

CONJECTURES THAT SOLVABLE GROUPS WHOSE CHARACTER GRAPHS HAVING DIAMETER 3 SATISFY

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ABSTRACT. In this note, we prove that Gluck's conjecture, Isaacs-Navarro-Wolf Conjecture and Taketa's inequality are true for solvable groups whose character graphs having diameter 3.

1. Introduction

Let G be a finite group and $\text{cd}(G)$ be the set of irreducible character degrees of G . In [9], character graph related to $\text{cd}(G)$, denoted by $\Delta(G)$, was defined. It plays an important role in the study of the influence of $\text{cd}(G)$ on the structure of G . The vertex set for $\Delta(G)$ is $\rho(G)$, which is the set of all the prime divisors of $\text{cd}(G)$, and there is an edge between two vertices p and q if pq divides some degree in $\text{cd}(G)$. For finite solvable group G , there are many fruitful research results on $\Delta(G)$. One of the most important results for $\Delta(G)$ belongs to Pálffy's Three Prime Theorem, see [10], which saying that for any three vertices of $\Delta(G)$, there exists at least one edge adjacent to two of them. By using Three Prime Theorem, for every solvable group G , one can easily get that $\Delta(G)$ has at most two connected components, and when $\Delta(G)$ is connected, the diameter is no more than 3. After that, Mark L. Lewis classified solvable groups whose character graphs have two components, see [6]. He also gave the first example of solvable group whose character graph having diameter 3, see [7]. Furthermore, the authors independently proved in [11] and [1] respectively that solvable group whose character graph having diameter 3 has a unique normal non-abelian Sylow subgroup.

There are many open problems on character theory of finite solvable groups. Write $b(G)$ for the largest irreducible character degree of G , $F(G)$ is the Fitting subgroup of G . Gluck conjectured in [2] that the inequality $|G : F(G)| \leq b(G)^2$ is true for solvable groups, which is known as Gluck Conjecture. Another

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is known as Isaacs-Navarro-Wolf Conjecture. As usual, write $\text{Irr}(G)$ as the set of complex irreducible characters of G . Write $V(G) = \langle g \in G \mid \chi(g) \neq 0, \forall \chi \in \text{Irr}(G) \rangle$, which is called the strongly vanishing-off subgroup of G . Isaacs-Navarro-Wolf Conjecture said that $V(G) \leq F(G)$ is true for solvable group G . The last conjecture here is a long standing conjecture by G. Seitz, saying that if G is a solvable group, then $dl(G) \leq |\text{cd}(G)|$, where $dl(G)$ is the derived length of G , which is now known as Taketa's inequality.

Some results have been achieved on these conjectures. For example, it was proved that Gluck Conjecture, Isaacs-Navarro-Wolf Conjecture and Takeata's inequality are true for non-connected solvable groups in [3] and [5] respectively. Bases on these results, in this paper, we prove that these conjectures are also true for solvable groups whose character graphs having diameter 3.

2. Preliminary

In this part, we fix some notations that used in finite solvable groups whose character graphs having diameter 3 and list some theorems that will be used in the next part.

Let G be a solvable group and $\Delta(G)$ has diameter 3. We use the partition of $\rho(G)$ as the author did in Definition 1.5 of [11]. Select $p_1, p_4 \in \rho(G)$ such that they have distance 3. Let ρ_4 be the set of all vertices that are distance 3 from the vertex p_1 . Clearly, $p_4 \in \rho_4$. Let ρ_3 be the set of all vertices that are distance 2 from the vertex p_1 . Let ρ_2 be the set of all vertices that are adjacent to vertex p_1 and adjacent to some prime in ρ_3 . And ρ_1 be the vertices consisting of p_1 and those that are adjacent to p_1 and not adjacent to anything in ρ_3 . Clearly, $\rho(G) = \rho_1 \cup \rho_2 \cup \rho_3 \cup \rho_4$.

The next two results tell us the normal structure of G when $\Delta(G)$ having diameter 3. By using them, we can change the diameter 3 situation to the non-connected situation.

Theorem 2.1 ([11, Theorem 3]). *Let G be a solvable group with $\Delta(G)$ having diameter 3. Then G has a normal non-abelian Sylow p -subgroup for exactly one prime p and $p \in \rho_3$.*

Corollary 2.2. *Let G be a solvable group with $\Delta(G)$ having diameter 3. And suppose that P is the unique normal non-abelian Sylow p -subgroup of G . Then $\Delta(G/P')$ is non-connected.*

Proof. First we claim that if G/P' has normal Sylow q -subgroup QP'/P' , where Q is some Sylow q -subgroup of G , $q \neq p$, then $Q \triangleleft G$. Suppose $QP'/P' \triangleleft G/P'$. Note that $P/P' \triangleleft G/P'$, we have $QP = QP'P \triangleleft G$ and QP'/P' and P/P' centralizes each other. In particular, Q centralizes P/P' and so $P = C_P(Q)P'$. Since $P' \leq \Phi(P)$, we have $P = C_P(Q)$ and so P and Q centralizes each other. Observe that $Q \text{ char } QP \triangleleft G$, it follows that $Q \triangleleft G$.

