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PRICING SYMMETRIC TYPE OF POWER QUANTO OPTIONS

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Abstract. We derive full closed-form expressions for the prices of European symmetric power quanto call options with four different forms of terminal payoffs under the assumption of the classical lognormal asset price and exchange rate model.

1. Introduction

A power option differs from standard vanilla option in the sense that the payoff function is not linear but raised to some positive power in the underlying spot. Through its non-linear payoff, a power option can hedge non-linear price risks. For example, if an importer earns profit by a percentage mark-up on imported products, the exchange rate change will lead to a price change, which in turn will affect demand volumes. The importer can hedge a risk of non-linearly decreasing earnings by purchasing a power option that provides a leveraged payout.

A quanto option is a cross-currency option where the underlying asset is denominated in a currency other than the currency in which the option is settled. If investors were to invest directly in a foreign stock index, they would expose themselves to risks in that foreign index as well as risks due to the fluctuations in the currency exchange rate. A quanto option can create greater liquidity in smaller or riskier markets by removing currency risk for overseas investors.

There are several types of power options and quanto options. In general, there are two types of power options which are asymmetric and symmetric power options (see page 87 of [9]). With an asymmetric power option, the underlying S_T and the strike K of a standard option payoff function are individually raised to the n-th power. In the symmetric type, the entire vanilla option payoff is raised to the n-th power so that put and call display the same

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payoff shapes. Meanwhile, quanto options can have four different forms of terminal payoffs: a foreign equity option converted to domestic currency, a foreign equity option struck in domestic currency, a foreign equity option struck in predetermined domestic currency or an FX option denoted in domestic currency.

So far, many researches have been done on power option pricing and quanto option pricing under the classical Black-Scholes [1] framework or under more sophisticated volatility assumptions. To mention some basic references, Heynen and Kat [3] obtained closed-form formulas for the prices of general power options under the Black-Scholes framework and suggested various hedging methods. Tompkins [8] focused on pricing and hedging of power options both theoretically and from the market point of view by exhibiting leverage nature of power options. Wystup [10] obtained various quanto option price formula including quanto forward option and quanto digital option under the Black-Scholes framework and introduced three vega positions on hedging of quanto options together with the application to performance linked deposit. The textbook of Wystup [9] contains general theory and application of quanto options and the textbook of Kwok [4] contains detailed explanation of four different terminal payoff types of quanto options.

Power options and quanto options are among the popular exotic options in the currency-related markets. Their contribution to reduction of risk encourages participation in these markets. For that reason, the combination of power option and quanto option can be considered on its optimal valuation. Among the very few existing researches related to power quanto option pricing is a research work on pricing power exchange option by Blenman and Clark [2] where the authors combine power option and Margrabe [7] type exchange option. Recently, Lee et al. [6] derived closed-form expressions for asymmetric power quanto call options price under the classical lognormal asset price and exchange rate model.

In this paper, we derive full closed-form expressions for the price of symmetric power quanto call options with four different forms of terminal payoff under the assumption of lognormal asset price and exchange rate. Due to the binomial term $(S_T - K)^n$, the general value formula derivation for symmetric power quanto options is more complicated than that of asymmetric ones. The binomial terms involve foreign exchange rate as well as underlying asset price in valuation of symmetric power option, which makes us follow delicate measure theoretic approaches.

In Section 2, we specify the dynamics of the processes of underlying asset price and exchange rate in the risk-neutral world. In Section 3, we specify four different forms of payoff at maturity and obtain the analytic expressions for the price of each symmetric power quanto call option together with another symmetric type of power quanto call options involving similar binomial terms in maturity payoffs.

2. Risk-neutral quanto dynamics

Let r_f and r_d denote the constant foreign and domestic riskless rates, respectively, and let q denote the dividend yield rate of a certain foreign asset. We assume that S_t is the asset price in foreign currency with the constant volatility σ_S, and V_t is the exchange rate in foreign currency per unit of the domestic currency with the constant volatility σ_V. Let the risk-neutral dynamics of S_t and V_t in foreign currency be governed by

$$\begin{cases} dS_t = (r_f - q)\, S_t dt + \sigma_S S_t dB_t^{\mathbb{Q}^f}, \\ dV_t = (r_f - r_d)\, V_t dt + \sigma_V V_t dW_t^{\mathbb{Q}^f}, \end{cases} \tag{2.1}$$

where $B_t^{\mathbb{Q}^f}$ and $W_t^{\mathbb{Q}^f}$ are two standard Brownian motions in foreign currency with the correlation ρ so that $dB_t^{\mathbb{Q}^f} dW_t^{\mathbb{Q}^f} = \rho dt$.

