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STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY AN ADDITIVE FRACTIONAL BROWNIAN SHEET

Oussama El Barrimi and Youssef Ouknine

Abstract. In this paper, we show the existence of a weak solution for a stochastic differential equation driven by an additive fractional Brownian sheet with Hurst parameters $H, H' > 1/2$, and a drift coefficient satisfying the linear growth condition. The result is obtained using a suitable Girsanov theorem for the fractional Brownian sheet.

1. Introduction

First introduced by Kolmogorov [10] and further studied by Mandelbrot and Van Ness in [11], the fractional Brownian motion (fBm for short) is a Gaussian process with stationary increments and long range dependence property, that have attracted many authors in various scientific areas, such as econometrics, network traffic analysis, telecommunications, and so on. Some surveys and comprehensive literature concerning fractional Brownian motion could be found in [13].

Regarding the fBm's generalization to the two parameters case, the fractional Brownian sheet to be defined more precisely in the sequel, appears to be a suitable candidate. Recently there has been a growing interest in studying stochastic differential equations governed by fractional Brownian sheet (see [6], [16]). This interest comes from the large number of applications of the fractional Brownian sheet in practice.

Let us now consider the following stochastic differential equation

$$X_{t,s} = z + \int_0^t \int_0^s b(v,u;X_{v,u})dudv + B^{H,H'}_{t,s}, \tag{1.1}$$

where $b : [0,T]^2 \times \mathbb{R} \to \mathbb{R}$ is a Borel function, and $B^{H,H'}$ is a fractional Brownian sheet with Hurst parameters $H, H' > 1/2$.

In the singular case $H, H' \leq 1/2$, by applying Girsanov theorem, the authors in [6] have proved that the equation (1.1) admits a unique weak solution, if the

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coefficient b satisfies the linear growth condition

$$|b(t,s;x)| \leq C(1+|x|), \quad C > 0. \tag{1.2}$$

The motivation for the present paper is to construct a weak solution for the equation (1.1) when $H, H' > 1/2$. It is well known that in this case, the application of Girsanov theorem requires more regularity assumptions on the drift coefficient (in the one parameter case, the Hölder continuity property is needed to ensure the existence of a weak solution, see [14] for details). This is due to the fact that the more regular is the noise, the more regularity assumptions are needed on the drift coefficient. In this paper, we will see that even when $H, H' > 1/2$, we can construct a weak solution to (1.1) assuming only that the drift b satisfies the linear growth condition (1.2). The idea behind our proof is as follows. We will use a transformation formula (see Corollary A.3) for fractional Brownian sheet, which changes fractional Brownian sheet of Hurst parameters $H, H' > 1/2$ into some integral with respect to a fractional Brownian sheet with Hurst parameters $H, H' \leq 1/2$. Consequently, the application of Girsanov theorem can be dealt with as in the case $H, H' \leq 1/2$.

The paper is organized as follows. Section 2 contains some preliminaries on the fractional Brownian sheet. In Section 3, we present the main result which is stated in Theorem 3.1. The technical ingredients needed for the proofs are established in the Appendix.

2. Preliminaries

In this section, we review some of the basic concepts of the fractional Brownian sheet, some properties and tools used in the proofs.

A fractional Wiener random field $B^{H,H'} = \{B^{H,H'}_{t,s}, t, s \in [0,T]\}$ referred to as fractional Brownian sheet, stands for a two-parameter centered Gaussian process with covariance function given by

$$\begin{aligned} R_{H,H'}(t,s;t',s') &= E(B^{H,H'}_{t,s} B^{H,H'}_{t',s'}) \\ &= \frac{1}{4}\left(t^{2H} + (t')^{2H} - |t'-t|^{2H}\right)\left(s^{2H'} + (s')^{2H'} - |s-s'|^{2H'}\right), \end{aligned}$$

where $H, H' \in (0,1)$. According to [3], [9], the fractional Brownian sheet vanishes almost surely on the axes and admits a continuous version. Note that in case $H, H' = 1/2$ one gets a standard Brownian sheet.

