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BOUNDEDNESS OF THE STRONG MAXIMAL OPERATOR WITH THE HAUSDORFF CONTENT

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Abstract. Let n be the spatial dimension. For d, $0 < d \leq n$, let H^d be the d-dimensional Hausdorff content. The purpose of this paper is to prove the boundedness of the dyadic strong maximal operator on the Choquet space $L^p(H^d, \mathbb{R}^n)$ for $\min(1, d) < p$. We also show that our result is sharp.

1. Introduction

The purpose of this paper is to prove the boundedness of the strong maximal function on the Choquet spaces. For a locally integrable function f on $\mathbb{R}^n$, the strong maximal operator $\mathcal{M}_S$ is defined by

$$\mathcal{M}_S f(x) := \sup_R \mathbf{1}_R(x) ⨍_R |f(y)| \, \mathrm{d}y,$$

where the supremum is taken over all rectangles in $\mathbb{R}^n$ whose sides are parallel to the coordinate axes and the barred integral $⨍_R f \, \mathrm{d}x$ stands for the usual integral average of f over R. $\mathbf{1}_R$ denotes the characteristic function of R. As usual, we can reduce the problem to the dyadic situation. We denote by $\mathcal{D}(\mathbb{R})$ the family of all dyadic intervals in $\mathbb{R}$, that is,

$$\mathcal{D}(\mathbb{R}) = \{2^k(m + [0, 1)) : k, m \in \mathbb{Z}\}.$$

Then elements of $\mathcal{R} = \mathcal{D}(\mathbb{R}) \times \mathcal{D}(\mathbb{R}) \times \cdots \times \mathcal{D}(\mathbb{R}) = \{\prod_{k=1}^n I_k : I_k \in \mathcal{D}(\mathbb{R})\}$ are called the dyadic rectangles. On the other hand, we denote the usual dyadic cubes by $\mathcal{D}(\mathbb{R}^n)$, i.e.,

$$\mathcal{D}(\mathbb{R}^n) = \{2^k(m + [0, 1)^n) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}.$$

We define the dyadic strong maximal function by

$$M_S f(x) = \sup_{R \in \mathcal{R}} \mathbf{1}_R(x) ⨍_R |f(y)| \, \mathrm{d}y,$$

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where the supremum is taken over all dyadic rectangles in $\mathcal{R}$.

If $E \subset \mathbb{R}^n$ and $0 < d \leq n$, then the d-dimensional Hausdorff content H^d of E is defined by

$$H^d(E) := \inf \sum_{j=1}^{\infty} l(Q_j)^d,$$

where the infimum is taken over all coverings of E by countable families of dyadic cubes Q_j and $l(Q)$ denotes the side length of the cube Q. In [2], for the Hardy-Littlewood maximal operator M, Orobitg and Verdera proved the strong type inequality

$$\int_{\mathbb{R}^n} (Mf)^p \, \mathrm{d}H^d \leq C \int_{\mathbb{R}^n} |f|^p \, \mathrm{d}H^d \tag{1.1}$$

for $d/n < p < \infty$, and the weak type inequality

$$\sup_{t>0} tH^d(\{x \in \mathbb{R}^n : \, Mf(x) > t\})^{1/p} \leq C \int_{\mathbb{R}^n} |f|^p \, \mathrm{d}H^d, \quad t > 0,$$

for $p = d/n$. Here, the integrals are taken in the Choquet sense, that is, the Choquet integral of $f \geq 0$ with respect to a set function $\mathcal{C}$ is defined by

$$\int_{\mathbb{R}^n} f \, \mathrm{d}\mathcal{C} := \int_0^{\infty} \mathcal{C}(\{x \in \mathbb{R}^n : \, f(x) > t\}) \, \mathrm{d}t.$$

Formerly, Adams proved the strong type estimate for $p = 1$ and $0 < d < n$ in [1] by using duality of BMO and the Hardy space H^1 among other things.

In this note, we prove the following strong type inequality for M_S.

Theorem 1.1. *Let $0 < d \leq n$. Then for $\min(1, d) < p < \infty$, we have*

$$\int_{\mathbb{R}^n} (M_S f)^p \, \mathrm{d}H^d \leq C \int_{\mathbb{R}^n} |f| \, \mathrm{d}H^d.$$

Moreover, the exponent p is sharp.

Remark 1.2. (1) Using the standard dyadic argument, we can prove the same inequality for $\mathcal{M}_S$. Further, one may expect to establish the weak type estimate for $p = \min(1, d)$. But we cannot prove it until now, and further refinement of the known proofs for the endpoint estimate for the strong maximal operator would be needed.