Then we obtain the following risk-neutral dynamics of (2.1) in domestic currency:

$$\begin{cases} dS_t = (r_f - q - \rho\sigma_S\sigma_V)\, S_t dt + \sigma_S S_t dB_t^{\mathbb{Q}^d}, \\ dV_t = (r_d - r_f)\, V_t dt + \sigma_V V_t dW_t^{\mathbb{Q}^d} \end{cases} \tag{2.2}$$

by the well-known standard procedure (see page 95 of [9] or Section 2 of [5]), where $B_t^{\mathbb{Q}^d}$ and $W_t^{\mathbb{Q}^d}$ are two correlated standard Brownian motions in domestic currency.

3. Symmetric power quanto option pricing

The following theorems in next subsections give the explicit formulas for the prices of European power quanto call options with constant foreign and domestic riskless rates according to four different forms of terminal payoff: a foreign equity option converted to domestic currency, a foreign equity option struck in domestic currency, a foreign equity option struck in predetermined domestic currency and an FX option denoted in domestic currency, respectively.

3.1. Type I

For positive integer n and foreign currency strike price K_f, the foreign equity power-n quanto call option (struck in foreign currency) converted to domestic currency has a maturity payoff given by

$$\begin{aligned} V_T \left\{\max\left(S_T - K_f, 0\right)\right\}^n &= V_T \left(S_T - K_f\right)^n \mathbb{1}_{\{S_T > K_f\}} \\ &= V_T \sum_{j=0}^{n} \binom{n}{j} S_T^{n-j} \left(-K_f\right)^j \mathbb{1}_{\{S_T > K_f\}}, \end{aligned} \tag{3.1}$$

where $\binom{n}{j} = \frac{n!}{j!(n-j)!}$.

Theorem 3.1. *Under the assumptions of* (2.2) *with* $n \in \mathbf{N}$*, the price of a European power-*n *quanto call option at time* t *in domestic currency with the payoff* (3.1) *is given by*

$$\begin{aligned} & C_1^{(n)}(t, S_t, V_t) \\ &= V_t e^{-r_f(T-t)} \sum_{j=0}^{n} \binom{n}{j} S_t^{n-j} (-K_f)^j e^{(n-j)\left\{r_f - q + \frac{(n-j-1)\sigma_S^2}{2}\right\}(T-t)} N\left(d^{(n-j)}\right), \end{aligned}$$

where

$$d^{(n-j)} = \frac{\ln \frac{S_t}{K_f} + \left\{r_f - q + \left(n - j - \frac{1}{2}\right)\sigma_S^2\right\}(T-t)}{\sigma_S \sqrt{T-t}}$$

and $N(\cdot)$ *denotes the cumulative distribution function for the standard normal distribution.*

Proof. We may write $C_1^{(n)}$ as

(3.2)

$$\begin{aligned} & C_1^{(n)}(t, S_t, V_t) \\ &= e^{-r_d(T-t)} \mathbb{E}_{\mathbb{Q}^d}\left[V_T \sum_{j=0}^{n} \binom{n}{j} S_T^{n-j} (-K_f)^j \mathbb{1}_{\{S_T > K_f\}} \middle| \mathcal{F}_t \right] \\ &= V_t e^{-r_f(T-t)} \sum_{j=0}^{n} \binom{n}{j} (-K_f)^j \mathbb{E}_{\mathbb{Q}^d}\left[e^{-\frac{\sigma_V^2}{2}(T-t) + \sigma_V\left(W_T^{\mathbb{Q}^d} - W_t^{\mathbb{Q}^d}\right)} S_T^{n-j} \mathbb{1}_{\{S_T > K_f\}} \middle| \mathcal{F}_t \right]. \end{aligned}$$

