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ON LACUNARY RECURRENCES WITH GAPS OF LENGTH FOUR AND EIGHT FOR THE BERNOULLI NUMBERS

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Abstract. The problem of finding fast computing methods for Bernoulli numbers has a long and interesting history. In this paper, the author provides new proofs for two lacunary recurrence relations with gaps of length four and eight for the Bernoulli numbers. These proofs invoked the fact that the nth powers of π^2, π^4 and π^8 can be expressed in terms of the nth elementary symmetric functions.

1. Introduction

The rational numbers B_n $(n = 0, 1, 2, \ldots)$ defined by the Laurent series expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \qquad |x| < 2\pi,$$

are well-known in literature as the Bernoulli numbers. Rewriting this relation as

$$\left(\sum_{n=1}^{\infty} \frac{x^n}{n!} \right) \left(\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \right) = x, \qquad |x| < 2\pi,$$

and considering the Cauchy product for series, we derive the classical recurrence relation for the Bernoulli numbers B_n $(n = 0, 1, 2, \ldots)$:

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = \delta_{0,n},$$

where $\delta_{i,j}$ is the Kronecker delta [8, p. 229]. In this relation nearly half the terms on the left hand side do not contribute anything since $B_{2k+1} = 0$ for $k > 0$. On the other hand, it is well-known that $B_0 = 1$, $B_1 = -1/2$ and $(-1)^{k+1} B_{2k} > 0$ for $k > 0$.

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The sequence of Bernoulli numbers is one of the most important number sequences in mathematics having extensive applications on many areas. Recurrence relations for the computation of these numbers have been the subject of a large number of papers. In 1986, Namias [13] conjectured that an infinite number of recurrence relations for Bernoulli numbers can be obtained. One year later, Belinfante [1] published an infinite number of recurrence relations for the Bernoulli numbers, namely

$$B_n = \frac{1}{m(1-m^n)} \sum_{k=0}^{n-1} m^k \binom{n}{k} B_k \sum_{j=1}^{m-1} j^{n-k}, \qquad m > 1.$$

In 1993, Howard [5] showed that this relation is not new; it is a special case of the multiplication theorem for Bernoulli polynomials: If n and m are positive integers with $m > 1$, then

$$B_n(mx) = \frac{1}{m^{n-1}} \sum_{j=0}^{m-1} B_n\left(x + \frac{j}{m}\right).$$

Very recently, Merca [11] uses the generating function for the Bernoulli polynomials to introduce a number of infinite families of linear recurrence relations for the Riemann zeta function at positive even integer arguments, $\zeta(2n)$. So by [11, Theorems 2.1 and 3.1], with x replaced by 0, we derive two infinite families of linear recurrence relations for the Bernoulli numbers:

$$\sum_{k=0}^{n} \left(\alpha^{n-k} - (-1)^k(\alpha-1)^{n-k}\right) \binom{n}{k} B_k = 0$$

and

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \alpha^{2k} B_{n-2k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{(\alpha-1)^k + (-1-\alpha)^k}{2} B_{n-k},$$

where n is a positive integer and α is a real or complex number.

Recurrence formulas of this kind have the disadvantage of demanding the previous knowledge of all (non-zero) $B_0, B_1, \ldots, B_{n-1}$ for the computation of the nth Bernoulli numbers. The problem of finding fast computing methods for Bernoulli numbers has a long and interesting history. Using different methods, Ramanujan [14], Lehmer [9], Riordan [15, pp. 138–140], Chellali [3], Yalavigi [18], Berndt [2], and Howard [6, 7] have all worked out lacunary recurrence formulas for the Bernoulli numbers. Ramanujan's paper [14] contains linear recurrence relations for Bernoulli numbers, where the indices have gaps of lengths 4, 6, 8, and 10. The following relations, due to Ramanujan,

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 2^{n-2k} \binom{2n+2}{4k+2} B_{2n-4k} = (-1)^{\lfloor n/2 \rfloor}(n+1) \tag{1}$$

is a classical example of a lacunary recurrence for the Bernoulli numbers. This recurrence relation has gaps of length four. For example, to compute B_{4n}, it is not necessary to know the non-zero values of B_j for all $j < 4n$; we need only to know the values of B_{4j}. We remark that this lacunary recurrence was obtained later by Lehmer [9] as a special case of a more general result. Another lacunary recurrence relation with gaps of length four derived by Lehmer in [9] is given by

$$\sum_{k=0}^{\lfloor n/2\rfloor} \left((-1)^k 2^{2k+1}+1\right)\binom{2n+4}{4k+4} B_{2n-4k} = \left((-1)^{\lfloor n/2\rfloor}2^{n+1}+1\right)\frac{n+2}{2}.$$