(2) We define the k-th variable maximal operator by

$$M_k f(x) = \sup_{I \in \mathcal{D}} \mathbf{1}_I(x_k) \fint_I |f(x_1, \ldots, y_k, \ldots, x_n)| \, \mathrm{d}y_k$$

for $1 \leq k \leq n$. That is, M_k is the operator defined on functions in $\mathbb{R}^n$ by letting the one-dimensional Hardy-Littlewood maximal operator acts on the k-th variable while keeping the remaining variables fixed. We first notice that the strong maximal operator is dominated pointwisely by an iterated maximal operator as follows

$$\mathcal{M}_S f(x) \leq M_n M_{n-1} \cdots M_1 f(x). \tag{1.2}$$

By Fubini's theorem for the Lebesgue measure $\mathrm{d}x$ and boundedness of M_k on $L^p(\mathbb{R}, \mathrm{d}x)$, we can get

$$\|\mathcal{M}_S f\|_{L^p(\mathbb{R}^n,\mathrm{d}x)} \le C\|f\|_{L^p(\mathbb{R}^n,\mathrm{d}x)}$$

for $p > 1$. However, we have not known whether the Fubini-type theorem holds or not for the Hausdorff content, this strategy does not work.

(3) Comparing Orobitg and Verdera's result (1.1), one may be wondering why the range $p, \min(1, d) < p < \infty$ in Theorem 1.1, does not depend on the spatial dimension n. We will give a remark on this point in the last section.

2. Lemmas

We begin to prove the following lemma. This is due to Orobitg and Verdera [2].

Lemma 2.1. *For any dyadic cube $Q \in \mathcal{D}(\mathbb{R}^n)$ and $\min(1, d) < p$, we have*

$$\int_{\mathbb{R}^n} M_S[\mathbf{1}_Q]^p \,\mathrm{d}H^d \le Cl(Q)^d.$$

Proof. Fix a dyadic interval $I \in \mathcal{D}(\mathbb{R})$. We define

$$\pi^0(I) := I,$$

and $\pi^j(I)$ denotes the smallest interval in $\mathcal{D}(\mathbb{R})$ containing $\pi^{j-1}(I)$ for $j = 1, 2, \dots$. We see $l(\pi^j(I)) = 2^j l(I)$. We denote by Pr_k, $k = 1, 2, \dots, n$ the projection on the x_k-axis. Obviously, $Q = \prod_{k=1}^n \mathrm{Pr}_k(Q)$. Further, we define

$$\mathcal{P}^m(Q) := \left\{ \prod_{k=1}^n \pi^{j_k}(\mathrm{Pr}_k(Q)) : \sum_{k=1}^n j_k = m \right\}, \quad m = 0, 1, 2, \dots.$$

In particular, we deduce that $\mathcal{P}^0(Q) = \{Q\}$, and that the number of elements in $\mathcal{P}^m(Q)$ is $\#\mathcal{P}^m(Q) = \binom{m+n-1}{n-1}$. Here, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Now we see that if $R \in \mathcal{P}^m(Q)$, then

$$\begin{aligned} |R| &= \prod_{k=1}^n l(\pi^{j_k}(\mathrm{Pr}_k(Q))) \\ &= \prod_{k=1}^n 2^{j_k} l(\mathrm{Pr}_k(Q)) \\ &= 2^{\sum_{k=1}^n j_k} |Q| = 2^m |Q|. \end{aligned}$$

This implies that the rectangle R in $\mathcal{P}^m(Q)$ contains the original cube Q and its volume is just $|Q|$ times 2^m. Moreover, we set

$$B_m := \bigcup_{R \in \mathcal{P}^m(Q)} R, \quad m = 0, 1, 2, \dots.$$

By definition, we have

$$\begin{aligned}|B_m| &\le \sum_{R\in\mathcal{P}^m(Q)} |R| \\ &= \#\mathcal{P}^m(Q)\cdot 2^m|Q| \\ &= \binom{m+n-1}{n-1}2^m|Q|,\end{aligned}$$

and this implies that B_m can be covered by at most $\binom{m+n-1}{n-1}2^m$ cubes Q. Now, we can show that

$$M_S[\mathbf{1}_Q](x) = \mathbf{1}_Q(x) + \sum_{m=1}^{\infty} 2^{-m}\mathbf{1}_{B_m\setminus B_{m-1}}(x).$$

Indeed, if $m = 0$ and $x \in Q$, then obviously $M_S[\mathbf{1}_Q](x) = 1$. If $m \ge 1$ and $x \in B_m \setminus B_{m-1}$, then there exists $R \in \mathcal{P}^m$ containing x, and for all $k; 0 \le k \le m-1$, and any $R' \in \mathcal{P}^k$, x does not belong to R'. Thus,

$$M_S[\mathbf{1}_Q](x) = \frac{|Q\cap R|}{|R|} = \frac{|Q|}{|R|} = \frac{1}{2^m}.$$