For a new risk-neutral probability measure $\tilde{\mathbb{Q}}^d$, the Radon-Nykodým derivative of $\tilde{\mathbb{Q}}^d$ with respect to $\mathbb{Q}^d$ is defined by

$$\left. \frac{d\tilde{\mathbb{Q}}^d}{d\mathbb{Q}^d} \right|_{\mathcal{F}_t} = e^{-\frac{\sigma_V^2}{2} t + \sigma_V W_t^{\mathbb{Q}^d}}.$$

Then Girsanov's theorem implies that

$$B_t^{\tilde{\mathbb{Q}}^d} = B_t^{\mathbb{Q}^d} - \rho \sigma_V t \tag{3.3}$$

is again a $\tilde{\mathbb{Q}}^d$-standard Brownian motion. Note then that the $\tilde{\mathbb{Q}}^d$-dynamic of S_t is given by

$$dS_t = (r_f - q) S_t dt + \sigma_S S_t dB_t^{\tilde{\mathbb{Q}}^d} \tag{3.4}$$

from (2.2) and (3.3). Thus, (3.2) becomes

(3.5)

$$C_1^{(n)}(t, S_t, V_t) = V_t e^{-r_f(T-t)} \sum_{j=0}^{n} \binom{n}{j} (-K_f)^j \mathbb{E}_{\tilde{\mathbb{Q}}^d}\left[S_T^{n-j} \mathbb{1}_{\{S_T > K_f\}} \middle| \mathcal{F}_t \right].$$

Likewise, for a new risk-neutral probability measure $\hat{\mathbb{Q}}^d$, the Radon-Nykodým derivative of $\hat{\mathbb{Q}}^d$ with respect to $\tilde{\mathbb{Q}}^d$ is defined by

$$\frac{d\hat{\mathbb{Q}}^d}{d\tilde{\mathbb{Q}}^d} = \frac{S_T^{n-j}}{\mathbb{E}_{\tilde{\mathbb{Q}}^d}\left[S_T^{n-j} \middle| \mathcal{F}_t \right]} \tag{3.6}$$

on $\mathcal{F}_T$. On the other hand, the dynamics of S_t^{n-j} under the measure $\tilde{\mathbb{Q}}^d$ is given by

$$dS_t^{n-j} = (n-j)\left\{ r_f - q + \frac{(n-j-1)\,\sigma_S^2}{2} \right\} S_t^n dt + (n-j)\,\sigma_S S_t^{n-j} dB_t^{\tilde{\mathbb{Q}}^d} \tag{3.7}$$

from (3.4). Then Girsanov's theorem implies from (3.3) that

$$\begin{aligned} B_t^{\hat{\mathbb{Q}}^d} &= B_t^{\tilde{\mathbb{Q}}^d} - (n-j)\,\sigma_S t \\ &= B_t^{\mathbb{Q}^d} - \{(n-j)\,\sigma_S + \rho\sigma_V\}\, t \end{aligned} \tag{3.8}$$

is again a $\hat{\mathbb{Q}}^d$-standard Brownian motion. Moreover, the $\hat{\mathbb{Q}}^d$-dynamics of S_t is given by

$$dS_t = \left\{ r_f - q + (n-j)\,\sigma_S^2 \right\} S_t dt + \sigma_S S_t dB_t^{\hat{\mathbb{Q}}^d} \tag{3.9}$$

from (2.2) and (3.8). Finally, (3.5) becomes

$$\begin{aligned} &C_1^{(n)}(t, S_t, V_t) \\ &= V_t e^{-r_f(T-t)} \sum_{j=0}^{n} \binom{n}{j} (-K)^j\, \mathbb{E}_{\tilde{\mathbb{Q}}^d}\left[S_T^{n-j} \middle| \mathcal{F}_t \right] \hat{\mathbb{Q}}\,(S_T > K_f) \\ &= V_t e^{-r_f(T-t)} \sum_{j=0}^{n} \binom{n}{j} S_t^{n-j} (-K_f)^j\, e^{(n-j)\left\{ r_f - q + \frac{(n-j-1)\sigma_S^2}{2} \right\}(T-t)} N\left(d^{(n-j)}\right) \end{aligned}$$

from (3.7) and (3.9), where

$$d^{(n-j)} = \frac{\ln \frac{S_t}{K_f} + \left\{ r_f - q + \left(n - j - \frac{1}{2}\right)\sigma_S^2 \right\}(T-t)}{\sigma_S \sqrt{T-t}}.$$

