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RINGS AND MODULES CHARACTERIZED BY OPPOSITES OF FP-INJECTIVITY

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Abstract. Let R be a ring with unity. Given modules M_R and ${}_RN$, M_R is said to be absolutely ${}_RN$-pure if $M \otimes N \to L \otimes N$ is a monomorphism for every extension L_R of M_R. For a module M_R, the subpurity domain of M_R is defined to be the collection of all modules ${}_RN$ such that M_R is absolutely ${}_RN$-pure. Clearly M_R is absolutely ${}_RF$-pure for every flat module ${}_RF$, and that M_R is FP-injective if the subpurity domain of M is the entire class of left modules. As an opposite of FP-injective modules, M_R is said to be a *test for flatness by subpurity* (*or t.f.b.s. for short*) if its subpurity domain is as small as possible, namely, consisting of exactly the flat left modules. Every ring has a right *t.f.b.s.* module. R_R is t.f.b.s. and every finitely generated right ideal is finitely presented if and only if R is right semihereditary. A domain R is Prüfer if and only if R is t.f.b.s. The rings whose simple right modules are t.f.b.s. or injective are completely characterized. Some necessary conditions for the rings whose right modules are t.f.b.s. or injective are obtained.

1. Introduction and preliminaries

Throughout, R will denote an associative ring with identity, and modules will be unital right modules unless otherwise stated. As usual, we denote by R-Mod and Mod-R the categories of left and right modules, respectively. Following [7], given a right module M and a left module N, M is said to be absolutely N-pure if $M \otimes N \to L \otimes N$ is a monomorphism for every extension L of M. The absolutely pure domain of a left module N is defined as the collection of all right modules M such that M is absolutely N-pure. Absolutely pure domain of any module consists of the class of FP-injective modules. A left module N is said to be *f-indigent* if its absolutely pure domain is exactly the class of FP-injective right modules.

In this paper, we investigate the *subpurity domain* of a right module M as the collection of all left modules N such that M is absolutely N-pure. Flat left modules are contained in the subpurity domain of any right module. In

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this setting, a right module M is FP-injective if and only if its subpurity domain consists of the entire class *R-Mod*. From this point of view, a reasonable opposite to FP-injectivity in this context is obtained by considering modules whose subpurity domain consists only of flat left modules. A right module M is called *test for flatness by subpurity* (*t.f.b.s.*) if the subpurity domain of M is exactly the class of flat left modules.

The paper is inspired by similar ideas and notions that have been appeared in several papers, for example [1, 2, 4, 5, 7]. A module M is said to be A-subinjective if for every extension B of A any homomorphism $\varphi : A \to M$ can be extended to a homomorphism $\phi : B \to M$ (see [4]). It is easy to see that M is injective if and only if M is A-subinjective for each module A. A module M is called *indigent* if M is subinjective relative to only injective modules. In [1], a module A is said to be a *test for injectivity by subinjectivity* (*or t.i.b.s.*) if whenever a module M is A-subinjective implies M is injective.

The paper is organized as follows.

In Section 2, we prove some properties of subpurity domains and absolutely pure domains. The ring is regular if and only if the subpurity domain of any right module is closed under homomorphic images. The ring is right semihereditary if and only if the absolutely pure domain of any left module is closed under homomorphic images. The subpurity domain of any flat right module is closed under submodules.

In Section 3, we prove that every ring has a right t.f.b.s. module. All right modules are $t.f.b.s.$ if and only if the ring is von Neumann regular. The ring is t.f.b.s. as a right module and right S-ring (i.e., finitely generated right ideals are finitely presented) if and only if R is right semihereditary. Hence a commutative domain R is Prüfer if and only if R is t.f.b.s. The rings whose simple right modules are t.f.b.s. or FP-injective are completely characterized.

For a ring R, by Lemma 5.1, each right module is t.f.b.s. or FP-injective if and only if each left module is f-indigent or flat. In Section 4, For a right Noetherian ring whose right modules are t.f.b.s. or injective some necessary conditions are proved.

For a ring R and a right module M, $E(M)$, $\mathrm{Rad}(M)$, $\mathrm{Soc}(M)$, $Z(M)$ will respectively denote the injective hull, Jacobson radical, socle, singular submodule of M. The character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ will be denoted by M^+. By $N \leq M$ we mean that N is a submodule of M. For additional terminology, concepts and results not mentioned here, we refer the reader to [3], [10] and [12].

