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ON CONGRUENCES INVOLVING THE GENERALIZED CATALAN NUMBERS AND HARMONIC NUMBERS

Sibel Koparal and Neşe Ömür

Abstract. In this paper, we prove some congruences involving the generalized Catalan numbers and harmonic numbers modulo p^2, one of which is

$$\sum_{k=1}^{p-1} k^2 B_{p,k} B_{p,k-d} \equiv 4\,(-1)^d \left\{ \frac{1}{3} d\left(2d^2+1\right)\left(4pH_d-1\right) \right. \\ \left. -p\left(\frac{26}{9}d^3+\frac{4}{3}d^2+\frac{7}{9}d+\frac{1}{2}\right)\right\} \pmod{p^2},$$

where a prime number $p > 3$ and $1 \le d \le p$.

1. Introduction

The Catalan numbers have important applications in combinatorics, number theory and the analysis algorithms in [5, 6, 8, 10].

In [10], Shapiro gave the generalized Catalan numbers $B_{n,k}$ as follows:

$$B_{n,k} = \frac{k}{n}\binom{2n}{n-k},\ 0 \le k \le n.$$

The numbers $B_{n,k}$ are the entries of the Catalan triangles and satisfy the recurrence relation

$$B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1}, \quad k \ge 2$$

with the initial conditions $B_{n,0} = 0 = B_{n,n+m}$, $m \ge 1$. The numbers $B_{n,k}$ are not known as Catalan numbers. However they have several applications [3, 7, 10].

Note that for $k = 1$, $B_{n,1}$ are the well known Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \ge 1.$$

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Harmonic numbers are those rational numbers given by

$$H_0 = 0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{Z}^+,$$

and for $m \in \mathbb{Z}^+$, harmonic numbers of order m are those rational numbers

$$H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}, \quad n \in \mathbb{Z}^+.$$

Some interesting properties of harmonic numbers are given as follows [1,2,4,11]:

$$\sum_{k=1}^{n-1} \frac{k^{\underline{m}}}{n-k} H_k = n^{\underline{m}} \left(H_n \left(H_n - H_m \right) - H_{n,2} + H_{m,2} \right), \tag{1}$$

$$\sum_{k=1}^{n-1} k^{\underline{m}} H_{n-k} = n^{\underline{m}} \frac{n+1}{m+1} \left(H_{n+1} - H_{m+1} \right), \tag{2}$$

$$\sum_{k=1}^{n} k^{\underline{m}} H_k = n^{\underline{m}} \frac{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right), \tag{3}$$

$$\sum_{k=1}^{n} \frac{H_k}{k} = \frac{1}{2} \left(H_n^2 + H_{n,2} \right), \tag{4}$$

where $x^{\underline{m}}$ stands for the falling factorial defined by

$$x^{\underline{m}} = x \left(x - 1 \right) \cdots \left(x - m + 1 \right).$$

In [12], Sun showed that for any odd prime p and $0 \leq k < p$,

$$(-1)^k \binom{p-1}{k} = \prod_{0<j\leq k} \left(1 - \frac{p}{j} \right) \equiv 1 - pH_k \pmod{p^2}. \tag{5}$$

In [6], Gutiérrez et al. gave some identities involving well-known Catalan numbers. For example,

$$\sum_{i=1}^{m} B_{n,i} B_{n,n+i-m} \left(n + 2i - m \right) = \left(n+1 \right) C_n \binom{2(n-1)}{m-1}, \ m \leq n,$$

$$\sum_{i=1}^{n} i \left(B_{n,i} \right)^2 = \frac{n \left(n+1 \right)}{2} C_n C_{n-1}.$$

In [8], Miana and Romero showed the following identity that for $1 \leq m \leq n$,

$$\sum_{i=1}^{m} B_{n,i} B_{n,n+i-m} \left(n + 2i - m \right)^3 = \binom{2n}{n} \binom{2(n-1)}{m-1} \left(n^2 + 4n - 2nm + m^2 \right).$$