It is a well known fact that a fractional Brownian motion $B^H = \{B^H_t, t \in [0,T]\}$, $H \in (0,1)$, admits the following integral representation (see [1], [2], [12])

$$B^H_t = \int_0^t K_H(t,s) dB_s,$$

B being a standard Brownian motion and $K_H(t,s)$ a square integrable Volterra kernel given by
(2.1)

$$K_H(t,s)=\left[c_H(t-s)^{H-\frac{1}{2}}+c_H(\frac{1}{2}-H)\int_s^t(1-(\frac{s}{u})^{\frac{1}{2}-H})(u-s)^{H-\frac{3}{2}}du\right]\mathbf{1}_{[0,t]}(s),$$

where

$$c_H=\left[\frac{2H(\Gamma(\frac{3}{2}-H))}{\Gamma(\frac{1}{2}+H)\Gamma(2-2H)}\right]^{1/2} \tag{2.2}$$

is the normalizing constant. From (2.1) we have

$$\frac{\partial K_H}{\partial t}(t,s)=c_H\left(\frac{1}{2}-H\right)\left(\frac{s}{t}\right)^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}}.$$

Appealing to [4], one can consider the following representation in law of the fractional Brownian sheet

$$B^{H,H'}_{t,s}=\int_0^t\int_0^s K_H(t,t')K_{H'}(s,s')dW_{t',s'},$$

where W is a standard Brownian sheet. The covariance kernel of the fractional Brownian sheet can be written as

$$R_{H,H'}(t,s;t',s')=\int_0^{t\wedge t'}\int_0^{s\wedge s'}K_{H,H'}(t,s;v,u)K_{H,H'}(t',s';v,u)dudv,$$

where $K_{H,H'}$ is the square integrable Volterra kernel;

$$K_{H,H'}(t,s;t',s')=K_H(t,t')K_{H'}(s,s'),$$

and K_H is the kernel defined by (2.1).

Take now $0<H,H'\le 1/2$, and let ζ be the set of step functions on $[0,T]^2$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of ζ with respect to the scalar product

$$\langle\mathbf{1}_{[0,t]\times[0,s]},\mathbf{1}_{[0,t']\times[0,s']}\rangle_{\mathcal{H}}=R_{H,H'}(t,s;t',s').$$

The mapping $\mathbf{1}_{[0,t]\times[0,s]}\to B^{H,H'}_{t,s}$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian subspace associated with $B^{H,H'}$, such isometry is denoted by $\varphi\to B^{H,H'}(\varphi)$.

We introduce also the linear operator $K^*_{H,H'}$ from ζ to $L^2([0,T]^2)$ defined by

$$\begin{aligned}(K^*_{H,H'}\varphi)(t,s)&=K_{H,H'}(T,T;t,s)\varphi(t,s)\\&\quad+\int_t^T\int_s^T(\varphi(v,u)-\varphi(t,s))\frac{\partial^2K_{H,H'}}{\partial v\partial u}(v,u;t,s)dudv.\end{aligned}$$

For any pair of step functions φ and ψ in ζ we have

$$\langle K^*_{H,H'}\varphi,K^*_{H,H'}\psi\rangle_{L^2([0,T]^2)}=\langle\varphi,\psi\rangle_{\mathcal{H}}.$$

This is an immediate consequence of the fact that

$$(K^*_{H,H'}\mathbf{1}_{[0,t]\times[0,s]})(v,u) = K_{H,H'}(t,s;v,u).$$

Accordingly, the operator $K^*_{H,H'}$ provides an isometry between ζ and $L^2([0,T]^2)$ that can be extended to the Hilbert space $\mathcal{H}$.

Define the process $W = \{W_{t,s}, t, s \in [0,T]\}$ by

$$W_{t,s} = B^{H,H'}((K^*_{H,H'})^{-1}\mathbf{1}_{[0,t]\times[0,s]}).$$

Then W is a standard Brownian sheet, moreover $B^{H,H'}$ has the integral representation

$$B^{H,H'}_{t,s} = \int_0^t \int_0^s K_{H,H'}(t,s;v,u)dW_{v,u}.$$

We need also to define an isomorphism $K_{H,H'}$ from $L^2([0,T]^2)$ onto

$$I^{H+\frac{1}{2},H'+\frac{1}{2}}(L^2)$$

associated with the kernel $K_{H,H'}(t,s;v,u)$ in terms of the fractional integrals as follows:

$$(K_{H,H'}\varphi)(t,s) = I^{2H,2H'} t^{\frac{1}{2}-H} s^{\frac{1}{2}-H'} I^{\frac{1}{2}-H,\frac{1}{2}-H} t^{H-\frac{1}{2}} s^{H'-\frac{1}{2}}\varphi, \quad \varphi \in L^2([0,T]^2).$$