Now, we have

$$M_S[\mathbf{1}_Q](x)^p = \mathbf{1}_Q(x) + \sum_{m=1}^{\infty} 2^{-mp}\mathbf{1}_{B_m\setminus B_{m-1}}(x),$$

and hence

$$\begin{aligned}\int_{\mathbb{R}^n} M_S[\mathbf{1}_Q]^p \,\mathrm{d}H^d &\le l(Q)^d + \sum_{m=1}^{\infty} 2^{-mp}H^d(B_m\setminus B_{m-1}) \\ &\le l(Q)^d + \sum_{m=1}^{\infty} 2^{-mp}H^d(B_m).\end{aligned}$$

Case $d \ge 1$: We notice $p > 1$. By the previous observation, we can cover B_m by $\binom{m+n-1}{n-1}2^m$ copies of cubes Q so that

$$\begin{aligned}\int_{\mathbb{R}^n} M_S[\mathbf{1}_Q]^p \,\mathrm{d}H^d &\le l(Q)^d + \sum_{m=1}^{\infty} 2^{-mp}\binom{m+n-1}{n-1}2^m l(Q)^d \\ &\le l(Q)^d + l(Q)^d\sum_{m=1}^{\infty}\frac{(m+n-1)^{n-1}}{(n-1)!}2^{(1-p)m}\end{aligned}$$

and hence by d'Alembert's criterion the last series converges as $1-p<0$.

Case $d < 1$: We notice $p > d$. Covering B_m by one large cube $\widetilde{Q}$ whose side length is $2^m l(Q)$, we have

$$\int_{\mathbb{R}^n} M_S[\mathbf{1}_Q]^p \,\mathrm{d}H^d \le l(Q)^d + \sum_{m=1}^{\infty} 2^{-mp}2^{md}l(Q)^d$$

$$= l(Q)^d + l(Q)^d \sum_{m=1}^{\infty} 2^{(d-p)m},$$

and the last series converges as $d - p < 0$. This completes the proof of the lemma. □

3. Proof of Theorem 1.1

The proof is due to [2]. We may assume that $f \geq 0$. For each integer k, let $\{Q_j^k\}_j$ be a family of nonoverlapping dyadic cubes Q_j^k such that

$$\{x \in \mathbb{R}^n : 2^k < f(x) \leq 2^{k+1}\} \subset \bigcup_j Q_j^k$$

and

$$\sum_j l(Q_j^k)^d \leq 2H^d(\{x \in \mathbb{R}^n : 2^k < f(x) \leq 2^{k+1}\}).$$

Set $g = \sum_k 2^{p(k+1)} \mathbf{1}_{A_k}$, where $A_k = \bigcup_j Q_j^k$. Thus, $f^p \leq g$.

Assume first that $d < 1$ and $1 \leq p$. Then

$$(M_S f)^p \leq M_S[f^p] \leq M_S[g] \leq \sum_k 2^{p(k+1)} \sum_j M_S[\mathbf{1}_{Q_j^k}].$$

By Lemma 2.1,

$$\begin{aligned}
\int_{\mathbb{R}^n} (M_S f)^p \,\mathrm{d}H^d &\leq \sum_k 2^{p(k+1)} \sum_j \int_{\mathbb{R}^n} M_S[\mathbf{1}_{Q_j^k}] \,\mathrm{d}H^d \\
&\leq C \sum_k 2^{p(k+1)} \sum_j l(Q_j^k)^d \\
&\leq C \sum_k 2^{p(k+1)} H^d(\{x \in \mathbb{R}^n : 2^k < f(x) \leq 2^{k+1}\}) \\
&\leq C \sum_k \frac{2^{2p}}{2^p - 1} \int_{2^{(k-1)p}}^{2kp} H^d(\{x \in \mathbb{R}^n : f(x)^p > t\}) \,\mathrm{d}t \\
&\leq C \int_{\mathbb{R}^n} f^p \,\mathrm{d}H^d,
\end{aligned}$$

which proves this case.

Assume now that $d < p < 1$. Since $f \leq \sum_k 2^{k+1} \mathbf{1}_{A_k}$,

$$M_S f \leq \sum_k 2^{k+1} \sum_j M_S[\mathbf{1}_{Q_j^k}].$$

We have that, since $p < 1$,

$$(M_S f)^p \leq \sum_k 2^{p(k+1)} \sum_j M_S[\mathbf{1}_{Q_j^k}]^p$$

and, hence,

$$\int_{\mathbb{R}^n} (M_S f)^p \, \mathrm{d}H^d \leq C \sum_k 2^{(k+1)p} \sum_j l(Q_j^k)^d \leq C \int_{\mathbb{R}^n} f^p \, \mathrm{d}H^d.$$

Finally, if we assume $d \geq 1$, then since $p > 1$, so we have nothing to prove. This completes the proof of the inequality in Theorem 1.1.

In the next section, we discuss the sharpness of the exponent p.