2. The notion of subpurity domain of a module

In this section we investigate some properties of subpurity domains. First, we recall the characterization of FP-injective modules. Note that FP-injective modules are also known as absolutely pure modules in the literature.

Proposition 2.1 ([8, 6.2.3])**.** *The following statements are equivalent for a right module N.*

(1) *N is FP-injective.*
(2) *$N \otimes M \to E(N) \otimes M$ is a monomorphism for any finitely presented left module M.*
(3) *$N \otimes M \to E(N) \otimes M$ is a monomorphism for each left module M.*
(4) $\mathrm{Ext}^1_R(F, N) = 0$ *for each finitely presented right module F.*

Definition 2.2. (1) Given a right module M and a left module N, M is absolutely N-pure if for every right module K with $M \leq K$ the map $i \otimes 1_N : M \otimes N \to K \otimes N$ is a monomorphism, where $i : M \to K$ is the inclusion map and 1_N is the identity map on N.

(2) The *subpurity domain* of a module M_R, denoted as $\mathcal{S}p(M)$, is defined to be the collection of all modules ${}_RN$ such that M is absolutely N-pure. That is,

$$\mathcal{S}p(M) = \{N \in R\text{-}Mod \,|\, M \text{ is absolutely } N\text{-pure}\}.$$

The proof of the following is standard, so the proof is omitted here.

Lemma 2.3. *The following statements are equivalent for a right module M and a left module N.*

(1) *M is absolutely N-pure.*
(2) *For every right R-module K and essential submodule M of K, the sequence $0 \to M \otimes N \to K \otimes N$ is exact.*
(3) *There is an FP-injective extension Q of M such that $0 \to M \otimes N \to Q \otimes N$ is exact.*
(4) *The sequence $0 \to M \otimes N \to E(M) \otimes N$ is exact.*
(5) *The sequence $0 \to M \otimes N \to E \otimes N$ is exact for some injective extension E of M.*

Proposition 2.4. *The following properties hold for a right module M.*

(1) *M is absolutely $\oplus_{j \in I} N_j$-pure if and only if M is absolutely N_j-pure for each $j \in I$.*
(2) *If K is a pure submodule of the left module N, then M is absolutely N-pure if and only if M is absolutely K-pure and absolutely N/K-pure.*

Proof. (1) We have $M \otimes (\oplus_{j \in I} N_j) \cong \oplus_{j \in I}(M \otimes N_j)$. Therefore $i \otimes 1_{\oplus_{j \in I} N_j} : M \otimes (\oplus_{j \in I} N_j) \to E(M) \otimes (\oplus_{j \in I} N_j)$ is a monomorphism if and only if $i \otimes 1_{N_j}$ is a monomorphism for each $j \in I$. This proves (1).

(2) By [7, Proposition 2.5]. □

Proposition 2.5. $\bigcap_{M \in Mod\text{-}R} \mathcal{S}p(M) = \{N \in R\text{-}Mod \,|\, N \text{ is flat}\}$.

Proof. Let $N \in \bigcap_{M \in Mod\text{-}R} \mathcal{S}p(M)$. Then $N \in \mathcal{S}p(I)$ for each right ideal I of R, i.e., $I \otimes N \to R \otimes N$ is a monomorphism. Hence N is flat by [12, Proposition 3.53]. The reverse containment is obvious. □

Proposition 2.6. *The following properties hold for any right module M and left module N.*

(1) *$\oplus_{i=1}^{n} M_i$ is absolutely N-pure if and only if M_i is absolutely N-pure for each $i = 1, 2, \ldots, n$.*

(2) *If R is right Noetherian and I is any index set, then $\oplus M_i$ is absolutely N-pure if and only if M_i is absolutely N-pure for each $i \in I$.*

Proof. (1) Set $M = \oplus_{i=1}^{n} M_i$ and suppose that M is absolutely N-pure. We have $E(M) = \oplus_{i=1}^{n} E(M_i)$ and $(\oplus_{i=1}^{n} M_i) \otimes N \cong \oplus_{i=1}^{n}(M_i \otimes N)$. Then $(\oplus_{i=1}^{n} M_i) \otimes N \to (\oplus_{i=1}^{n} E(M_i)) \otimes N$ is a monomorphism if and only if $M_i \otimes N \to E(M_i) \otimes N$ is a monomorphism for each $i = 1, \ldots, n$. Therefore M is absolutely N-pure if and only if M_i is absolutely N-pure for each $i = 1, \ldots, n$.