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In [9], Ömür and Koparal obtained various congruences involving the numbers $B_{p,k-d}$. For example, for $1 \le d \le p-1$,

$$\sum_{k=1}^{p-1} B_{p,k}B_{p,k-d} \equiv 4\left(-1\right)^{d}\left(-2d + p\left(8d\left(H_d - 1\right) + \frac{2}{d} - 3\right)\right) \pmod{p^2}$$

and for $1 \le d \le p$,

$$\sum_{k=1}^{p-1} kB_{p,k}B_{p,k-d} \equiv 2\left(-1\right)^{d}\left(-2d^2 + p\left(1 + 8d^2\left(H_d - 1\right) - 4d\right)\right) \pmod{p^2}.$$

In this paper, we prove some congruences involving the generalized Catalan numbers and harmonic numbers modulo p^2, one of which is

$$\sum_{k=1}^{p-1} k^2 B_{p,k}B_{p,k-d} \equiv 4\left(-1\right)^{d}\left\{\frac{1}{3}d\left(2d^2+1\right)\left(4pH_d - 1\right) - p\left(\frac{26}{9}d^3 + \frac{4}{3}d^2 + \frac{7}{9}d + \frac{1}{2}\right)\right\} \pmod{p^2},$$

where a prime number $p > 3$ and $1 \le d \le p$.

2. Some congruences involving the generalized Catalan numbers and harmonic numbers

In this section, firstly, we will continue our work with the following lemmas.

Lemma 2.1. *Let $p > 3$ be a prime number. For $0 \le m < p-1$ and $1 \le d \le p-1$, we have*

$$\sum_{k=d}^{p-1} k^{\underline{m}} H_{k-d} \equiv (p+d-1)^{\underline{m}} \frac{p+d}{p\,(m+1)} - (d-1)^{\underline{m}} \frac{d}{m+1}\left(\frac{1}{p} + H_d - H_{m+1}\right) \pmod{p}.$$

Proof. Observed that

$$\sum_{k=1}^{p+d-2} k^{\underline{m}} H_{p+d-1-k} = \sum_{k=1}^{d-1} k^{\underline{m}} H_{p+d-1-k} + \sum_{k=d}^{p-1} k^{\underline{m}} H_{p+d-1-k} + \sum_{k=p}^{p+d-2} k^{\underline{m}} H_{p+d-1-k}.$$

From the congruences for $1 \le k,\ m < p$,

$$H_{p-k} \equiv H_{k-1} \pmod{p},\ (p+k)^{\underline{m}} \equiv k^{\underline{m}} \pmod{p} \text{ and}$$
$$H_{p+k} \equiv H_k + \frac{1}{p} \pmod{p}, \tag{6}$$

we have

$$\sum_{k=1}^{p+d-2} k^{\underline{m}} H_{p+d-1-k} \equiv \sum_{k=1}^{d-1} k^{\underline{m}} \left(H_{d-1-k} + \frac{1}{p} \right) + \sum_{k=d}^{p-1} k^{\underline{m}} H_{k-d} + \sum_{k=0}^{d-2} k^{\underline{m}} H_{d-1-k}$$
$$= 2\sum_{k=1}^{d-2} k^{\underline{m}} H_{d-1-k} + \frac{1}{p}\sum_{k=1}^{d-1} k^{\underline{m}} + \sum_{k=d}^{p-1} k^{\underline{m}} H_{k-d} \pmod{p}.$$

Considering (2) and (6), we write

$$\sum_{k=d}^{p-1} k^{\underline{m}} H_{k-d} \equiv \sum_{k=1}^{p+d-2} k^{\underline{m}} H_{p+d-1-k} - 2\sum_{k=1}^{d-2} k^{\underline{m}} H_{d-1-k} - \frac{1}{p}\sum_{k=1}^{d-1} k^{\underline{m}}$$
$$\equiv -(d-1)^{\underline{m}} \frac{d}{m+1} \left(H_d - H_{m+1} \right) + (p+d-1)^{\underline{m}} \frac{p+d}{p\,(m+1)}$$
$$- \frac{1}{p}\sum_{k=1}^{d-1} k^{\underline{m}} \pmod{p}.$$

It follows from the sum $\sum\limits_{k=a}^{n} k^{\underline{m}} = \frac{1}{m+1}\left((n+1)n^{\underline{m}} - a(a-1)^{\underline{m}} \right)$ that

$$\sum_{k=d}^{p-1} k^{\underline{m}} H_{k-d}$$
$$\equiv (p+d-1)^{\underline{m}} \frac{p+d}{p\,(m+1)} - (d-1)^{\underline{m}} \frac{d}{m+1} \left(\frac{1}{p} + H_d - H_{m+1} \right) \pmod{p}.$$