Note that, for $\varphi \in L^2([0,T]^2)$, $I^{\alpha,\beta}$ is the left-sided fractional Riemann Liouville integral of order (α,β) defined by

$$I^{\alpha,\beta}\varphi(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y (x-u)^{\alpha-1}(y-v)^{\beta-1}\varphi(u,v)dudv, \tag{2.3}$$

where Γ is the Gamma function (see [5] for the one parameter case). We note that (2.3) can be written as follows:

$$I^{\alpha,\beta}\varphi(x,y) = I^{\alpha}(I^{\beta}\varphi(\cdot,y))(x).$$

The inverse of $K_{H,H'}$ is given by

$$(K^{-1}_{H,H'}\varphi)(t,s) = t^{\frac{1}{2}-H} s^{\frac{1}{2}-H'} D^{\frac{1}{2}-H,\frac{1}{2}-H'} t^{H-\frac{1}{2}} s^{H'-\frac{1}{2}} D^{2H,2H'}\varphi,$$

where, for $\varphi \in I^{H+\frac{1}{2},H'+\frac{1}{2}}(L^2)$, $D^{\alpha,\beta}$ is the left-sided Riemann-Liouville derivative of order (α,β) defined by

$$D^{\alpha,\beta}\varphi(x,y) = \frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)} \frac{\mathrm{d}^2}{\mathrm{d}x\mathrm{d}y} \int_0^x \int_0^y \frac{\varphi(u,v)}{(x-u)^{\alpha}(y-v)^{\beta}} dudv.$$

If φ vanishes on the axes and is absolutely continuous, it can be proved that

$$(K^{-1}_{H}\varphi)(t,s) = t^{H-\frac{1}{2}} s^{H'-\frac{1}{2}} I^{\frac{1}{2}-H,\frac{1}{2}-H'} t^{\frac{1}{2}-H} s^{\frac{1}{2}-H'} \frac{\mathrm{d}^2\varphi}{\mathrm{d}t\mathrm{d}s}. \tag{2.4}$$

3. Main result

In this section, by using a transformation formula for fractional Brownian sheet established in Corollary A.3, we construct a weak solution to the equation (1.1) when the Hurst parameters $H, H' > 1/2$. To do so we shall follow the approach developed in [14] for the one-parameter case, where the authors use a suitable Girsanov's theorem.

By a weak solution to (1.1) we mean a sextuple $(\Omega, \mathcal{F}, \{\mathcal{F}_{t,s}\}_{t,s\in[0,T]}, P, X, B^{H,H'})$ such that

(1) $(\Omega,\mathcal{F},P)$ is a probability space equipped with the filtration $\{\mathcal{F}_{t,s}\}_{t,s\in[0,T]}$ that satisfies the usual conditions,
(2) X is an $\{\mathcal{F}_{t,s}\}_{t,s\in[0,T]}$ adapted process, and $B^{H,H'}$ is an $\{\mathcal{F}_{t,s}\}_{t,s\in[0,T]}$ fractional Brownian sheet,
(3) X and $B^{H,H'}$ satisfy (1.1).

Theorem 3.1. *Suppose that $b(t, s; x)$ satisfies the linear growth condition (1.2). Let $B^{H,H'}$ be a fractional Brownian sheet with Hurst parameters $H, H' > 1/2$. Then the equation (1.1) has a weak solution.*

Proof. Let us first define $l_{t,s} = -b(t, s; B^{H,H'}_{t,s} + z)$. Set $\widetilde{B}^{H,H'}_{t,s} = B^{H,H'}_{t,s} + \int_0^t \int_0^s l_{v,u} du dv$. By mean of the integral representation of the fractional Brownian sheet and the transformation formula presented in Corollary A.3, we get that

$$
\begin{aligned}
\widetilde{B}^{H,H'}_{t,s} =& B^{H,H'}_{t,s} + \int_0^t \int_0^s l_{v,u} du dv \\
=& \int_0^t \int_0^s l_{v,u} du dv + C_{H,H'} \int_0^t \int_0^s (t-v)^{2H-1}(s-u)^{2H'-1} dB^{1-H,1-H'}_{u,v} \qquad (3.1)\\
=& \int_0^t \int_0^s C_{H,H'}(t-v)^{2H-1}(s-u)^{2H'-1} \\
& \left[C^{-1}_{H,H'}(t-v)^{1-2H}(s-u)^{1-2H'} l_{v,u} du dv + dB^{1-H,1-H'}_{v,u} \right].
\end{aligned}
$$