(2) Since R is Noetherian, $E(M) = \oplus_{i \in I} E(M_i)$. The rest of the proof is similar to that of (1). □

The following is a consequence of Proposition 2.6(1).

Corollary 2.7. $\mathcal{S}p(\oplus_{i=1}^{n} M_i) = \cap_{i=1}^{n} \mathcal{S}p(M_i)$.

In general the subpurity domain is not closed under submodules. For flat modules we have the following.

Proposition 2.8. *Let F be a flat right module. Suppose that F is absolutely M-pure for some left module M. Then F is absolutely K-pure for any submodule K of M. In other words, the subpurity domain of any flat right module is closed under submodules.*

Proof. Let K be a submodule of M. We have the commutative diagram:

$$\begin{array}{ccc} F \otimes K & \xrightarrow{h} & E(F) \otimes K \\ \downarrow f & & \downarrow t \\ F \otimes M & \xrightarrow{g} & E(F) \otimes M \end{array}$$

induced by the inclusions $F \to E(F)$ and $K \to M$. Since F is flat and absolutely M-pure, the maps f and g are monomorphisms. Then by the commutativity of the diagram $gf = th$ is a monomorphism. Then h is a monomorphism, and so F is absolutely K-pure. □

Proposition 2.9. *A ring R is regular if and only if the subpurity domain of each right module is closed under homomorphic images.*

Proof. Necessity is clear, since every right and left module is flat over regular rings. Conversely suppose the subpurity of any right module is closed under homomorphic images. Since flat left modules are contained in the subpurity domain of any right module, the hypothesis implies that homomorphic images of flat left modules are closed under homomorphic images. This implies that every left module is flat, because every left module is a homomorphic image of a flat (projective) left module. Therefore R is a regular ring. □

In [7], for a left module N, the author investigate the absolutely pure domain of N as the collection of right modules M such that M is absolutely N-pure. In the following result we characterize when the absolutely pure domain is closed under quotient modules.

Proposition 2.10. *A ring R is right semihereditary if and only if whenever a right module M is absolutely N-pure for some left module N, then M/K is absolutely N-pure for each $K \leq M$.*

Proof. Suppose R is right semihereditary and suppose a right module M is absolutely N-pure for some left module N. Let $K \leq M$ and let E be the injective hull of M. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & M/K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & E/K & \longrightarrow & 0 \end{array}$$

with f is an isomorphism. Applying $- \otimes N$ to the diagram above gives the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K \otimes N & \longrightarrow & M \otimes N & \longrightarrow & M/K \otimes N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f\otimes 1_N} & & \downarrow{\scriptstyle g\otimes 1_N} & & \downarrow{\scriptstyle h\otimes 1_N} & & \\ 0 & \longrightarrow & K \otimes N & \longrightarrow & E \otimes N & \longrightarrow & E/K \otimes N & \longrightarrow & 0 \end{array}$$

Since $f \otimes 1_N$ and $g \otimes 1_N$ is a monomorphism, $h \otimes 1_N$ is a monomorphism by the Five Lemma. On the other hand, E/K is FP-injective by the semihereditary condition (see [11, Theorem 2]). Hence M/K is absolutely N-pure by Lemma 2.3.

Conversely, let M be an FP-injective right module. Then M is absolutely N-pure for every left module N. By the hypothesis M/K is also absolutely N-pure for each $K \leq M$ and left module N. Then M/K is FP-injective by Proposition 2.3. Hence R is right semihereditary by [11, Theorem 2]. □

The following proposition summarizes various known results that will be used in the sequel.