□

Now, we consider another symmetric type power quanto option with the maturity payoff given by

$$V_T \max\{(S_T - K_f)^n, 0\}. \tag{3.10}$$

This power quanto option's maturity payoff coincides with the above defined powered option for odd exponents. For even exponents, i.e., $n = 2L$ for any $L \in \mathbf{N}$, (3.10) can be rewritten as

$$V_T (S_T - K_f)^{2L} = V_T \sum_{j=0}^{2L} \binom{2L}{j} S_T^{2L-j} (-K_f)^j. \tag{3.11}$$

Thus, we have the following option pricing formula:

$$\begin{aligned}
&C_1^{(n)}(t,S_t,V_t)\\
&= e^{-r_d(T-t)}\mathbb{E}_{\mathbb{Q}^d}\left[V_T \sum_{j=0}^{2L} \binom{2L}{j} S_T^{2L-j}(-K_f)^j \middle| \mathcal{F}_t \right]\\
&= V_t e^{-r_f(T-t)} \sum_{j=0}^{2L} \binom{2L}{j} (-K_f)^j\, \mathbb{E}_{\tilde{\mathbb{Q}}^d}\left[S_T^{2L-j} \middle| \mathcal{F}_t \right]\\
&= V_t e^{-r_f(T-t)} \sum_{j=0}^{2L} \binom{2L}{j} (-K_f)^j\, S_t^{2L-j} e^{(2L-j)\left\{r_f - q + \frac{(2L-j-1)\sigma_S^2}{2}\right\}(T-t)}
\end{aligned}$$

by substituting $2L$ for n from (3.7).

3.2. Type II

For positive integer n and domestic currency strike price K_d, the foreign equity power-n quanto call option in domestic currency has a maturity payoff given by

$$\begin{aligned}
\{\max(V_T S_T - K_d, 0)\}^n &= (V_T S_T - K_d)^n\, \mathbb{1}_{\{V_T S_T > K_d\}}\\
&= \sum_{j=0}^{n} \binom{n}{j} V_T^{n-j} S_T^{n-j}(-K_d)^j\, \mathbb{1}_{\{V_T S_T > K_d\}}. \qquad (3.12)
\end{aligned}$$

Theorem 3.2. *Under the assumptions of* (2.2) *with* $n \in \mathbf{N}$, *the price of a European power-n quanto call option at time t in domestic currency with the payoff* (3.12) *is given by*

$$\begin{aligned}
&C_2^{(n)}(t,S_t,V_t)\\
&= e^{-r_d(T-t)} \sum_{j=0}^{n} \binom{n}{j} (-K_d)^j V_t^{n-j} S_t^{n-j} e^{(n-j)\left\{r_d - q + \frac{(n-j-1)\left(\sigma_S^2+\sigma_V^2+2\rho\sigma_S\sigma_V\right)}{2}\right\}(T-t)} N\left(d^{(n-j)}\right),
\end{aligned}$$

where

$$d^{(n-j)} = \frac{\ln \frac{V_t S_t}{K_d} + \left\{r_d - q + \left(n-j-\frac{1}{2}\right)\left(\sigma_S^2+\sigma_V^2+2\rho\sigma_S\sigma_V\right)\right\}(T-t)}{\sqrt{\left(\sigma_S^2+\sigma_V^2+2\rho\sigma_S\sigma_V\right)(T-t)}}.$$

