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BICOMPRESSIBLE SURFACES AND INCOMPRESSIBLE SURFACES

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Dedicated to Professor Tsuyoshi Kobayashi on the occasion of his 60th birthday

Abstract. We give new evidence that "complicated" Heegaard surfaces behave like incompressible surfaces. More precisely, suppose that a closed connected orientable 3-manifold M contains a closed connected incompressible surface F which separates M into two (connected) components M_1 and M_2. Let S be a Heegaard surface of M. Our result is that if the Hempel distance of S is at least four, then S is isotoped so that $S \cap M_i$ is incompressible for each $i = 1, 2$.

1. Introduction

When one studies topological and geometrical properties of 3-manifolds, sub-manifolds of codimension one, which are so-called surfaces, play crucial roles. If surfaces as sub-manifolds are *incompressible* (see below), then such 3-manifolds are extremely well understood. Another good point is that if an irreducible 3-manifold contains two such surfaces with non-empty intersection, then they are isotoped to be in a good position, which means that their intersection is essential in both surfaces.

Though *bicompressible surfaces* (see below) have properties opposite to incompressible surfaces, they are often useful to study 3-manifolds. One of the most important examples of bicompressible surfaces is a *Heegaard surface* of a closed orientable 3-manifold, which divides the 3-manifold into two handlebodies. Unlike incompressible surfaces, it is not always true that two Heegaard surfaces are isotoped to be in a good position. However, it is proved in [7] that if two Heegaard surfaces are "complicated", then they are isotoped to be in a good position. This implies that complicated Heegaard surfaces behave like incompressible surfaces. We here use (Hempel) *distance* in [3], given by a non-negative integer (see below), to measure the complexity of Heegaard surfaces.

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Then we can restate that Heegaard surfaces of distance at least two behave like incompressible surfaces.

There is another reason supporting the opinion above. Let F be a closed connected incompressible surface in a closed orientable 3-manifold M that separates M into two components, say M_1 and M_2. Let S be another closed connected separating surface in M which has non-empty intersection with F. It follows by the definition of incompressibility that if S is also incompressible, then S is isotoped so that $S \cap M_i$ is incompressible for each $i = 1, 2$. A conclusion similar to the above is obtained by Kobayashi-Qiu [6]: if S is a Heegaard surface of distance at least two, then S is isotoped so that $S \cap M_i$ is incompressible for *some* $i = 1, 2$.

In this paper, we obtain the completely same conclusion by imposing more rigorous requirements: if S is a Heegaard surface of distance at least *four*, then S is isotoped so that $S \cap M_i$ is incompressible for *each* $i = 1, 2$. We give a stronger version of the result in the following section. In the interest of fairness, the author adds the reference that a parallel result is obtained in [4] around the same time.

2. Definitions and results

Throughout this paper, we work in piecewise linear category. Let M be a compact connected orientable 3-manifold. We say that two surfaces properly embedded in M have *non-empty intersection* if they cannot be isotoped to be disjoint.

The key tool in this paper is a well-known ambient isotopy, which is called an *isotopy of type A* by Jaco [5, Chapter II]. Let S and F be surfaces properly embedded in M such that S and F intersect transversely. Suppose that there is a disk δ in M which satisfies the following three conditions: (i) $\delta \cap (S \cup F) = \partial\delta$, (ii) $\partial\delta \cap F$ is a single arc, say α, and (iii) $\partial\delta \cap S$ is a single arc, say β. Then we can consider an ambient isotopy of M so that $\beta \subset S$ goes to α along δ and is further pushed out (cf. Fig. 1). Let S' be the resulting surface obtained from S. Then we say that S' is obtained from S by an *isotopy of type A* along δ.

Suppose that S is a compact orientable surface properly embedded in M. We say that a simple closed curve or a simple arc properly embedded in S is *inessential* in S if it cuts off a disk from S. A simple closed curve or a simple arc is said to be *essential* in S if it is not inessential in S. The surface S is said to be *compressible* in M if there is an embedded disk D in M such that $D \cap S = \partial D$ and ∂D is essential in S. Such a disk D is called a *compressing disk* for S. We say that S is *incompressible* in M if S is not compressible in M. If S cuts off a sub-manifold M' from M and no compressing disks for S are contained in M', then S is said to be *incompressible toward* M'. Suppose that S is connected and separates M into two components V and W. Then S is *bicompressible* if there are compressing disks for S in both V and W. Given a closed bicompressible surface, say S again, let $\mathcal{C}_V$ and $\mathcal{C}_W$ be the sets of