Proposition 2.11. *Let R be a ring and M, N be right modules. The following are hold.*

(1) [11, Theorem 3] *R is right Noetherian if and only if each FP-injective right module is injective.*
(2) [9, Proposition 2.3] *If R is nonsingular commutative ring, then all nonsingular modules are flat if and only if R is semihereditary.*
(3) [8, Theorem 3.2.10] *M is flat if and only if M^+ is injective.*
(4) [8, Theorem 3.2.16] *If R is right Noetherian, M is injective if and only if M^+ is flat.*

(5) [8, Theorem 3.2.11] *If M is finitely presented, then*

$$M \otimes_R N^+ \cong \mathrm{Hom}_R(M_R, N_R)^+.$$

(6) [11, Theorem 3] *A ring R is right Noetherian if and only if every FP-injective right module is injective.*

3. Test modules for flatness

It is clear that a right R-module M is FP-injective if and only if $\mathcal{S}p(M) = R\text{-}Mod$. On the other hand, it makes sense to consider the opposite case: What are the modules whose subpurity domain is as small as possible? It is clear that, $\mathcal{S}p(M)$ consists of the class of left flat modules.

Definition 3.1. A right R-module M is called *test for flatness by subpurity* (*t.f.b.s.*) if $\mathcal{S}p(M_R)$ consists of only flat left R-modules.

Proposition 3.2. *The following hold for a right R-module M.*

(1) *If M has a pure submodule N which is t.f.b.s., then M is t.f.b.s.*
(2) *If M is t.f.b.s., then $M \oplus N$ is t.f.b.s. for any module N.*
(3) *If A be an FP-injective right module, then $M \oplus A$ is t.f.b.s. if and only if M is t.f.b.s.*
(4) *M is t.f.b.s. if and only if M^n is t.f.b.s.*
(5) *If M is flat and t.f.b.s., then submodules of flat left modules are flat.*

Proof. (1) Let M be an absolutely A-pure module. Then $g : M \otimes A \to E(M) \otimes A$ is a monomorphism. As N is pure in M the map $f : N \otimes A \to M \otimes A$ is also a monomorphism. Now the map $gf : N \otimes A \to E(M) \otimes A$ is a monomorphism. Then N is absolutely A-pure, and so A is flat, because N is t.f.b.s. Hence M is t.f.b.s.

(2) is clear by Corollary 2.7.

(3) follows from the equality $\mathcal{S}p(M \oplus A) = \mathcal{S}p(M) \cap \mathcal{S}p(A) = \mathcal{S}p(M)$.

(4) follows from $\mathcal{S}p(M^n) = \cap \mathcal{S}p(M) = \mathcal{S}p(M)$.

(5) is clear by Proposition 2.8. □

Proposition 3.3. *Every ring has a $t.f.b.s.$ module.*

Proof. Let R be a ring and $N = \oplus I$, where I ranges among finitely generated right ideals of R. Assume that a right R-module N is absolutely A-pure. Since I is a direct summand of N, I is absolutely A-pure. So the map $I \otimes A \to R \otimes A$ is a monomorphism. Therefore A is flat by [12, Proposition 3.53]. □

We have seen that each ring has a $t.f.b.s.$ module. The rings over which each module is $t.f.b.s.$ are as follows.

Proposition 3.4. *The following statements are equivalent for a ring R.*

(1) *R is von Neumann regular.*
(2) *Every right R-module is t.f.b.s.*
(3) *There exists an FP-injective right t.f.b.s. R-module.*

Proof. (1) ⇒ (2) Suppose M is regular and let M be a right R-module. Since R is regular every right module is flat, in particular $E(M)/M$ is flat. Then M is a pure submodule of $E(M)$. So that M is FP-injective. Therefore $\mathcal{S}p(M) = R\text{-}Mod$, i.e., M is t.f.b.s.

(2) ⇒ (3) Clear.

(3) ⇒ (1) Let M be an absolutely pure t.f.b.s. right R-module. Since M is FP-injective, $\mathcal{S}p(M) = R\text{-}Mod$. But M is t.f.b.s., hence every left R-module is flat. This implies R is a von Neumann regular ring. □

Definition 3.5. A ring R is called a right S-ring if every finitely generated flat right ideal is projective. Right coherent rings, right semihereditary rings, local rings and semiperfect rings are examples of right S-rings.

Theorem 3.6. *A ring R is right t.f.b.s. and a right S-ring if and only if R is right semihereditary.*

Proof. Suppose R_R is t.f.b.s. Then every left ideal of R is flat by Proposition 3.2(5). Hence every right ideal of R is flat by [10, Lemma 4.66]. Now the right S-ring condition implies that every finitely generated right ideal of R is projective. Therefore, R is right semihereditary.