Thus, we complete the proof. □

Lemma 2.2. *Let p be an odd prime. For $1 < d \le p-1$, we have*

$$\sum_{k=d}^{p-1} \frac{1}{k} H_{k-d} \equiv H_{p+d-1}^2 - H_{p+d-1,2} - 2H_{d-1}^2 + 2H_{d-1,2} - \frac{2}{p} H_{d-1} \pmod{p}.$$

Proof. We write

$$\sum_{k=1}^{p+d-2} \frac{1}{k} H_{p+d-k-1} = \sum_{k=1}^{d-1} \frac{1}{k} H_{p+d-k-1} + \sum_{k=d}^{p-1} \frac{1}{k} H_{p+d-k-1} + \sum_{k=p}^{p+d-2} \frac{1}{k} H_{p+d-k-1}.$$

From the congruences $H_{p+k} \equiv H_k + \frac{1}{p} \pmod{p}$ and $H_{p-k} \equiv H_{k-1} \pmod{p}$ for $1 \le k < p$, we get

$$\sum_{k=1}^{p+d-2} \frac{1}{k} H_{p+d-k-1} \equiv \sum_{k=1}^{d-1} \frac{1}{k} \left(H_{d-k-1} + \frac{1}{p} \right) + \sum_{k=d+1}^{p-1} \frac{1}{k} H_{k-d}$$
$$+ \frac{H_{d-1}}{p} + \sum_{k=1}^{d-2} \frac{1}{p+k} H_{d-k-1} \pmod{p}.$$

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By the congruence $\frac{1}{p+k}\equiv\frac{1}{k} \pmod p$ for $1\leq k<p$, we have

$$\sum_{k=1}^{p+d-2}\frac{1}{k}H_{p+d-k-1}\equiv 2\sum_{k=1}^{d-2}\frac{1}{k}H_{d-k-1}+\sum_{k=d+1}^{p-1}\frac{1}{k}H_{k-d}+2\frac{H_{d-1}}{p} \pmod p.$$

Considering the sum

$$\sum_{k=1}^{n-1}\frac{1}{k}H_{n-k}=H_n^2-H_{n,2} \tag{7}$$

from (1), we get

$$H_{p+d-1}^2-H_{p+d-1,2}\equiv 2\left(H_{d-1}^2-H_{d-1,2}\right)+\sum_{k=d+1}^{p-1}\frac{1}{k}H_{k-d}+2\frac{H_{d-1}}{p} \pmod p.$$

Thus the desired result is obtained. □

Lemma 2.3. *For $n\in\mathbb{N}$, we have*

$$\sum_{k=1}^{n}(-1)^k k^2H_k=(-1)^n\binom{n+1}{2}H_n+(-1)^n\frac{n}{4}+\frac{1}{8}\left(1-(-1)^n\right).$$

Proof. Observed that

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k k^2H_k&=\sum_{k=1}^{n}(-1)^k k^2\sum_{j=1}^{k}\frac{1}{j}=\sum_{j=1}^{n}\frac{1}{j}\sum_{k=j}^{n}(-1)^k k^2\\&=\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{n}(-1)^k k^2-\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{j-1}(-1)^k k^2.\end{aligned}$$

Applying $\sum\limits_{k=1}^{n}(-1)^k k^2=(-1)^n\binom{n+1}{2}$, we write

$$\sum_{k=1}^{n}(-1)^k k^2H_k=(-1)^n\binom{n+1}{2}H_n+\frac{1}{2}\sum_{j=1}^{n}(-1)^j(j-1).$$

By the sums

$$\sum_{k=1}^{n}(-1)^k k=-\frac{1}{4}+\frac{(-1)^n}{2}\left(n+\frac{1}{2}\right)\quad\text{and}\quad\sum_{k=1}^{n}(-1)^k=\frac{(-1)^n-1}{2},$$

we conclude the proof. □

Now we give some congruences involving harmonic numbers in the following theorems.