The formulas become more complex, and difficult to obtained and write down, as the gaps increase in size. By Ramanujan's paper [14], it is an easy exercise to derive the following complete set of lacunary recurrence formulas with gaps of length eight:

$$\sum_{k=0}^{n}(-1)^k\alpha_{4k+2}\binom{8n+4}{8k+4}B_{8n-8k} = (-1)^n(2n+1)\alpha_{4n+2},$$

$$\sum_{k=0}^{n}(-1)^k\alpha_{4k+2}\binom{8n+6}{8k+4}B_{8n+2-8k} = (-1)^n\frac{4n+3}{2}\alpha_{4n+3},$$

$$\sum_{k=0}^{n}(-1)^k\alpha_{4k+2}\binom{8n+8}{8k+4}B_{8n+4-8k} = (-1)^{n+1}\sqrt{2}(n+1)\beta_{4n+3},$$

$$\sum_{k=0}^{n}(-1)^k\alpha_{4k+2}\binom{8n+10}{8k+4}B_{8n+6-8k} = (-1)^n\frac{4n+5}{2\sqrt{2}}\beta_{4n+4},$$

where

$$\alpha_n = \left(1+\frac{1}{\sqrt{2}}\right)^n + \left(1-\frac{1}{\sqrt{2}}\right)^n \quad \text{and} \quad \beta_n = \left(1+\frac{1}{\sqrt{2}}\right)^n - \left(1-\frac{1}{\sqrt{2}}\right)^n.$$

A more concise example for the lacunary recurrence relation with gaps of length eight is given by Lehmer in [9]:

$$\sum_{k=0}^{\lfloor n/4\rfloor} 2^{n+1-2\lfloor\frac{n+1}{4}\rfloor-2k} a_{4k+2}\binom{2n+4}{8k+4}B_{2n-8k} = (-1)^{\lfloor n/2\rfloor}(n+2)a_{n+2},$$

where

$$a_n + 34a_{n-4} + a_{n-8} = 0,$$

with the initial condition

$$\{a_n\}_{0\leqslant n\leqslant 7} = \{2, 0, 3, 10, 14, -12, -99, -338\}.$$

In this paper, motivated by these results, we shall provide new proofs for two lacunary recurrence relations for the Bernoulli numbers. The first has gaps of length four.

Theorem 1.1. *For $n \geqslant 0$,*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \left(2^{2n-2k} - 2^{2k+1}\right) \binom{2n+2}{4k+2} B_{2n-4k} = (-1)^{n+1}(n+1).$$

The second recurrence is more involved and has gaps of length eight.

Theorem 1.2. *For $n \geqslant 0$,*

$$\sum_{k=0}^{\lfloor n/4 \rfloor} (-1)^k b_{4k+2} (2^{2n-2k} - 2^{6k+1}) \binom{2n+4}{8k+4} B_{2n-8k}$$
$$= (-1)^{n+1}(n+2)\frac{b_{2n+3} - c_{2n+3}}{8},$$

where

$$b_n = 2b_{n-1} + b_{n-2} \quad \text{and} \quad c_n = 2c_{n-1} - 3c_{n-2},$$

with the initial conditions

$$b_0 = b_1 = 1 \quad \text{and} \quad c_0 = c_1 = 1.$$

The recurrence formula in Theorems 1.1 and 1.2 are not new, they already appear in [9], where Lehmer [9] defines the sequence $\{R_n\}_{n \geqslant 0}$ by:

$$\sum_{n=0}^{\infty} R_n \frac{x^n}{n!} = \frac{xe^x}{e^{2x} - 1} = \frac{e^x}{e^x - 1} - \frac{1}{2}\frac{2x}{e^{2x} - 1}.$$

It is clear that

$$R_n = (1 - 2^{n-1})B_n.$$

So Theorem 1.1 is exactly formula 3 of [9, Eq. (15), p. 643] and Theorem 1.2 is exactly recurrence formula 3 with gaps of 8 on page 645 of [9].