Conversely, assume that R is right semihereditary. Then R is a right S-ring. To prove that R_R is t.f.b.s., suppose R is absolutely A-pure. Let I be a finitely generated right ideal of R. Then $R^m = I \oplus K$, because R is right semihereditary. So I is absolutely A-pure by Proposition 2.6(1). Hence $0 \to I \otimes A \to R \otimes A$ is a monomorphism, and so A is flat. This gives that R_R is t.f.b.s. □

By [13, Corollary 3.1], a commutative ring is an S-ring if and only if $\operatorname{ann}(I)$ is finitely generated for each finitely generated flat ideal I. Therefore any commutative domain is an S-ring. The following is now clear by Theorem 3.6.

Corollary 3.7. *A commutative domain is Prüfer if and only if it is t.f.b.s.*

Corollary 3.8. *Let R be a semiperfect ring. Then the following are equivalent.*

(1) *R_R is t.f.b.s.*
(2) *${}_RR$ is t.f.b.s.*
(3) *R is semihereditary.*

There are t.f.b.s. modules which are not t.i.b.s. For example any Prüfer domain is t.f.b.s. by Corollary 3.7, but R is not t.i.b.s. unless it is Dedekind by [1, Corollary 20].

Proposition 3.9. *If N is right t.i.b.s., then N is right t.f.b.s.*

Proof. Let M be an arbitrary left R-module, and suppose that the exact sequence $0 \to N \otimes M \to E(N) \otimes M$ is a monomorphism. Then $(E(N) \otimes M)^+ \to (N \otimes M)^+$ is epic. By the First Adjoint Isomorphism Theorem, we get the following diagram:

$$\begin{array}{ccccc}
(E(N)\otimes M)^{+} & \longrightarrow & (N\otimes M)^{+} & \longrightarrow & 0\\
\downarrow \cong & & \downarrow \cong & & \\
\mathrm{Hom}(E(N),M^{+}) & \longrightarrow & \mathrm{Hom}(N,M^{+}) & \longrightarrow & 0
\end{array}$$

Hence M^+ is N-subinjective, and since N is t.i.b.s., M^+ is injective. Therefore, M is flat by Proposition 2.11(3), and so N is t.f.b.s. □

There are t.f.b.s. modules which are not t.i.b.s., for example, every semihereditary ring is *t.f.b.s.* as a right module over itself. On the other hand, by [1, Theorem 19], R_R is t.i.b.s. if and only if R is right hereditary and right Noetherian.

In searching the converse of Proposition 3.9, we have the following.

Proposition 3.10. *Let R be a right Noetherian ring. If M is a t.f.b.s. right R-module and $E(M)$ is finitely generated, then M is right t.i.b.s.*

Proof. Suppose a right module N is M-subinjective, i.e., the sequence

$$\mathrm{Hom}_R(E(M),N)\to \mathrm{Hom}_R(M,N)\to 0$$

is epic. Then we get the following commutative diagram:

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathrm{Hom}_R(M,N)^{+} & \longrightarrow & \mathrm{Hom}_R(E(M),N)^{+}\\
 & & \downarrow \cong & & \downarrow \cong\\
0 & \longrightarrow & M\otimes_R N^{+} & \longrightarrow & E(M)\otimes_R N^{+}
\end{array}$$

whose columns are isomorphisms by Proposition 2.11(5). Since M is t.f.b.s., N^+ is flat, and so N is injective by the Noetherianity of R. □

Proposition 3.11. *The following are equivalent for a ring R.*

(1) *R_R is t.f.b.s. and Noetherian.*
(2) *R_R is t.i.b.s.*

Proof. (1) ⇒ (2) Suppose that R_R is t.f.b.s. Since R is right Noetherian, it is right S-ring. So R_R is right semihereditary by Theorem 3.6. Then R is right hereditary since R is Noetherian. Hence R_R is t.i.b.s. by [1, Theorem 19].

(2) ⇒ (1) By Proposition 3.9 R_R is t.f.b.s., and R_R is Noetherian by [1, Theorem 19]. □

4. Rings whose simple modules are FP-injective or t.f.b.s.

In this section, we characterize the rings over which each simple right module is t.i.b.s. or injective, and the right Noetherian rings whose simple modules are *t.f.b.s.* or injective.