For given t, s in (3.1) and any $0 \le v \le t$, $0 \le u \le s$, let

$$
\begin{aligned}
\widetilde{B}^{1-H,1-H'}_{v,u} &= B^{1-H,1-H'}_{v,u} + \int_0^v \int_0^u C^{-1}_{H,H'}(t-y)^{1-2H}(s-x)^{1-2H'} l_{y,x} dx dy \\
&= \int_0^v \int_0^u K_{1-H,1-H'}(v, u; y, x) dW_{y,x} \\
&\quad + \int_0^v \int_0^u C^{-1}_{H,H'}(t-y)^{1-2H}(s-x)^{1-2H'} l_{y,x} dx dy \\
&= \int_0^v \int_0^u K_{1-H,1-H'}(v, u; y, x) \Big[K^{-1}_{1-H,1-H'}
\end{aligned}
$$

$$\left(\int_0^{\cdot}\int_0^{\cdot} C_{H,H'}^{-1}(t-n)^{1-2H}(s-m)^{1-2H'} l_{n,m} dmdn\right)$$
$$(y,x)dxdy + dW_{y,x}\Bigg].$$

Let ξ be the Doléans-Dade exponential defined by

$$\xi_{T,T} \triangleq \exp\left\{-\int_0^T\int_0^T g_{y,x}\,\mathrm{d}W_{y,x} - \frac{1}{2}\int_0^T\int_0^T g_{y,x}^2\,\mathrm{d}x\mathrm{d}y\right\},$$

where

$$g_{y,x} = K_{1-H,1-H'}^{-1}\left(\int_0^{\cdot}\int_0^{\cdot} C_{H,H'}^{-1}(T-n)^{1-2H}(T-m)^{1-2H'} l_{n,m} dmdn\right)(y,x).$$

Using the transformation formula, we know that $(\widetilde{B}_{t,s}^{H,H'})_{0\le t\le T, 0\le s\le T}$ is an $\mathcal{F}_{t,s}^{B^{H,H'}}$ fractional Brownian motion with Hurst parameters $H, H' \in (\frac{1}{2}, 1)$ with respect to the new measure $\widetilde{\mathbb{P}}$ defined by $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \xi_{T,T}$, if $(\widetilde{B}_{t,s}^{1-H,1-H'})_{0\le t\le T, 0\le s\le T}$ is an $\mathcal{F}_{t,s}^{B^{1-H,1-H'}}$ fractional Brownian motion with Hurst parameters $(1-H, 1-H')$ under the measure $\widetilde{\mathbb{P}}$.

Next, if we want to show that $(\widetilde{B}_{t,s}^{1-H,1-H'})_{0\le t\le T, 0\le s\le T}$ is an $\mathcal{F}_{t,s}^{B^{1-H,1-H'}}$ fractional Brownian motion with Hurst parameters $(1-H, 1-H')$ under the measure $\widetilde{\mathbb{P}}$, we need first to verify that the conditions imposed in Girsanov theorem [6, Theorem 3] hold. To do this, we will apply the same arguments as in [14]. From (2.4) and the linear growth property of b, we obtain for $y, x \in [0,T]^2$

$$\begin{aligned}
|g_{y,x}| &= \Bigg| y^{\frac{1}{2}-H} x^{\frac{1}{2}-H'} I^{H-\frac{1}{2}, H'-\frac{1}{2}} y^{H-\frac{1}{2}} x^{H'-\frac{1}{2}} C_{H,H'}^{-1}(T-y)^{1-2H}(T-x)^{1-2H'} \\
&\quad |b(y,x;B_{y,x}^{H,H'}+z)|\Bigg| \\
&= \frac{1}{\Gamma\left(H-\frac{1}{2}\right)\Gamma\left(H'-\frac{1}{2}\right)} C_{H,H'}^{-1} y^{\frac{1}{2}-H} x^{\frac{1}{2}-H'} \\
&\quad \int_0^y\int_0^x (T-v)^{1-2H}(T-u)^{1-2H'} v^{H-\frac{1}{2}} u^{H'-\frac{1}{2}} |b(v,u;B_{v,u}^{H,H'}+z)| \\
&\quad (y-v)^{\frac{-3}{2}+H}(x-u)^{\frac{-3}{2}+H'} dudv \\
&\le \frac{C(1+|z|+\|B^{H,H'}\|_\infty)}{\Gamma\left(H-\frac{1}{2}\right)\Gamma\left(H'-\frac{1}{2}\right)} C_{H,H'}^{-1} y^{\frac{1}{2}-H} x^{\frac{1}{2}-H'} \\
&\quad \int_0^y\int_0^x (T-v)^{1-2H}(T-u)^{1-2H'} v^{H-\frac{1}{2}} u^{H'-\frac{1}{2}} \\
&\quad (y-v)^{\frac{-3}{2}+H}(x-u)^{\frac{-3}{2}+H'} dudv,
\end{aligned}$$