Related to Theorem 1.2, we remark that b_n is the first differences of the Pell numbers, i.e.,

$$b_n = P_{n+1} - P_n.$$

It is well-known that the Pell numbers

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$$

satisfy the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2}$$

with $P_0 = 0$ and $P_1 = 1$. It is an easy exercise to prove that

$$b_n = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2} \quad \text{and} \quad c_n = \frac{(1+\sqrt{-2})^n + (1-\sqrt{-2})^n}{2}.$$

2. Proof of Theorem 1.1

In order to prove this theorem, we consider that the nth powers of π^2 and π^4 can be expressed in terms of the nth elementary symmetric function [10,12] as follows:

$$e_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right)=\frac{\pi^{2n}}{(2n+1)!},\qquad n\geqslant 0 \tag{2}$$

and

$$e_n\left(\frac{1}{1^4},\frac{1}{2^4},\frac{1}{3^4},\dots\right)=\frac{2^{2n+1}\cdot\pi^{4n}}{(4n+2)!},\qquad n\geqslant 0. \tag{3}$$

On the other hand, according to Merca [10], the Riemann zeta function with even arguments $\zeta(2n)$ can be expressed in terms of the nth complete homogeneous symmetric function of the numbers $\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots$, as follows:

$$h_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right)=2\left(1-\frac{2}{2^{2n}}\right)\zeta(2n),\quad n\geqslant 0. \tag{4}$$

These relations allow us to consider the following two identities of formal power series in t:

$$\sum_{n=0}^{\infty}e_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right)t^n=\prod_{n=1}^{\infty}\left(1+\frac{t}{n^2}\right)$$

and

$$\sum_{n=0}^{\infty}h_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right)t^n=\prod_{n=1}^{\infty}\left(1-\frac{t}{n^2}\right)^{-1}.$$

We have

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}(-1)^n e_n\left(\frac{1}{1^4},\frac{1}{2^4},\frac{1}{3^4},\dots\right)t^{2n}\right)\left(\sum_{n=0}^{\infty}h_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right)t^n\right)\\&=\prod_{n=1}^{\infty}\left(1+\frac{t}{n^2}\right)\\&=\sum_{n=0}^{\infty}e_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right)t^n.\end{aligned}$$

Considering the well-known Cauchy products of two power series, we can write

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}(-1)^n e_n\left(\frac{1}{1^4},\frac{1}{2^4},\frac{1}{3^4},\dots\right)t^{2n}\right)\left(\sum_{n=0}^{\infty}h_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right)t^n\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k e_k\left(\frac{1}{1^4},\frac{1}{2^4},\frac{1}{3^4},\dots\right)h_{n-2k}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right)\right)t^n.\end{aligned}$$

So we deduce the identity

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k e_k\left(\frac{1}{1^4},\frac{1}{2^4},\frac{1}{3^4},\ldots\right)h_{n-2k}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right)=e_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right).$$

Taking into account the relations (2)-(4) and the famous formula for the even-argument ζ-values [17]

$$\zeta(2n)=(-1)^{n+1}\frac{(2\pi)^{2n}}{2\cdot(2n)!}B_{2n},$$

we obtain the identity

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^{n+1-k}\frac{2^{2n-2k+1}}{(4k+2)!(2n-4k)!}\left(1-\frac{2}{2^{2n-4k}}\right)B_{2n-4k}=\frac{1}{(2n+1)!}.$$

The proof follows easily.

3. Proof of Theorem 1.2

The proof of this theorem is quite similar to that of Theorem 1.1. In addition, we consider [12] that the nth power of π^8 can be expressed in terms of the nth elementary symmetric function of the numbers $\frac{1}{1^8},\frac{1}{2^8},\frac{1}{3^8},\ldots$, as follows:

$$e_n\left(\frac{1}{1^8},\frac{1}{2^8},\frac{1}{3^8},\ldots\right)=\frac{2^{6n+3}\cdot\pi^{8n}}{(8n+4)!}b_{4n+2}. \tag{5}$$

We have

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty}(-1)^n e_n\left(\frac{1}{1^8},\frac{1}{2^8},\frac{1}{3^8},\ldots\right)t^{4n}\right)\left(\sum_{n=0}^{\infty}h_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right)t^n\right)\\
&=\prod_{n=1}^{\infty}\left(1-\frac{t^4}{n^8}\right)\left(1-\frac{t}{n^2}\right)^{-1}\\
&=\prod_{n=1}^{\infty}\left(1+\frac{t^2}{n^4}\right)\left(1+\frac{t}{n^2}\right)\\
&=\left(\sum_{n=0}^{\infty}e_n\left(\frac{1}{1^4},\frac{1}{2^4},\frac{1}{3^4},\ldots\right)t^{2n}\right)\left(\sum_{n=0}^{\infty}e_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right)t^n\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\lfloor n/2\rfloor}e_k\left(\frac{1}{1^4},\frac{1}{2^4},\frac{1}{3^4},\ldots\right)e_{n-2k}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right)\right)t^n.
\end{aligned}$$