Remark 4.1. Let R be a right Noetherian ring and M be a right R-module. Let Γ be an ascending chain of injective submodules of M. Then $\cup_{E\in\Gamma}E$ is injective by [12, Exercise 2.31 and Theorem 4.10]. Hence, by Zorn's Lemma, M contains a largest injective submodule, and so M can be written as $M = K \oplus N$ where K is injective and N has no nonzero injective submodule.

Theorem 4.2. *The following are equivalent for a ring R.*

(1) *Every simple right module is t.i.b.s. or injective.*
(2) *R is a right V-ring or R is right Noetherian and every simple right module is t.f.b.s. or FP-injective.*
(3) *R is a right V-ring or $R \cong A \times B$, where A is right Artinian with a unique non-injective simple right R-module and $\mathrm{Soc}(A_A)$ is homogeneous and B is semisimple.*

Proof. (1) $\Rightarrow$ (2) Suppose that R is not a right V-ring and let T be a simple module which is not injective. Let us show that R is right Noetherian. Since T is finitely generated, arbitrary direct sum of injective modules is T-subinjective. So R is right Noetherian. T is t.i.b.s. by the hypothesis. Then T is t.f.b.s. by Proposition 3.9.

(2) $\Rightarrow$ (3) Suppose that every simple right module is t.f.b.s. or injective. Then there exits a non-injective simple right R-module T, which is t.f.b.s. by the hypothesis. Let U be a simple right R-module which is not isomorphic to T. Then $\mathrm{Hom}(T,U) = 0$. Hence, by Proposition 2.11(5), $T \otimes U^+ \cong \mathrm{Hom}(T,U)^+ = 0$. This means that, T is absolutely U^+-pure. Since T is t.f.b.s., U^+ is flat. Thus U is injective by Proposition 2.11(4). This implies that, T is the unique non-injective simple right R-module up to isomorphism. We shall prove that R is right semiartinian. Suppose there is a non-zero right R-module M such that $\mathrm{Soc}(M) = 0$. Let N be a submodule of M. Then $\mathrm{Hom}(T,N) = 0$, and so $0 = \mathrm{Hom}(T,N)^+ \cong T \otimes N^+$ by Proposition 2.11(5). That is, T is absolutely N^+-pure. Since T is t.f.b.s., N^+ is flat. Hence N is injective again by Proposition 2.11(4). Therefore N is a direct summand of M, and so M is semisimple. This is a contradiction. Hence R is right semiartinian, and R is right Artinian by the right Noetherian assumption. Let $R_R = e_1R \oplus \cdots \oplus e_tR \oplus e_{t+1}R \oplus \cdots \oplus e_nR$, where $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents. Without loss of generality we can assume that $e_{t+1}R, \ldots, e_nR$ are the injective minimal right ideals of R. Set $A = e_1R \oplus \cdots \oplus e_tR$ and $B = e_{t+1}R \oplus \cdots \oplus e_nR$. Then B is a two sided ideal of R and $\mathrm{Hom}(A,B) \cong \oplus_{i=1}^{t} \oplus_{j=t+1}^{n} \mathrm{Hom}(e_iR, e_jR) = 0$. Otherwise we have $\mathrm{Hom}(e_iR, e_jR) \neq 0$ for some $1 \leq i \leq t$ and $t+1 \leq j \leq n$, which implies $e_iR/e_iJ \cong e_jR$, and so $e_iR \cong e_jR$, a contradiction because e_iR is not injective. Thus A is a two sided ideal, and $R = A \oplus B$ is a ring direct sum.