Theorem 2.4. *Let p be an odd prime. For $1<d<p$, we have*

$$\sum_{k=1}^{p-1}\frac{B_{p,k}B_{p,k-d}}{k}\equiv 4(-1)^d\left\{H_d+H_{d-1}+\frac{2}{p}+p\left(4H_d^2+4H_{d-1}^2\right.\right.$$

$$-2H_{d,2}-2H_{d-1,2}-H_{p+d}^2-H_{p+d-1}^2\big)\big\} \pmod{p^2},$$

and for $3<d\leq p$,

$$\sum_{k=1}^{p-1}(-1)^k k^2 B_{p,k}B_{p,k-d}$$
$$\equiv -4d+4p\left\{1+\frac{(-1)^d}{2}+\binom{d+1}{2}\left(H_{\lfloor\frac{d-1}{2}\rfloor}+H_{\lfloor\frac{p+d-1}{2}\rfloor}\right)\right.$$
$$\left.-\binom{d}{2}\left(H_{\lfloor\frac{d-2}{2}\rfloor}+H_{\lfloor\frac{p+d-2}{2}\rfloor}\right)\right\} \pmod{p^2}.$$

Proof. From $B_{p,k-d}=-B_{p,d-k}$ for $k<d$, we write

$$\sum_{k=1}^{p-1}\frac{B_{p,k}B_{p,k-d}}{k}=\sum_{k=1}^{d}\frac{B_{p,k}B_{p,k-d}}{k}+\sum_{k=d+1}^{p-1}\frac{B_{p,k}B_{p,k-d}}{k}$$
$$=-\sum_{k=1}^{d-2}\frac{B_{p,k}B_{p,d-k}}{k}+\sum_{k=d+1}^{p-1}\frac{B_{p,k}B_{p,k-d}}{k}$$
$$-\frac{1}{p^2}\binom{2p}{p-1}\binom{2p}{p-d+1}.$$

For $1<k\leq p$ and $d+2\leq k$, using the congruence

$$B_{p,k-d}\equiv 2(-1)^{d-k-1}\left(1-p\left(H_{k-d}+H_{k-1-d}\right)\right) \pmod{p^2}, \tag{8}$$

in [9], we have

$$\sum_{k=1}^{p-1}\frac{B_{p,k}B_{p,k-d}}{k}$$
$$\equiv 4(-1)^d\left\{p\sum_{k=1}^{d-2}\frac{H_k+H_{k-1}}{k}+p\sum_{k=1}^{d-2}\frac{H_{d-k}+H_{d-k-1}}{k}\right.$$
$$\left.+\sum_{k=d+1}^{p-1}\frac{1}{k}-\sum_{k=1}^{d-2}\frac{1}{k}-p\sum_{k=d+1}^{p-1}\frac{H_{k-d}+H_{k-d-1}}{k}-p\sum_{k=d+1}^{p-1}\frac{H_k+H_{k-1}}{k}\right\}$$
$$-\frac{1}{p^2}\binom{2p}{p-1}\binom{2p}{p-d+1} \pmod{p^2}.$$

Using (4), (7) and Lemma 2.2, we write

$$\sum_{k=1}^{p-1}\frac{B_{p,k}B_{p,k-d}}{k}$$
$$\equiv 4(-1)^d\left\{pH_{d-2}^2+p\left(H_d^2+H_{d-1}^2-H_{d,2}-H_{d-1,2}-\frac{1}{d-1}\right)+H_{p-1}-H_d-H_{d-2}\right.$$
$$-p\left(H_{p+d-1}^2+H_{p+d}^2-H_{p+d-1,2}-H_{p+d,2}-2\left(H_{d-1}^2+H_d^2-H_{d-1,2}-H_{d,2}\right)\right)$$

$$+2(H_{d-1}+H_d)-p(H_{p-1}^2-H_d^2)\} - \frac{1}{p^2}\binom{2p}{p-1}\binom{2p}{p-d+1} \pmod{p^2}.$$

Since the congruences $H_{p+k,2} \equiv H_{k,2} + \frac{1}{p^2} \pmod p$ for $1 \le k \le p-1$ and $H_{p-1} \equiv 0 \pmod{p^2}$, we have

$$\begin{aligned}&\sum_{k=1}^{p-1} \frac{B_{p,k}B_{p,k-d}}{k}\\ &\equiv 4\,(-1)^d \left\{\frac{2}{p} - H_{d-2} + 2H_{d-1} + H_d + p\left(H_{d-2}^2 + 4H_d^2 - \frac{1}{d-1}\right)\right.\\ &\quad \left. -p\left(H_{p+d-1}^2 - 3H_{d-1}^2 + 2H_{d-1,2} + H_{p+d}^2 + 2H_{d,2}\right)\right\}\\ &\quad - \frac{1}{p^2}\binom{2p}{p-1}\binom{2p}{p-d+1} \pmod{p^2}.\end{aligned}$$