where $\Gamma(\cdot)$ is the standard Gamma function.

Then, using Hölder's inequality and direct calculation, we have for $1 < q < \frac{2}{3-2H'}$

$$\int_0^x (T-u)^{1-2H'} u^{H'-\frac{1}{2}} (x-u)^{\frac{-3}{2}+H'} du$$
$$\leq \left(\int_0^x \left(u^{H'-\frac{1}{2}} (x-u)^{\frac{-3}{2}+H'} \right)^q du \right)^{\frac{1}{q}} \left(\int_0^x (T-u)^{\frac{q(1-2H')}{q-1}} du \right)^{\frac{q-1}{q}}$$
$$= x^{2H'-2+\frac{1}{q}} B^{\frac{1}{q}} (qH' - \frac{q}{2} + 1, qH' - \frac{3q}{2} + 1) \left(\frac{T^{2-2H'-\frac{1}{q}} - (T-x)^{2-2H'-\frac{1}{q}}}{|\frac{2q-2qH'-1}{q-1}|^{\frac{q-1}{q}}} \right),$$

where $B(\cdot,\cdot)$ is the standard Beta function. As a result, we have for $1 < p < \frac{2}{3-2H}$ and $1 < q < \frac{2}{3-2H'}$

$$|g_{y,x}| \leq C_{H,H'}^{-1} \frac{C(1+|z|+\|B^{H,H'}\|_\infty)}{\Gamma\left(H-\frac{1}{2}\right)\Gamma\left(H'-\frac{1}{2}\right)} y^{H-\frac{3}{2}+\frac{1}{p}} x^{H'-\frac{3}{2}+\frac{1}{q}}$$
$$\times B^{\frac{1}{q}} (qH' - \frac{q}{2} + 1, qH' - \frac{3q}{2} + 1)$$
$$\times B^{\frac{1}{p}} (pH - \frac{p}{2} + 1, pH - \frac{3p}{2} + 1) \left(\frac{T^{2-2H'-\frac{1}{q}} + (T-x)^{2-2H'-\frac{1}{q}}}{|\frac{2q-2qH'-1}{q-1}|^{\frac{q-1}{q}}} \right)$$
$$\left(\frac{T^{2-2H-\frac{1}{p}} + (T-y)^{2-2H-\frac{1}{p}}}{|\frac{2p-2pH-1}{p-1}|^{\frac{p-1}{p}}} \right).$$

Therefore, we obtain

$$\int_0^T \int_0^T |g_{y,x}|^2 dxdy$$
$$\leq 4 \left[C_{H,H',p,q} \frac{(1+|z|+\|B^{H,H'}\|_\infty)}{\Gamma\left(H-\frac{1}{2}\right)\Gamma\left(H'-\frac{1}{2}\right)} \right]^2 B^{\frac{2}{p}} (pH - \frac{p}{2} + 1, pH - \frac{3p}{2} + 1)$$
$$\times B^{\frac{2}{q}} (qH' - \frac{q}{2} + 1, qH' - \frac{3q}{2} + 1) \int_0^T \int_0^T y^{2H-3+\frac{2}{p}} x^{2H'-3+\frac{2}{q}}$$
$$\left[T^{4-4H'-\frac{2}{q}} + (T-x)^{4-4H'-\frac{2}{q}} \right] \left[T^{4-4H-\frac{2}{p}} + (T-y)^{4-4H-\frac{2}{p}} \right] dxdy.$$