(3) $\Rightarrow$ (1) If R is a right V-ring, then (1) follows. Assume $R \cong A \times B$, where A is right Artinian with a unique non-injective simple right R-module and $\mathrm{Soc}(A_A)$ is homogeneous and B is semisimple. Let T be the unique noninjective simple right R-module and M be a T-subinjective right R-module. We shall

prove that M is injective. Since R is right Artinian, $M = E \oplus N$ for some injective submodule E and a submodule N which does not contain non-zero injective submodule by Remark 4.1. Suppose that $N \neq 0$. Note that $\mathrm{Soc}(N) \neq 0$ by the Artinian assumption. Let S be a simple submodule of N. Since S is not injective, $S \cong T$. Let $f : T \to N$ be a non-zero homomorphism. Since M is T-subinjective and N is a direct summand of M, N is T-subinjective. Therefore, f extends to a homomorphism $g : E(S) \to N$. As f is one to one and S essential in $E(S)$, g is one to one. Therefore $g(E(S))$ is a nonzero injective submodule of N, a contradiction. Hence we must have $N = 0$ and so M is injective. Therefore T_R is t.i.b.s. □

Over a von Neumann regular ring, every simple right module is t.f.b.s. by Proposition 3.4. Thus the rings whose simple right modules are *t.f.b.s.* or injective need not be right Noetherian.

By [1, Proposition 25], every nonzero cyclic right module is t.i.b.s. if and only if R is semisimple Artinian. It is natural ask what are the rings whose simple right modules are t.i.b.s.? Theorem 4.2 in hand, we have the following.

Corollary 4.3. *Every simple module is a t.i.b.s. if and only if R is semisimple Artinian or right Artinian with a unique simple module.*

5. Rings whose modules are FP-injective or t.f.b.s.

In this section, we shall prove some necessary conditions for the right Noetherian rings whose right modules are t.f.b.s. or FP-injective.

The following lemma is clear from the definitions, so its proof is omitted.

Lemma 5.1. *The following conditions are equivalent for a ring R.*

(1) *Every right R-module is t.f.b.s. or FP-injective.*
(2) *If A_R is absolutely ${}_RB$-pure then A_R is absolutely pure or ${}_RB$ is flat.*
(3) *Every left R-module is flat or f-indigent.*

Lemma 5.2. *Let R be a right Noetherian right V-ring. Suppose every (cyclic) right module is t.f.b.s. or injective. Then $R \cong A \times B$, where B is semisimple and A is right SI with* $\mathrm{Soc}(A_A) = 0$.

Proof. By the hypothesis, R_R is t.f.b.s. or injective. First suppose R_R is t.f.b.s. Then R_R is hereditary by Theorem 3.6. We shall prove that every cyclic singular right module is injective. Let K_R be cyclic singular right R module. Since R_R is nonsingular, then $\mathrm{Hom}(K, R) = 0$. Hence, by Proposition 2.11(5), $\mathrm{Hom}(K, R)^+ \cong K \otimes R^+ = 0$. This means that K is absolutely R^+-pure. Therefore R^+ is flat or K is injective by Lemma 5.1. Since R is right Noetherian and non-injective, R^+ is not flat. So, K is injective. Hence R is a right SI-ring. Since R is a right Noetherian right V-ring, all semisimple modules are injective, so $\mathrm{Soc}(R_R)$ is injective. Then $R = A \oplus \mathrm{Soc}(R_R)$. Set $B = \mathrm{Soc}(R_R)$. Then B is a two sided ideal of R and $\mathrm{Hom}(A, B) = 0$. Otherwise, we have $\mathrm{Hom}(A, B) \neq 0$ which implies $\frac{A}{K} \cong S$ for $K \leq A$ and simple ideal S. This gives $A \cong K \oplus S$, a contradiction because $\mathrm{Soc}(A_A)=0$. Thus A is a two sided ideal and $R\cong A\times B$ is a ring direct sum.

If R_R is not t.f.b.s., then R is right injective. So R is right QF. Hence R is semisimple Artinian. This completes the proof. □

In [7, Theorem 4.2], the author prove some necessary conditions for a two sided Noetherian ring over which each right module is flat or f-indigent. In light of Lemma 5.1 the following corresponding result is a slight generalization of [7, Theorem 4.2] to right Noetherian rings.