On the other hand, we can write

$$\left(\sum_{n=0}^{\infty}(-1)^n e_n\left(\frac{1}{1^8},\frac{1}{2^8},\frac{1}{3^8},\ldots\right)t^{4n}\right)\left(\sum_{n=0}^{\infty}h_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right)t^n\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\lfloor n/4 \rfloor} (-1)^k e_k \left(\frac{1}{1^4}, \frac{1}{2^4}, \frac{1}{3^4}, \ldots \right) h_{n-4k} \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \ldots \right) \right) t^n.$$

In this way, we deduce the identity

$$\sum_{k=0}^{\lfloor n/4 \rfloor} (-1)^k e_k \left(\frac{1}{1^4}, \frac{1}{2^4}, \frac{1}{3^4}, \ldots \right) h_{n-4k} \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \ldots \right)$$
$$= \sum_{k=0}^{\lfloor n/2 \rfloor} e_k \left(\frac{1}{1^4}, \frac{1}{2^4}, \frac{1}{3^4}, \ldots \right) e_{n-2k} \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \ldots \right),$$

that can be rewritten as

$$\sum_{k=0}^{\lfloor n/4 \rfloor} (-1)^{n+1-k} b_{4k+2} \frac{2^{2n-2k+2} - 2^{6k+3}}{(8k+4)!(2n-8k)!} B_{2n-8k}$$
$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^{2k}}{(4k+2)!(2n-4k+1)!}$$

or

$$\sum_{k=0}^{\lfloor n/4 \rfloor} (-1)^{n+1-k} b_{4k+2} (2^{2n-2k} - 2^{6k+1}) \binom{2n+4}{8k+4} B_{2n-8k}$$
$$= \frac{n+2}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{2k} \binom{2n+3}{4k+2}.$$

Now we consider the multisection formula first published by Simpson (see [4, Ch. 16], [15, Ch. 4, S. 4.3] and [16]) as early as 1759. If

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

is a finite or infinite series, then for $0 \leqslant r < n$ the sum

$$a_r x^r + a_{r+n} x^{r+n} + a_{r+2n} x^{r+2n} + \cdots$$

is given by

$$\sum_{k \geqslant 0} a_{r+kn} x^{r+kn} = \frac{1}{n} \sum_{k=0}^{n-1} z^{-kr} f(z^k x), \tag{6}$$

where $z = e^{\frac{2\pi i}{n}}$ is the nth root of 1.

Applying the multisection formula (6) to

$$f(x) = (1 + \sqrt{2}x)^p = \sum_{k=0}^{p} \binom{p}{k} \left(\sqrt{2}x \right)^k,$$

we obtain

$$\sum_{k\geqslant 0}\binom{p}{4k+2}\left(\sqrt{2}x\right)^{4k+2}=\frac{1}{4}\sum_{k=0}^{3}e^{-k\pi i}\left(1+e^{\frac{k\pi i}{2}}\sqrt{2}x\right)^{p}.$$

The case $x=1$ and $p=2n+3$ of this relation read as

$$\begin{aligned}&\sum_{k=0}^{\lfloor n/2\rfloor}2^{2k+1}\binom{2n+3}{4k+2}\\&=\frac{1}{4}\left((1+\sqrt{2})^{2n+3}-(1+\sqrt{-2})^{2n+3}+(1-\sqrt{2})^{2n+3}-(1-\sqrt{-2})^{2n+3}\right)\\&=\frac{1}{2}\left(b_{2n+3}-c_{2n+3}\right).\end{aligned}$$

The proof is finished.

4. Concluding remarks

The relationships of the elementary symmetric functions to the natural powers of π^2, π^4 and π^8 have been used in this paper to provide new proofs for two lacunary recurrence relations for the Bernoulli numbers with gaps of length four and eight. Theorem 1.1 can be combined to Ramanujan's recurrence formula (1) to get a different lacunary recurrence formula for the Bernoulli numbers with gaps of length four.

Corollary 4.1. *For $n\geqslant 0$,*

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k2^{2k+1}\binom{2n+2}{4k+2}B_{2n-4k}=\left((-1)^{\lfloor n/2\rfloor}2^n+(-1)^n\right)(n+1).$$

We note that, when the problem of finding a closed form for

$$e_n\left(\frac{1}{1^{2k}},\frac{1}{2^{2k}},\frac{1}{3^{2k}},\ldots\right)$$

in terms of π^{2kn} and the problem of finding a closed form for

$$h_n\left(\frac{1}{1^{2k}},\frac{1}{2^{2k}},\frac{1}{3^{2k}},\ldots\right)$$

in terms of $\zeta(2kn)$ for arbitrary k will be solved, then further, stronger lacunary recurrences for the Bernoulli numbers will follow by the methods used in this article.

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