Proof. We may write $C_2^{(n)}$ as

$$\begin{aligned}
C_2^{(n)}(t,S_t,V_t) &= e^{-r_d(T-t)}\mathbb{E}_{\mathbb{Q}^d}\left[\sum_{j=0}^{n} \binom{n}{j} \hat{S}_T^{n-j}(-K_d)^j \mathbb{1}_{\{\hat{S}_T > K_d\}} \middle| \mathcal{F}_t \right]\\
&= e^{-r_d(T-t)} \sum_{j=0}^{n} \binom{n}{j} (-K_d)^j\, \mathbb{E}_{\mathbb{Q}^d}\left[\hat{S}_T^{n-j} \mathbb{1}_{\{\hat{S}_T > K_d\}} \middle| \mathcal{F}_t \right], \qquad (3.13)
\end{aligned}$$

where $\hat{S}_T = V_T S_T$. We note that the $\mathbb{Q}^d$-dynamics of $\hat{S}_T$ is given by

$$d\hat{S}_t = (r_d - q)\,\hat{S}_t dt + \sigma_S \hat{S}_t dB_t^{\mathbb{Q}^d} + \sigma_V \hat{S}_t dW_t^{\mathbb{Q}^d}$$

from (2.2), which can be rewritten as

$$d\hat{S}_t = (r_d - q)\,\hat{S}_t dt + \sqrt{\sigma_S^2 + \sigma_V^2 + 2\rho\sigma_S\sigma_V}\,\hat{S}_t dZ_t^{\mathbb{Q}^d}, \tag{3.14}$$

where $Z_t^{\mathbb{Q}^d}$ is a new $\mathbb{Q}^d$-standard Brownian motion. For a new risk-neutral probability measure $\tilde{\mathbb{Q}}^d$, the Radon-Nykodým derivative of $\tilde{\mathbb{Q}}^d$ with respect to $\mathbb{Q}^d$ is defined by

$$\frac{d\tilde{\mathbb{Q}}^d}{d\mathbb{Q}^d} = \frac{\hat{S}_T^{n-j}}{\mathbb{E}_{\mathbb{Q}^d}\left[\hat{S}_T^{n-j}\middle|\mathcal{F}_t\right]} \tag{3.15}$$

on $\mathcal{F}_T$. On the other hand, the $\mathbb{Q}^d$-dynamics of $\hat{S}_t^{n-j}$ is given by

$$\begin{aligned} d\hat{S}_t^{n-j} = {} & (n-j)\left\{r_d - q + \frac{(n-j-1)\left(\sigma_S^2 + \sigma_V^2 + 2\rho\sigma_S\sigma_V\right)}{2}\right\}\hat{S}_t^{n-j} dt \\ & + (n-j)\sqrt{\sigma_S^2 + \sigma_V^2 + 2\rho\sigma_S\sigma_V}\,\hat{S}_t^{n-j} dZ_t^{\mathbb{Q}^d}, \end{aligned} \tag{3.16}$$

from (3.14), and hence, (3.15) can be rewritten as

$$\left.\frac{d\tilde{\mathbb{Q}}^d}{d\mathbb{Q}^d}\right|_{\mathcal{F}_t} = e^{-\frac{(n-j)^2\left(\sigma_S^2+\sigma_V^2+2\rho\sigma_S\sigma_V\right)}{2}t+(n-j)\sqrt{\sigma_S^2+\sigma_V^2 t+2\rho\sigma_S\sigma_V}Z_t^{\mathbb{Q}^d}}.$$

Thus, Girsanov's theorem implies that

$$Z_t^{\tilde{\mathbb{Q}}^d} = Z_t^{\mathbb{Q}^d} - (n-j)\sqrt{\sigma_S^2 + \sigma_V^2 + 2\rho\sigma_S\sigma_V}\,t \tag{3.17}$$

is again a standard $\tilde{\mathbb{Q}}^d$-Brownian motion. Moreover, the $\tilde{\mathbb{Q}}^d$-dynamics of $\hat{S}_t$ is given by

$$\begin{aligned} d\hat{S}_t = {} & \left\{r_d - q + (n-j)\left(\sigma_S^2 + \sigma_V^2 + 2\rho\sigma_S\sigma_V\right)\right\}\hat{S}_t dt \\ & + \sqrt{\sigma_S^2 + \sigma_V^2 + 2\rho\sigma_S\sigma_V}\,\hat{S}_t Z_t^{\tilde{\mathbb{Q}}^d} \end{aligned} \tag{3.18}$$