Theorem 5.3. *Let R be a right Noetherian ring. Suppose that every right R-module is t.f.b.s. or injective. Then $R\cong A\times B$, where B is semisimple, and*

(1) *A is right hereditary right Artinian serial with homogeneous socle, $J(A)^2=0$ and A has a unique noninjective simple right A-module, or;*
(2) *A is a QF-ring that is isomorphic to a matrix ring over a local ring, or;*
(3) *A is right SI with $\mathrm{Soc}(A_A)=0$.*

Proof. Suppose that every right module is t.f.b.s. or injective. Then R is a right V-ring or $R\cong A\times B$, where A is right Artinian with a unique non-injective simple right module and $\mathrm{Soc}(A_A)$ is homogeneous and B is semisimple by Theorem 4.2. Assume that R is not a V-ring, then A is right Artinian. So $A_A=e_1A\oplus e_2A\oplus\cdots\oplus e_nA$, where $e_1,\dots,e_n$ are primitive orthogonal idempotents. By the hypothesis and the said property A has a unique noninjective minimal right ideal, say T, up to isomorphism. Also any simple right ideal which is not isomorphic to T is injective. Therefore for each $1\le i\le n$, e_iA/e_iJ is injective or isomorphic to T. If e_iA/e_iJ is injective, then $\mathrm{Hom}(e_iA,T)=0$. Then $\mathrm{Hom}(e_iA,T)^+\cong e_iA\otimes T^+=0$ by Proposition 2.11(5). Therefore, e_iA is absolutely T^+-pure, since T^+ is not flat, e_iA is injective by Lemma 5.1. In this case, we have $Soc(e_iA)\cong T$, by injectivity of e_iA. As the ring is right Artinian, $Soc(e_iA)$ is essential in e_iA. So there is a submodule $X\le e_iA$ such that $X/Soc(e_iA)$ is singular. By singularity, $X/Soc(e_iA)$ is not isomorphic to T, hence it must be injective, and so it is a direct summand of $e_iA/Soc(e_iA)$. Since e_iA is local, $e_iA/Soc(e_iA)$ is indecomposable. Therefore $X/Soc(e_iA)=e_iA/Soc(e_iA)$, and so the composition length of e_iA is 2.

Now, if $e_iA/e_iJ\cong T$, then e_iA/e_iJ is projective. So e_iJ is a direct summand of e_iA. But e_iA is local, so e_iA must be simple.

As a consequence, A is a direct sum of right ideals which are simple or injective with composition length 2. Hence by [6, 13.5] we obtain A is serial and $J(A)^2=0$. Now, by the hypothesis, R_R is hereditary or injective. If R is hereditary, we obtain (1). If R is injective, then R is right QF by the Noetherian assumption. Then $e_iA\cong e_jA$ for each i and j. That is, $A\cong(eA)^n$ for some local idempotent e in A. In conclusion we obtain (2).

If R is a right V-ring, then (3) follows by Lemma 5.2. This completes the proof. □

References

[1] R. Alizade, E. Büyükaşık, and N. Er, *Rings and modules characterized by opposites of injectivity*, J. Algebra **409** (2014), 182–198.
[2] R. Alizade and Y. Durğun, *Test modules for flatness*, Rend. Semin. Mat. Univ. Padova **137** (2017), 75–91.
[3] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, second edition, Graduate Texts in Mathematics, **13**, Springer-Verlag, New York, 1992.
[4] P. Aydoğdu and S. R. López-Permouth, *An alternative perspective on injectivity of modules*, J. Algebra **338** (2011), 207–219.
[5] P. Aydoğdu and B. Saraç, *On Artinian rings with restricted class of injectivity domains*, J. Algebra **377** (2013), 49–65.
[6] N. V. Dung, D. V. Hunh, P. F. Simith, and R. Wisbauer, *Extending modules*, Pitman Research Notes in Mathematics Series, **313**, Longman Scientific & Technical, Harlow, 1994.
[7] Y. Durğun, *An alternative perspective on flatness of modules*, J. Algebra Appl. **15** (2016), no. 8.
[8] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, De Gruyter Expositions in Mathematics, **30**, Walter de Gruyter & Co., Berlin, 2000.
[9] K. R. Goodearl, *Singular torsion and the splitting properties*, American Mathematical Society, Providence, RI, 1972.
[10] T. Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics, **189**, Springer-Verlag, New York, 1999.
[11] C. Megibben, *Absolutely pure modules*, Proc. Amer. Math. Soc. **26** (1970), 561–566.
[12] J. J. Rotman, *An Introduction to Homological Algebra*, Pure and Applied Mathematics, **85**, Academic Press, Inc., New York, 1979.
[13] W. V. Vasconcelos, *On finitely generated flat modules*, Trans. Amer. Math. Soc. **138** (1969), 505–512.

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