For $1 < d \le p$, taking the congruence

$$\begin{aligned}&\frac{d-1}{p^2}\binom{2p}{p-d+1}\binom{2p}{p-1}\\ &\equiv 4\,(-1)^d\left(1 + \frac{p}{d-1} + \frac{2p}{d} - p - 2pH_d\right) \pmod{p^2},\end{aligned} \tag{9}$$

in [9], we obtain the desired result. With the help of Lemma 2.3, the proof of the other result is given. □

For example, for $p \ge 5$ and $d = 4$ in Theorem 2.4, we obtain the congruences

$$\sum_{k=1}^{p-1} \frac{B_{p,k}B_{p,k-4}}{k} \equiv 4\left(\frac{47}{12} + \frac{2}{p} + p\left(\frac{1817}{72} - H_{p+4}^2 - H_{p+3}^2\right)\right) \pmod{p^2}$$

and

$$\sum_{k=1}^{p-1} (-1)^k\, k^2 B_{p,k}B_{p,k-4} \equiv -16 + 4p\left\{\frac{11}{2} + 10H_{\lfloor \frac{p+3}{2} \rfloor} - 6H_{\lfloor \frac{p+2}{2} \rfloor}\right\} \pmod{p^2}.$$

Theorem 2.5. *Let $p > 3$ be a prime number. For $0 < d \le p$ and $0 < m < p-1$, we have*

$$\begin{aligned}&\sum_{k=1}^{p-1} k^{\underline{m}} B_{p,k}B_{p,k-d}\\ &\equiv \frac{4\,(-1)^d}{m+1}\left\{p\,(m-1)!\,(-1)^m - (p+d)\,(p+d-1)^{\underline{m}} - (p+d+1)\,(p+d)^{\underline{m}}\right.\\ &\quad +2p\,(2H_d - H_m)\left((d+1)\,d^{\underline{m}} + d(d-1)^{\underline{m}}\right)\\ &\quad \left. -p\left(\frac{8d}{(m+1)}(d-1)^{\underline{m}} + \frac{m+1}{m}\left(d^{\underline{m}} + (d-1)^{\underline{m}}\right)\right)\right\} \pmod{p^2}.\end{aligned}$$

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Proof. Observed that

$$\sum_{k=1}^{p-1} k^{\underline{m}} B_{p,k} B_{p,k-d} = \sum_{k=1}^{d} k^{\underline{m}} B_{p,k} B_{p,k-d} + \sum_{k=d+1}^{p-1} k^{\underline{m}} B_{p,k} B_{p,k-d}.$$

From $B_{p,k-d} = -B_{p,d-k}$ for $k < d$, we get

$$\begin{aligned}\sum_{k=1}^{p-1} k^{\underline{m}} B_{p,k} B_{p,k-d} = &-\sum_{k=1}^{d-2} k^{\underline{m}} B_{p,k} B_{p,d-k} + \sum_{k=d+1}^{p-1} k^{\underline{m}} B_{p,k} B_{p,k-d} \\ &- \frac{d-1}{p^2} (d-1)^{\underline{m}} \binom{2p}{p-d+1} \binom{2p}{p-1}.\end{aligned}$$

It follows from (8) that

$$\begin{aligned}&\sum_{k=1}^{p-1} k^{\underline{m}} B_{p,k} B_{p,k-d} \\ &\equiv 4(-1)^d \left\{ -\sum_{k=1}^{d-2} k^{\underline{m}} + p \sum_{k=1}^{d-2} k^{\underline{m}} (H_k + H_{k-1}) + p \sum_{k=1}^{d-2} k^{\underline{m}} (H_{d-k} + H_{d-k-1}) \right. \\ &\qquad \left. + \sum_{k=d+1}^{p-1} k^{\underline{m}} - p \sum_{k=d+1}^{p-1} k^{\underline{m}} (H_{k-d} + H_{k-d-1}) - p \sum_{k=d+1}^{p-1} k^{\underline{m}} (H_k + H_{k-1}) \right\} \\ &\qquad - \frac{d-1}{p^2} (d-1)^{\underline{m}} \binom{2p}{p-d+1} \binom{2p}{p-1} \pmod{p^2}.\end{aligned}$$

