \lambda\zeta)$, $\lambda \in B_1$. Then calculating $\Delta F(0)$ with chain rule gives the first equality, and the Taylor expansion gives the second equality of (a).

(b) follows from (a) and the fact $\Delta_{tan} = \Delta - \Delta_{rad}$.

(c) also follows from a direct calculation using (a) and (b).

(d) Using the unitary invariance of σ with the obvious equality

$$\Delta f(a)=\lim_{\rho\to 0}\frac{4n}{\rho^2}\int_S\{f(a+\rho\eta)-f(a)\}\,d\sigma(\eta),$$

that $\Delta f(Ua)=\Delta(f\circ U)(a)$ follows immediately. Noting that $a=r\zeta$ in the integral representation of Δ_{rad} in (a), $\Delta_{rad}f(Ua)=\Delta_{rad}(f\circ U)(a)$ follows. So is Δ_{tan}. Applying the identity $U\varphi_a U^{-1}=\varphi_{Ua}$, which is obvious from (1.1), to the representation

$$\widetilde{\Delta} f(a)=\lim_{\rho\to 0}\frac{4n}{\rho^2}\int_S\{f\circ\varphi_a(\rho\eta)-f(a)\}\,d\sigma(\eta)$$

(see [3, 4.1.3]), that $\widetilde{\Delta} f(Ua)=\widetilde{\Delta}(f\circ U)(a)$ follows.

(e) Direct calculation gives (e). See [1] Theorem 2.1. □

Proof of Lemma 2.2. (a) Let $a=r\zeta$. Let $f\in C^2(B)$ and let $F(\lambda)=f\circ\varphi_a(\lambda\zeta)$, $\lambda\in B_1$. Let Δ_1 denote the one variable Δ. Then by definition,

$$\widetilde{\Delta}_{rad}f(a)=\Delta_1 F(0).$$

By the chain rule,

$$\begin{aligned}\Delta_1F(\lambda)&=4\frac{\partial^2}{\partial\lambda\partial\bar{\lambda}}\left[f\circ\varphi_a(\lambda\zeta)\right]\\&=4\sum_{j,k=1}^{n}\left(D_k\bar{D}_jf\right)\circ\varphi_a(\lambda\zeta)\ \frac{\partial}{\partial\lambda}\left[\varphi_a(\lambda\zeta)\right]_k\frac{\partial}{\partial\bar{\lambda}}\left[\bar{\varphi}_a(\lambda\zeta)\right]_j.\end{aligned}$$

Let $s=\sqrt{1-|a|^2}$. Then by the definition of φ_a,

$$\varphi_a(\lambda\zeta)=a-s\lambda\zeta+\frac{s}{1+s}\lambda|a|a+R(\lambda,a)$$

with λ-degree of $R(\lambda,a)\geq 2$ (see [3, p. 48]). Hence

$$\begin{aligned}\frac{\partial}{\partial\lambda}\left[\varphi_a(\lambda\zeta)\right]_k\Big|_{\lambda=0}&=-s\zeta_k+\frac{s}{1+s}|a|^2\zeta_k;\\\frac{\partial}{\partial\bar{\lambda}}\left[\bar{\varphi}_a(\lambda\zeta)\right]_j\Big|_{\lambda=0}&=-s\bar{\zeta}_j+\frac{s}{1+s}|a|^2\bar{\zeta}_j.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\Delta_1F(0)&=4\sum_{j,k=1}^{n}\left(D_k\bar{D}_jf\right)(a)\ \left\{-s\zeta_k+\frac{s}{1+s}|a|^2\zeta_k\right\}\left\{-s\bar{\zeta}_j+\frac{s}{1+s}|a|^2\bar{\zeta}_j\right\}\\&=4s^4\sum_{j,k=1}^{n}\zeta_k\bar{\zeta}_jD_k\bar{D}_jf(a),\end{aligned}$$

which gives the equality (a).

(b) By (a),

$$\widetilde{\Delta}_{tan}f(a)=\widetilde{\Delta}f(a)-\widetilde{\Delta}_{rad}f(a)$$

$$= 4(1-|a|^2)\sum_{j,k=1}^{n}(\delta_{k,j}-a_k\bar{a}_j)\,D_k\bar{D}_jf(a)$$

$$-\,4(1-|a|^2)^2\sum_{j,k=1}^{n}\zeta_k\bar{\zeta}_jD_k\bar{D}_jf(a)$$

$$= 4(1-|a|^2)\sum_{j,k=1}^{n}(\delta_{k,j}-\zeta_k\bar{\zeta}_j)D_k\bar{D}_jf(a),$$

which gives (b). □

Now, Theorem 1.2 is straightforward from Lemma 2.1 and Lemma 2.2.

3. Remarks

Theorem 1.4 says that pluri-harmonic Bloch functions can be characterized in terms of the radial growth or tangential growth of the invariant Laplacian only. See [1, 2] for the definition of pluri-harmonic Bloch function.

This type of characterization can be expanded (possibly through Theorem 1.2(d)) also to some function classes involved with surface or volume integrals.

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