from (3.14) and (3.17). Finally, (3.13) becomes

$$\begin{aligned} & C_2^{(n)}(t, S_t, V_t) \\ & = e^{-r_d(T-t)}\sum_{j=0}^{n}\binom{n}{j}(-K_d)^j\,\mathbb{E}_{\mathbb{Q}^d}\left[\hat{S}_T^{n-j}\middle|\mathcal{F}_t\right]\tilde{\mathbb{Q}}^d\left(\hat{S}_T > K_d\right) \\ & = e^{-r_d(T-t)}\sum_{j=0}^{n}\binom{n}{j}(-K_d)^j\,V_t^{n-j}S_t^{n-j}e^{(n-j)\left\{r_d-q+\frac{(n-j-1)\left(\sigma_S^2+\sigma_V^2+2\rho\sigma_S\sigma_V\right)}{2}\right\}(T-t)}N\left(d^{(n-j)}\right) \end{aligned}$$

from (3.16) and (3.18), where

$$d^{(n-j)} = \frac{\ln \frac{V_t S_t}{K_d} + \left\{ r_d - q + \left(n - j - \frac{1}{2} \right) \left(\sigma_S^2 + \sigma_V^2 + 2\rho\sigma_S\sigma_V \right) \right\} (T-t)}{\sqrt{\left(\sigma_S^2 + \sigma_V^2 + 2\rho\sigma_S\sigma_V \right) (T-t)}}. \quad \square$$

Again, we consider another symmetric type power quanto option in a similar type with the maturity payoff given by

$$\max\left\{ \left(V_T S_T - K_d \right)^n, 0 \right\}. \tag{3.19}$$

This power quanto option's maturity payoff coincides with the above defined powered option for odd exponents. For even exponents, i.e., $n = 2L$ for any $L \in \mathbf{N}$, (3.19) can be rewritten as

$$\left(V_T S_T - K_d \right)^{2L} = \sum_{j=0}^{2L} \binom{2L}{j} V_T^{2L-j} S_T^{2L-j} \left(-K_d \right)^j. \tag{3.20}$$

Thus, we have the following option pricing formula:

$$\begin{aligned}
& C_2^{(n)}\left(t, S_t, V_t \right) \\
&= e^{-r_d(T-t)} \mathbb{E}_{\mathbb{Q}^d} \left[\sum_{j=0}^{2L} \binom{2L}{j} \hat{S}_T^{2L-j} \left(-K_d \right)^j \middle| \mathcal{F}_t \right] \\
&= e^{-r_d(T-t)} \sum_{j=0}^{2L} \binom{2L}{j} \left(-K_d \right)^j \mathbb{E}_{\mathbb{Q}^d} \left[\hat{S}_T^{2L-j} \middle| \mathcal{F}_t \right] \\
&= e^{-r_d(T-t)} \sum_{j=0}^{2L} \binom{2L}{j} \left(-K_d \right)^j V_t^{2L-j} S_t^{2L-j} e^{(2L-j)\left\{ r_d - q + \frac{(2L-j-1)\left(\sigma_S^2 + \sigma_V^2 + 2\rho\sigma_S\sigma_V \right)}{2} \right\}(T-t)}
\end{aligned}$$

by substituting $2L$ for n from (3.16).

3.3. Type III

For positive integer n and foreign currency strike price K_f, the foreign equity power-n quanto call option struck in predetermined domestic currency has a maturity payoff given by

$$\begin{aligned}
V_0 \left\{ \max\left(S_T - K_f, 0 \right) \right\}^n &= V_0 \left(S_T - K_f \right)^n \mathbb{1}_{\{S_T > K_f\}} \\
&= V_0 \sum_{j=0}^{n} \binom{n}{j} S_T^{n-j} \left(-K_f \right)^j \mathbb{1}_{\{S_T > K_f\}},
\end{aligned} \tag{3.21}$$

where V_0 is the some fixed exchange rate.