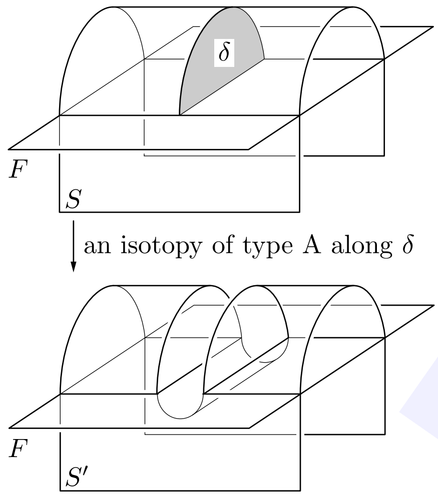

FIGURE 1

isotopy classes of the boundary components of compressing disks for S in V and W respectively. If the genus of S is at least two, then the *distance* $\mathrm{d}(S)$ is defined to be the minimal non-negative integer n so that there is a sequence of simple closed curves $c_0, \ldots, c_n$ in S such that $[c_0] \in \mathcal{C}_V$, $[c_n] \in \mathcal{C}_W$, and c_{i-1} is isotoped to be disjoint to c_i for any $i = 1, \ldots, n$. If the genus of S is one, then we define $\mathrm{d}(S) = -\infty$ for convenience.

Theorem 2.1. *Suppose that a compact connected orientable irreducible* 3-*manifold M contains a closed connected incompressible surface F which separates M into two components M_1 and M_2. Let S be a closed bicompressible surface in M which has non-empty intersection with F. If $\mathrm{d}(S) \geq 3$, then either*

(1) $\mathrm{d}(S) = 3$ *and S is isotoped so that $S \cap F$ is a simple closed curve which is essential in both S and F, and that $S \cap M_i$ is incompressible for some $i = 1, 2$, or*
(2) *S is isotoped so that $S \cap M_i$ is incompressible for each $i = 1, 2$.*

In the theorem above, suppose that M is closed and that S is a Heegaard surface. Then if M is reducible, we see that $\mathrm{d}(S) = 0$ by Haken's lemma [2]. It is also well-known that any closed incompressible surface has non-empty intersection with S. Hence we immediately have the following.

Corollary 2.2. *Suppose that a closed connected orientable* 3-*manifold M contains a closed connected incompressible surface F which separates M into two components M_1 and M_2. Let S be a Heegaard surface in M. If $\mathrm{d}(S) \geq 4$, then S is isotoped so that $S \cap M_i$ is incompressible for each $i = 1, 2$.*

3. Proof of Theorem 2.1

The special case of [1, Corollary 3.8] says that if $\mathrm{d}(S) \geq 2$, then S and F can be isotoped so that each component of $S \cap F$ is essential in both surfaces. Hence we may assume that $S \cap F$ is a collection of simple closed curves which are essential in both S and F under our assumption that $\mathrm{d}(S) \geq 3$. Suppose that the conclusion (2) in Theorem 2.1 does not hold, i.e., $S \cap M_i$ is compressible for some $i = 1, 2$. Let V and W be 3-manifolds obtained by cutting M along S.

Lemma 3.1. *Both $S \cap M_1$ and $S \cap M_2$ are incompressible toward $V \cap M_1$ and $V \cap M_2$, or toward $W \cap M_1$ and $W \cap M_2$ respectively.*

Proof. If $S \cap M_i$ admits compressing disks in both $V \cap M_i$ and $W \cap M_i$, then we have $\mathrm{d}(S) \leq 2$. Hence $S \cap M_i$ is incompressible toward $V \cap M_i$ or $W \cap M_i$. This implies that if $S \cap M_1$ or $S \cap M_2$ is incompressible in M_1 or M_2 respectively, then the lemma holds. The remaining case is that $S \cap M_1$ and $S \cap M_2$ are compressible in M_1 and M_2 respectively. If $S \cap M_1$ contains a compressing disk in $V \cap M_1$ ($W \cap M_1$ resp.) and $S \cap M_2$ contains a compressing disk in $W \cap M_2$ ($V \cap M_2$ resp.), then we have $\mathrm{d}(S) \leq 1$, a contradiction. Thus we have the desired conclusion. □

At this moment, S and F satisfy the following condition $(*)$ under the assumption that the conclusion (2) in Theorem 2.1 does not hold.

$(*)$ Each component of $S \cap F$ is essential in both surfaces, and S satisfies one of the three conditions (cf. Fig. 2):

- (i) $S \cap M_1$ admits a compressing disk in $V \cap M_1$ and is incompressible toward $W \cap M_1$, and $S \cap M_2$ is incompressible in M_2.
- (ii) $S \cap M_1$ is incompressible in M_1, and $S \cap M_2$ admits a compressing disk in $V \cap M_2$ and is incompressible toward $W \cap M_2$.
- (iii) Both $S \cap M_1$ and $S \cap M_2$ admit compressing disks in $V \cap M_1$ and $V \cap M_2$ respectively, and are incompressible toward $W \cap M_1$ and $W \cap M_2$ respectively.