According to (2), (3) and Lemma 2.1, using the sum

$$\sum_{k=a}^{n} k^{\underline{m}} = \frac{1}{m+1} \left((n+1) n^{\underline{m}} - a (a-1)^{\underline{m}} \right),$$

we obtain

$$\begin{aligned}&\sum_{k=1}^{p-1} k^{\underline{m}} B_{p,k} B_{p,k-d} \\ &\equiv 4(-1)^d \left\{ -\frac{(d-1)(d-2)^{\underline{m}}}{m+1} + 2p \left(\frac{(d-1)(d-2)^{\underline{m}}}{m+1} \left(H_{d-1} - \frac{1}{m+1} \right) \right) \right. \\ &+p \left(\frac{2(d+1) d^{\underline{m}}}{m+1} (H_{d+1} - H_{m+1}) + \frac{2d(d-1)^{\underline{m}}}{m+1} (H_d - H_{m+1}) - (d-1)^{\underline{m}} \right) \\ &-p \left(\frac{(p+d)(p+d-1)^{\underline{m}}}{p(m+1)} - \frac{d(d-1)^{\underline{m}}}{p(m+1)} + \frac{(p+d+1)(p+d)^{\underline{m}}}{p(m+1)} - \frac{(d+1) d^{\underline{m}}}{p(m+1)} \right) \\ &-p \left(\frac{2p(p-1)^{\underline{m}}}{m+1} \left(H_p - \frac{1}{m+1} \right) - 2 \frac{d+1}{m+1} d^{\underline{m}} \left(H_{d+1} - \frac{1}{m+1} \right) - \frac{(p-1)^{\underline{m}} - d^{\underline{m}}}{m} \right) \\ &\left. + \left(\frac{1}{m+1} \left(p(p-1)^{\underline{m}} - (d+1) d^{\underline{m}} \right) \right) - \frac{p}{m} (d-2)^{\underline{m}} \right\}\end{aligned}$$

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$$-\frac{d-1}{p^2}\,(d-1)^{\underline{m}}\binom{2p}{p-d+1}\binom{2p}{p-1}\pmod{p^2}.$$

From (5) and (9), we write

$$\sum_{k=1}^{p-1} k^{\underline{m}} B_{p,k}B_{p,k-d}$$
$$\equiv 4\,(-1)^d\left\{-\frac{(d-1)\,(d-2)^{\underline{m}}}{m+1}+2p\left(\frac{(d-1)\,(d-2)^{\underline{m}}}{m+1}\left(H_{d-1}-\frac{1}{m+1}\right)\right)\right.$$
$$+p\left(\frac{2\,(d+1)\,d^{\underline{m}}}{m+1}\,(H_{d+1}-H_{m+1})+\frac{2d(d-1)^{\underline{m}}}{m+1}\,(H_d-H_{m+1})-(d-1)^{\underline{m}}\right)$$
$$-\frac{p+d}{m+1}(p+d-1)^{\underline{m}}+\frac{d(d-1)^{\underline{m}}}{m+1}-\frac{p+d+1}{m+1}(p+d)^{\underline{m}}$$
$$-\frac{p}{m+1}m!(-1)^m-p\left(-2\frac{d+1}{m+1}d^{\underline{m}}\left(H_{d+1}-\frac{1}{m+1}\right)-\frac{1}{m}\left((-1)^m\,m!-d^{\underline{m}}\right)\right)$$
$$\left.-\,(d-1)^{\underline{m}}\left(\frac{p}{d-1}+\frac{2p}{d}+1-p-2pH_d\right)-\frac{p}{m}(d-2)^{\underline{m}}\right\}\pmod{p^2}.$$