Theorem 3.3. *Under the assumptions of* (2.2) *with* $n \in \mathbf{N}$, *the price of a European power-*n *quanto call option at time* t *in domestic currency with the payoff* (3.21) *is given by*

$$C_3^{(n)}\left(t, S_t \right)$$

$$= V_0 e^{-r_d(T-t)} \sum_{j=0}^{n} \binom{n}{j} (-K_f)^j S_t^{n-j} e^{(n-j)\left\{r_f - q - \rho\sigma_S\sigma_V + \frac{(n-j-1)\sigma_S^2}{2}\right\}(T-t)} N\left(d^{(n-j)}\right),$$

where

$$d^{(n-j)} = \frac{\ln \frac{S_t}{K_f} + \left[r_f - q + \left\{\left(n - j - \frac{1}{2}\right)\sigma_S - \rho\sigma_V\right\}\sigma_S\right](T-t)}{\sigma_S\sqrt{T-t}}.$$

Proof. We may write $C_3^{(n)}$ as

$$C_3^{(n)}(t, S_t) = V_0 e^{-r_d(T-t)} \mathbb{E}_{\mathbb{Q}^d}\left[\sum_{j=0}^{n}\binom{n}{j} S_T^{n-j}(-K_f)^j \mathbb{1}_{\{S_T > K_f\}}\middle| \mathcal{F}_t\right]$$

$$= V_0 e^{-r_d(T-t)} \sum_{j=0}^{n}\binom{n}{j}(-K_f)^j \mathbb{E}_{\mathbb{Q}^d}\left[S_T^{n-j}\mathbb{1}_{\{S_T > K_f\}}\middle| \mathcal{F}_t\right]. \tag{3.22}$$

For a new risk-neutral probability measure $\tilde{\mathbb{Q}}^d$, the Radon-Nykodým derivative of $\tilde{\mathbb{Q}}^d$ with respect to $\mathbb{Q}^d$ is defined by

$$\frac{d\tilde{\mathbb{Q}}^d}{d\mathbb{Q}^d} = \frac{S_T^{n-j}}{\mathbb{E}_{\mathbb{Q}^d}\left[S_T^{n-j}\middle| \mathcal{F}_t\right]} \tag{3.23}$$

on $\mathcal{F}_T$. On the other hand, the $\mathbb{Q}^d$-dynamics of S_t^{n-j} is given by

$$dS_t^{n-j} = (n-j)\left\{r_f - q - \rho\sigma_S\sigma_V + \frac{(n-j-1)\sigma_S^2}{2}\right\} S_t^{n-j} dt$$
$$+ (n-j)\sigma_S \S_t^{n-j} dB_t^{\mathbb{Q}^d} \tag{3.24}$$

from (2.2), and hence, (3.23) can be rewritten as

$$\left.\frac{d\tilde{\mathbb{Q}}^d}{d\mathbb{Q}^d}\right|_{\mathcal{F}_t} = e^{-\frac{(n-j)^2\sigma_S^2}{2}t + (n-j)\sigma_S B_t^{\mathbb{Q}^d}}.$$

Thus, Girsanov's theorem implies that

$$Z_t^{\tilde{\mathbb{Q}}^d} = Z_t^{\mathbb{Q}^d} - (n-j)\sigma_S t \tag{3.25}$$

is again a standard $\tilde{\mathbb{Q}}^d$-Brownian motion. Moreover, the $\tilde{\mathbb{Q}}^d$-dynamics of S_t is given by

$$dS_t = \left[r_f - q + \{(n-j)\sigma_S - \rho\sigma_V\}\sigma_S\right] S_t dt + \sigma_S S_t dB_t^{\tilde{\mathbb{Q}}^d} \tag{3.26}$$

from (2.2) and (3.25). Finally, (3.22) becomes

$$C_3^{(n)}(t, S_t)$$
$$= V_0 e^{-r_d(T-t)} \sum_{j=0}^{n}\binom{n}{j}(-K_f)^j \mathbb{E}_{\mathbb{Q}^d}\left[S_T^{n-j}\middle| \mathcal{F}_t\right] \tilde{\mathbb{Q}}(S_T > K_f)$$

$$= V_0 e^{-r_d(T-t)} \sum_{j=0}^{n} \binom{n}{j} (-K_f)^j S_t^{n-j} e^{(n-j)\left\{r_f - q - \rho\sigma_S\sigma_V + \frac{(n-j-1)\sigma_S^2}{2}\right\}(T-t)} N\left(d^{(n-j)}\right)$$

from (3.24) and (3.26), where

$$d^{(n-j)} = \frac{\ln\frac{S_t}{K_f} + \left[r_f - q + \left\{\left(n-j-\frac{1}{2}\right)\sigma_S - \rho\sigma_V\right\}\sigma_S\right](T-t)}{\sigma_S\sqrt{T-t}}. \qquad \square$$