By using the claim above and Itô's theorem, we have $\rho(G/P') = \rho(G) \setminus \{p\}$. Recall that $p_1, p_4 \in \rho(G)$ has distance 3 in $\Delta(G)$ and $p \in \rho_3$. Now $p_1, p_4 \in$

$\rho(G/P')$ as the vertices of $\Delta(G/P')$, which is a subgraph of $\Delta(G)$, the distance between them is no less than 3. So that $\Delta(G/P')$ either has diameter 3 or is non-connected. On the other hand, suppose G/P' has a normal non-abelian Sylow subgroup, then by using the claim above again, we know G has other normal non-abelian Sylow subgroup besides P , violating Theorem 2.1. So G/P' has no normal non-abelian Sylow subgroup. In particular, by using Theorem 2.1 again, it follows that $\Delta(G/P')$ can not have diameter 3. And so $\Delta(G/P')$ is non-connected. \square

The following two theorems are about non-connected solvable groups, which saying that they satisfy Gluck Conjecture, Isaacs-Navarro-Wolf Conjecture and Takeata's inequality.

Theorem 2.3 ([3, Theorems A and B]). *Let G be a solvable group whose character graph $\Delta(G)$ is non-connected. Then $|G : F(G)| \leq b(G)^2$ and $V(G) \leq F(G)$ hold.*

Observe that the character graph defined in [5] is different from here in this paper, but it is well known that they have the same number of components and so we state the Theorem A of [5] in the form of our graph here.

Theorem 2.4 ([5, Theorem A]). *Let G be a solvable group whose character graph $\Delta(G)$ is non-connected. Then $dl(G) \leq |cd(G)|$ holds.*

3. Main result

Theorem 3.1. *Let G be a solvable group whose character graph $\Delta(G)$ has diameter 3. Then*

- (1) G satisfies Gluck Conjecture, i.e., $|G : F(G)| \leq b(G)^2$;
- (2) G satisfies Isaacs-Navarro-Wolf Conjecture, i.e., $V(G) \leq F(G)$;
- (3) G satisfies Takeata's inequality, i.e., $dl(G) \leq |cd(G)|$.

Proof. Let G be a solvable group with $\Delta(G)$ having diameter 3. And suppose that P is the unique normal non-abelian Sylow subgroup of G . Note that $P' \leq \Phi(P)$, $P \triangleleft G$, we have $P' \leq \Phi(G)$. Consider $F(G/P')$. Suppose $F(G/P') = X/P'$. Clearly, $F(G) \leq X$ since $F(G)/P'$ is nilpotent. On the other hand, observe that $F(G/P'/\Phi(G/P')) = F(G/P')/\Phi(G/P') = X/P'/\Phi(G)/P' \cong X/\Phi(G)$ is nilpotent, so that $X/\Phi(G) \leq F(G/\Phi(G)) = F(G)/\Phi(G)$ and $X \leq F(G)$ follows. Now we have $X = F(G)$ and $F(G/P') = F(G)/P'$. Observe that by Corollary 2.2, $\Delta(G/P')$ is non-connected.

(1) Since $\Delta(G/P')$ is non-connected, using Theorem 2.3 to G/P' , we know $|G/P' : F(G/P')| \leq b(G/P')^2$. Recall that $F(G/P') = F(G)/P'$ and $b(G/P') \leq b(G)$, we have $|G : F(G)| = |G/P' : F(G/P')| \leq b(G/P')^2 \leq b(G)^2$, and $|G : F(G)| \leq b(G)^2$ follows.

(2) Using Theorem 2.3 to G/P' , we also have $V(G/P') \leq F(G/P')$. Observe that $F(G/P') = F(G)/P'$ and $V(G)P'/P' \leq V(G/P')$, we have $V(G)P'/P' \leq V(G/P') \leq F(G/P') = F(G)/P'$ and so $V(G) \leq F(G)$ holds.

(3) Since M-groups satisfy Taketa's inequality (see Theorem 5.12 in [4]). In particular, it holds for p -group P . And we have $dl(P) = dl(P') + 1 \leq |\text{cd}(P)| = |\text{cd}(P | P')| + 1$ and so $dl(P') \leq |\text{cd}(P | P')|$ follows. Applying Theorem 2.4 to G/P' , we get that $dl(G/P') \leq |\text{cd}(G/P')|$. Now $dl(G) = dl(G/P') + dl(P') \leq |\text{cd}(G/P')| + |\text{cd}(P | P')| \leq |\text{cd}(G/P')| + |\text{cd}(G | P')|$ since $|\text{cd}(P | P')| \leq |\text{cd}(G | P')|$. Observe that $|\text{cd}(G/P')| + |\text{cd}(G | P')| = |\text{cd}(G)|$ since $\text{cd}(G/P') \cap \text{cd}(G | P') = \emptyset$, we have $dl(G) \leq |\text{cd}(G)|$.

The proof is now completed. \square

Note. In [8], the authors gave a proof that G satisfy Taketa's inequality when $\Delta(G)$ having diameter 3. But our proof here is short.

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