If $\frac{1}{2} < H \leq \frac{3}{4}$ and $\frac{1}{2} < H' \leq \frac{3}{4}$, then $5-4H-\frac{2}{p} \geq 2-\frac{2}{p} > 0$ and $5-4H'-\frac{2}{q} \geq 2-\frac{2}{q} > 0$. Consequently, we have

$$\int_0^T \int_0^T |g_{y,x}|^2 dxdy$$
$$\leq 4 \left[C_{H,H',p,q} \frac{(1+|z|+\|B^{H,H'}\|_\infty)}{\Gamma\left(H-\frac{1}{2}\right)\Gamma\left(H'-\frac{1}{2}\right)} \right]^2 B^{\frac{2}{p}} (pH - \frac{p}{2} + 1, pH - \frac{3p}{2} + 1)$$

$$\times B^{\frac{2}{q}}(qH' - \frac{q}{2} + 1, qH' - \frac{3q}{2} + 1)$$
$$\left[\frac{T^{2-2H'}}{2H' - 2 + \frac{2}{q}} + T^{2-2H'} B(2H' - 2 + \frac{2}{q}, 5 - 4H' - \frac{2}{q})\right]$$
$$\left[\frac{T^{2-2H}}{2H - 2 + \frac{2}{p}} + T^{2-2H} B(2H - 2 + \frac{2}{p}, 5 - 4H - \frac{2}{p})\right].$$

On the other hand, if $\frac{3}{4} < H < 1$ and $\frac{3}{4} < H' < 1$, then, for $\frac{2}{5-4H} < p < \frac{2}{3-2H}$ and $\frac{2}{5-4H'} < q < \frac{2}{3-2H'}$, we conclude that $5 - 4H - \frac{2}{p} > 0$ and $5 - 4H' - \frac{2}{q} > 0$. Hence,

$$\int_0^T \int_0^T |g_{y,x}|^2 dx dy$$
$$\leq 4 \left[C_{H,H',p,q} \frac{(1 + |z| + \|B^{H,H'}\|_\infty)}{\Gamma\left(H - \frac{1}{2}\right)\Gamma\left(H' - \frac{1}{2}\right)}\right]^2 B^{\frac{2}{p}}(pH - \frac{p}{2} + 1, pH - \frac{3p}{2} + 1)$$
$$\times B^{\frac{2}{q}}(qH' - \frac{q}{2} + 1, qH' - \frac{3q}{2} + 1)$$
$$\left[\frac{T^{2-2H'}}{2H' - 2 + \frac{2}{q}} + T^{2-2H'} B(2H' - 2 + \frac{2}{q}, 5 - 4H - \frac{2}{q})\right]$$
$$\left[\frac{T^{2-2H}}{2H - 2 + \frac{2}{p}} + T^{2-2H} B(2H - 2 + \frac{2}{p}, 5 - 4H - \frac{2}{p})\right].$$

Finally, using a version of Novikov criterion for two-parameter processes, we can prove the fact that $E[\xi_{T,T}] = 1$. Indeed, it suffices to show that for some $\lambda > 0$

$$E\left[\exp\left(\lambda \int_0^T \int_0^T |g_{y,x}|^2 dx dy\right)\right] < \infty,$$

which is an immediate consequence of the exponential integrability of a Gaussian process (see [7]). This completes the proof. □

Remark 3.2. The uniqueness in law property can be deduced similarly as in [6, Theorem 5]. Indeed, it suffices to consider $(B^{H,H'}, X)$ to be a weak solution of the equation (1.1), and use the same approach investigated in the proof of the main theorem 3.1.

Appendix A. Appendix

The purpose of this section is to establish a transformation formula for a fractional Brownian sheet (see [8], [15] for the one parameter case). Let us first recall that

$$(K^*_{H,H'} \mathbf{1}_{[0,t]\times[0,s]})(t', s') = K_{H,H'}(t, s; t', s') = K_H(t, t') K_{H'}(s, s'),$$

where K_H, K'_H can be also defined by (see [5]):

$$\begin{cases} K_H(t,t') = \frac{c_H}{\Gamma(H+\frac{1}{2})}(t'-t)^{H-\frac{1}{2}} F(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}, \frac{t-t'}{t})\mathbf{1}_{[0,t]}(t'), \\ K_{H'}(s,s') = \frac{c'_H}{\Gamma(H'+\frac{1}{2})}(s'-s)^{H'-\frac{1}{2}} F(\frac{1}{2}-H', H'-\frac{1}{2}, H'+\frac{1}{2}, \frac{s-s'}{s})\mathbf{1}_{[0,s]}(s'), \end{cases}$$