4. Sharpness

In this section, we show that the condition $\min(1, d) < p$ in Theorem 1.1 is sharp. In particular, for some dyadic cube Q we show that

$$\int_{\mathbb{R}^n} M_S[\mathbf{1}_Q]^p \, \mathrm{d}H^d = \infty$$

if $p \leq \min(1, d)$.

Let $d < n$. Fix a dyadic cube Q as

$$Q = [0, l(Q)]^n.$$

That is, Q is the cube which is located in the first quadrant and one of its vertices is on the origin. We denote $F_0 := Q$, and

$$F_m := [0, 2^m l(Q)] \times [0, l(Q)]^{n-1}, \quad (m = 0, 1, 2, \dots).$$

For each m, the rectangle F_m is in $\mathcal{P}^m(Q)$ and contains the cube Q and side-lengths are $2^m l(Q)$ and $l(Q)$. We first observe

$$\begin{aligned}
\int_{\mathbb{R}^n} M_S[\mathbf{1}_Q]^p \, \mathrm{d}H^d &= p \int_0^\infty H^d(M_S[\mathbf{1}_Q] > t) t^{p-1} \, \mathrm{d}t \\
&= p \sum_{m=0}^\infty \int_{2^{-m-1}}^{2^{-m}} H^d(M_S[\mathbf{1}_Q] > t) t^{p-1} \, \mathrm{d}t \\
&\geq p \sum_{m=0}^\infty H^d(M_S[\mathbf{1}_Q] > 2^{-m}) \int_{2^{-m-1}}^{2^{-m}} t^{p-1} \mathrm{d}t \\
&= (1 - 2^{-p}) \sum_{m=1}^\infty H^d(B_{m-1}) 2^{-mp} \\
&\geq (1 - 2^{-p}) \sum_{m=1}^\infty H^d(F_{m-1}) 2^{-mp},
\end{aligned}$$

where we have used the fact that

$$\{x \in \mathbb{R}^n : M_S[\mathbf{1}_Q](x) > 2^{-m}\} = B_{m-1} \supset F_{m-1}$$

in the last two lines. To compute $H^d(F_{m-1})$, we need to find the infimum covering of F_{m-1} by the dyadic cubes in $\mathcal{D}(\mathbb{R}^n)$. It is easy to see that (see also

Remark 4.1 below)

$$
\begin{aligned}
H^d(F_{m-1}) &= \min_{0\le k\le m-1} 2^{m-1-k}(2^k l(Q))^d \\
&= l(Q)^d \min_{0\le k\le m-1} 2^{kd+m-1-k} \\
&= \begin{cases} 2^{m-1} l(Q)^d, & (1\le d<n), \\ 2^{(m-1)d} l(Q)^d, & (0<d<1). \end{cases}
\end{aligned}
$$

Case $1 \le d < n$: We have

$$
\begin{aligned}
\int_{\mathbb{R}^n} M_S[\mathbf{1}_Q]^p \,\mathrm{d}H^d &\ge (1-2^{-p}) \sum_{m=1}^{\infty} H^d(F_{m-1}) 2^{-mp} \\
&= (1-2^{-p}) \sum_{m=1}^{\infty} 2^{m-1} l(Q)^d 2^{-mp} \\
&= (1-2^{-p}) l(Q)^d \sum_{m=1}^{\infty} 2^{(1-p)m-1},
\end{aligned}
$$

then since $p \le 1$, the last series diverges.

Case $0 < d < 1$: We have

$$
\begin{aligned}
\int_{\mathbb{R}^n} M_S[\mathbf{1}_Q]^p \,\mathrm{d}H^d &\ge (1-2^{-p}) \sum_{m=1}^{\infty} H^d(F_{m-1}) 2^{-mp} \\
&= (1-2^{-p}) \sum_{m=1}^{\infty} 2^{(m-1)d} l(Q)^d 2^{-mp} \\
&= (1-2^{-p}) l(Q)^d \sum_{m=1}^{\infty} 2^{(d-p)m-d},
\end{aligned}
$$

then since $p \le d$, the last series also diverges.

Remark 4.1. We describe the reason why the range p in Theorem 1.1 does not depend on the dimension n. As mentioned above, we need to compute the Hausdorff content of the dyadic rectangle F_{m-1} and find the minimum covering of F_{m-1} by using the family of dyadic cubes. Actually, the covering $\{Q_j\}_j$ of F_{m-1} which minimizes $\sum_j l(Q_j)^d$ is different depending on d. That is, if $0 < d < 1$, the minimum is attained by one large cube whose sidelength is $2^{m-1} l(Q)$, and if $1 < d$, 2^{m-1} cubes $\{Q_j\}$, whose sidelengths are equal to $l(Q)$, attain the minimum. The border $d = 1$ does not depend on n, this is because p is independent of n.

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