We now take a compressing disk D for S in W such that $|D \cap F|$ is minimal among all the bicompressible surfaces which are isotopic to S and satisfy the condition $(*)$, where $|\cdot|$ means the number of connected components. Since F is incompressible, it follows from standard arguments that $D \cap F$ consists of arcs properly embedded in D. Let $\delta \subset D$ be an outermost disk component of D cut along $D \cap F$, i.e., δ is a disk component of D cut along $D \cap F$ such that $\partial\delta$ consists of two arcs α and β with $\alpha \in (D \cap F)$ and $\beta \subset \partial D$. Without loss of generality, we may assume that δ is contained in $W \cap M_1$. Then the following is proved in [6, Claim 2 in the proof of Proposition 2.6]:

Lemma 3.2 ([6, Claim 2 in the proof of Proposition 2.6])**.** *β is essential in $S \cap M_1$.*

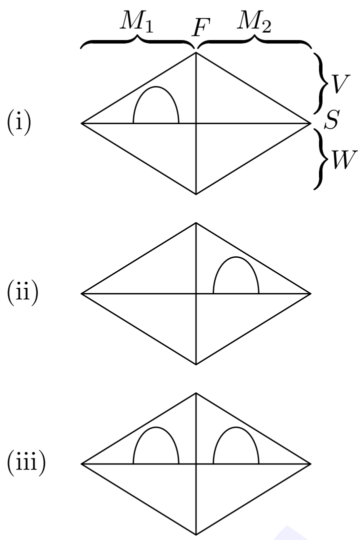

FIGURE 2

Let S' be a surface obtained from S by an isotopy of type A along δ. Let V' and W' be 3-manifolds obtained from V and W respectively by an influence of the isotopy above. We note that $M = V \cup_S W = V' \cup_{S'} W'$. Set $D' = D \cap W'$. Then D' is a compressing disk for S' in W', and $|D' \cap F| < |D \cap F|$. It follows from Lemma 3.2 that each component of $S' \cap F$ is essential in S'. The following is also obtained as [6, Claim 3 in the proof of Proposition 2.6]:

Lemma 3.3 ([6, Claim 3 in the proof of Proposition 2.6])**.** *Each component of $S' \cap F$ is essential in F.*

Hence the minimality of $|D \cap F|$ shows that (S', V', W') does not satisfy all the cases from (i) to (iii) in the condition $(*)$. We here give brief remarks that (a) if $S \cap M_1$ is incompressible in M_1, then so is $S' \cap M_1$, (b) if $S \cap M_1$ is compressible in M_1, then there are both possibilities that $S' \cap M_1$ is compressible or is incompressible, (c) if $S \cap M_2$ is compressible in M_2, then so is $S' \cap M_2$, and (d) if $S \cap M_2$ is incompressible in M_2, then there are both possibilities that $S' \cap M_2$ is compressible or is incompressible. Since $\mathrm{d}(S') = \mathrm{d}(S) \geq 3$ and hence S' also satisfies Lemma 3.1, we see from the arguments above that S satisfies the case (i) in the condition $(*)$ and $S' \cap M_2$ admits a compressing disk in $W' \cap M_2$ and is incompressible toward $V' \cap M_2$ (cf. Fig. 3).

Let c_V be the boundary of a compressing disk for $S \cap M_1$ in $V \cap M_1$ and c_W the boundary of a compressing disk for $S' \cap M_2$ in $W' \cap M_2$. If $|S \cap F| \geq 2$ and $|S' \cap F| \geq 2$, then there is a component of $S \cap F$ or $S' \cap F$ disjoint from $c_V \cup c_W$ under natural identification of S and S'. This implies that $\mathrm{d}(S) \leq 2$,

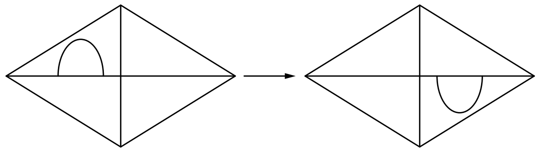

FIGURE 3

a contradiction. Hence we see that either $S \cap F$ or $S' \cap F$ is a simple closed curve which is essential in both surfaces. Let c_1 be a component of $S \cap F$ and c_2 a component of $S' \cap F$. Since we assume that $\mathrm{d}(S) \geq 3$, the sequence c_V, c_1, c_2, c_W gives $\mathrm{d}(S) = 3$ under natural identification of S and S'. Hence we have the conclusion (1) in Theorem 2.1.

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