$F(a,b,c,z)$ being the Gauss hypergeometric function and c_H is the constant defined in (2.2). We note that K_H can be also defined in term of fractional integral as (see [8])

$$K_H(t,t') = c_H {t'}^{\frac{1}{2}-H} \left(I^{H-\frac{1}{2}} .^{H-\frac{1}{2}} \right)(t').$$

As a result

$$\begin{aligned} &(K^*_{H,H'}\mathbf{1}_{[0,t]\times[0,s]})(t',s') \\ &= c_{H,H'}(t')^{\frac{1}{2}-H}(s')^{\frac{1}{2}-H'} \left(I^{H-\frac{1}{2},H'-\frac{1}{2}} .^{H-\frac{1}{2}} .^{H'-\frac{1}{2}} \right)(t',s'). \end{aligned}$$

Similarly $(K^*_{H,H'})^{-1}$ can be written as

$$\begin{aligned} &((K^*_{H,H'})^{-1}\mathbf{1}_{[0,t]\times[0,s]})(t',s') \\ &= c^{-1}_{H,H'}(t')^{\frac{1}{2}-H}(s')^{\frac{1}{2}-H'} \left(I^{\frac{1}{2}-H,\frac{1}{2}-H'} .^{H-\frac{1}{2}} .^{H'-\frac{1}{2}} \right)(t',s'). \end{aligned}$$

Lemma A.1. *Let $H, L \in (0,1)$, $H', L' \in (0,1)$. Then*

$$\begin{aligned} &((K^*_{L,L'})^{-1}K^*_{H,H'}\mathbf{1}_{[0,t]\times[0,s]})(t',s') \\ &= \frac{c_{H,H'}}{c_{L,L'}}(t')^{1-H-L}(s')^{1-H'-L'} \left(I^{H-L,H'-L'} .^{H+L-1} .^{H'+L'-1} \right)(t',s'). \end{aligned}$$

Moreover,

$$(K_{H,H'}K^*_{L,L'}K_{H,H'}\mathbf{1}_{[0,t]\times[0,s]})(t',s') = (K_{H,H'}\mathbf{1}_{[0,t]\times[0,s]})(t',s').$$

Proof. Similar to the proof of [8, Lemma 3.5] for the one parameter case. □

Theorem A.2. *Let $B^{H,H'}$ be a fractional Brownian sheet, and let $L, L' \in (0,1)$. Then there exists a unique (up to modification) fractional Brownian sheet $B^{L,L'}$ such that, we have a.s*

$$\begin{aligned} B^{H,H'}_{t,s} &= \frac{c_{H,H'}}{c_{L,L'}} \int_0^T \int_0^T v^{1-H-L} u^{1-H'-L'} \left(I^{H-L,H'-L'} .^{H+L-1} .^{H'+L'-1} \right)(v,u) \\ &= C_{H,H',L,L'} \int_0^t \int_0^s F(1-K'-L', H'-L', 1+H'-L', \frac{u-s}{u}) \\ &\qquad F(1-K-L, H-L, 1+H-L, \frac{v-t}{v})(t-v)^{H-L}(s-u)^{H'-L'} dB^{L,L'}_{u,v}. \end{aligned}$$

Proof. We follow the same arguments as in [8, Theorem 5.1], and we use Lemma A.1. □

Corollary A.3. *Let $B^{H,H'}$ be a fractional Brownian sheet. Then there exists a unique (up to modification) fractional Brownian sheet $B^{1-H,1-H'}$ such that*

$$B_{t,s}^{H,H'} = C_{H,H'} \int_0^t \int_0^s (t-v)^{2H-1}(s-u)^{2H'-1} dB_{u,v}^{1-H,1-H'}, a.s.$$

Proof. Set $L := 1 - H$ and $L' := 1 - H'$ in Theorem A.2. □

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Oussama El Barrimi
Department of Mathematics
Faculty of Sciences Semlalia
Cadi Ayyad University
2390 Marrakesh, Morocco
Email address: Oussama.elbarrimi@gmail.com

Youssef Ouknine
Department of Mathematics
Faculty of Sciences Semlalia
Cadi Ayyad University
2390 Marrakesh, Morocco
and
Hassan II Academy of Sciences and Technology Rabat
Morocco
and
Mohammed VI polytechnic University Ben Ghuerir
Morocco
Email address: Youssef.ouknine@um6p.ma