Using the identities $\binom{n}{k}=\frac{n}{k}\binom{n-1}{k-1}$ and $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, we have

$$\sum_{k=1}^{p-1} k^{\underline{m}} B_{p,k}B_{p,k-d}$$
$$\equiv\frac{4\,(-1)^d}{m+1}\left\{p\,(m-1)!\,(-1)^m-(p+d)\,(p+d-1)^{\underline{m}}-(p+d+1)\,(p+d)^{\underline{m}}\right.$$
$$+2p\,(2H_d-H_m)\left((d+1)\,d^{\underline{m}}+d(d-1)^{\underline{m}}\right)$$
$$\left.-p\left(\frac{8d}{(m+1)}(d-1)^{\underline{m}}+\frac{m+1}{m}\left(d^{\underline{m}}+(d-1)^{\underline{m}}\right)\right)\right\}\pmod{p^2}.$$

This completes the proof. □

Corollary 2.6. *Let $p>3$ be a prime number. For $0<d\le p$, we have*

$$\sum_{k=1}^{p-1} k^2 B_{p,k}B_{p,k-d}\equiv 4\,(-1)^d\left\{\frac{1}{3}d\left(2d^2+1\right)(4pH_d-1)\right.$$
$$\left.-p\left(\frac{26}{9}d^3+\frac{4}{3}d^2+\frac{7}{9}d+\frac{1}{2}\right)\right\}\pmod{p^2}.$$

Proof. When $m=2$ in Theorem 2.5, using the congruence

$$\sum_{k=1}^{p-1} kB_{p,k}B_{p,k-d}\equiv 2\,(-1)^d\left(-2d^2+p\left(1+8d^2\,(H_d-1)-4d\right)\right)\pmod{p^2}$$

in [9], the proof easily is obtained. □

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For example, for $p > 3$, $d = 2$ in Corollary 2.6, we have

$$\sum_{k=1}^{p-1} k^2 B_{p,k} B_{p,k-2} \equiv 2\,(11p - 12) \pmod{p^2}.$$

References

[1] H. Alzer, D. Karayannakis, and H. M. Srivastava, *Series representations for some mathematical constants*, J. Math. Anal. Appl. **320** (2006), no. 1, 145–162.

[2] J. Choi and H. M. Srivastava, *Some summation formulas involving harmonic numbers and generalized harmonic numbers*, Math. Comput. Modelling **54** (2011), no. 9-10, 2220–2234.

[3] E. Deutsch and L. Shapiro, *A survey of the Fine numbers*, Discrete Math. **241** (2001), no. 1-3, 241–265.

[4] D. H. Greene and D. E. Knuth, *Mathematics for the analysis of algorithms*, third edition, Progress in Computer Science and Applied Logic, **1**, Birkhäuser Boston, Inc., Boston, MA, 1990.

[5] V. J. W. Guo and J. Zeng, *Factors of binomial sums from the Catalan triangle*, J. Number Theory **130** (2010), no. 1, 172–186.

[6] J. M. Gutiérrez, M. A. Hernández, P. J. Miana, and N. Romero, *New identities in the Catalan triangle*, J. Math. Anal. Appl. **341** (2008), no. 1, 52–61.

[7] P. Hilton and J. Pedersen, *Catalan numbers, their generalization, and their uses*, Math. Intelligencer **13** (1991), no. 2, 64–75.

[8] P. J. Miana and N. Romero, *Computer proofs of new identities in the Catalan triangle*, in Proceedings of the "Segundas Jornadas de Teoría de Números", 203–208, Bibl. Rev. Mat. Iberoamericana, Rev. Mat. Iberoamericana, Madrid, 2008.

[9] N. Ömür and S. Koparal, *Some congruences involving numbers $B_{p,k-d}$*, Util. Math. **95** (2014), 307–317.

[10] L. W. Shapiro, *A Catalan triangle*, Discrete Math. **14** (1976), no. 1, 83–90.

[11] J. Spiess, *Some identities involving harmonic numbers*, Math. Comp. **55** (1990), no. 192, 839–863.

[12] Z.-W. Sun, *On harmonic numbers and Lucas sequences*, Publ. Math. Debrecen **80** (2012), no. 1-2, 25–41.

Sibel Koparal
Department of Mathematics
Kocaeli University
İzmit 41380 Kocaeli, Turkey
Email address: `sibel.koparal@kocaeli.edu.tr`

Neşe Ömür
Department of Mathematics
Kocaeli University
İzmit 41380 Kocaeli, Turkey
Email address: `neseomur@kocaeli.edu.tr`

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