Once again, we consider another symmetric type power quanto option in a similar type with the maturity payoff given by

$$V_0 \max\{(S_T - K_f)^n, 0\}. \tag{3.27}$$

This power quanto option's maturity payoff coincides with the above defined powered option for odd exponents. For even exponents, i.e., $n = 2L$ for any $L \in \mathbf{N}$, (3.27) can be rewritten as

$$V_0 (S_T - K_f)^{2L} = V_0 \sum_{j=0}^{2L} \binom{2L}{j} S_T^{2L-j} (-K_f)^j. \tag{3.28}$$

Thus, we have the following option pricing formula:

$$\begin{aligned}
&C_3^{(n)}(t, S_t) \\
&= V_0 e^{-r_d(T-t)} \mathbb{E}_{\mathbb{Q}^d}\left[\sum_{j=0}^{2L} \binom{2L}{j} S_T^{2L-j} (-K_f)^j \,\middle|\, \mathcal{F}_t\right] \\
&= V_0 e^{-r_d(T-t)} \sum_{j=0}^{2L} \binom{2L}{j} (-K_f)^j \,\mathbb{E}_{\mathbb{Q}^d}\left[S_T^{2L-j} \,\middle|\, \mathcal{F}_t\right] \\
&= V_0 e^{-r_d(T-t)} \sum_{j=0}^{2L} \binom{2L}{j} (-K_f)^j S_t^{2L-j} e^{(2L-j)\left\{r_f - q - \rho\sigma_S\sigma_V + \frac{(2L-j-1)\sigma_S^2}{2}\right\}(T-t)}
\end{aligned}$$

by substituting $2L$ for n from (3.24).

3.4. Type IV

For positive integer n and strike price on the exchange rate K_e, the FX power-n call option denoted in domestic currency is an equity-linked foreign exchange option which has a maturity payoff given by

$$\begin{aligned}
S_T\{\max(V_T - K_e, 0)\}^n &= S_T (V_T - K_e)^n \mathbb{1}_{\{V_T > K_e\}} \\
&= S_T \sum_{j=0}^{n} \binom{n}{j} V_T^{n-j} (-K_e)^j \mathbb{1}_{\{V_T > K_e\}}.
\end{aligned} \tag{3.29}$$

Theorem 3.4. *Under the assumptions of (2.2) with $n \in \mathbf{N}$, the price of a European power-n quanto call option at time t in domestic currency with the* *payoff* (3.29) *is given by*

$$\begin{aligned} & C_4^{(n)}(t, S_t, V_t) \\ = & \, S_t e^{(r_f - r_d - q - \rho\sigma_S\sigma_V)(T-t)} \\ & \times \sum_{j=0}^{n} \binom{n}{j} (-K_e)^j V_t^{n-j} e^{(n-j)\left\{r_d - r_f + \rho\sigma_S\sigma_V + \frac{(n-j-1)\sigma_V^2}{2}\right\}(T-t)} N\left(d^{(n-j)}\right), \end{aligned}$$

where

$$d^{(n-j)} = \frac{\ln \frac{V_t}{K_e} + \left\{r_d - r_f + \rho\sigma_S\sigma_V + + \left(n - j - \frac{1}{2}\right)\sigma_V^2\right\}(T-t)}{\sigma_V\sqrt{T-t}}.$$

Proof. We may write $C_4^{(n)}$ as

$$\begin{aligned} C_4^{(n)}(t, S_t, V_t) &= e^{-r_d(T-t)} \mathbb{E}_{\mathbb{Q}^d}\left[S_T \sum_{j=0}^{n} \binom{n}{j} V_T^{n-j} (-K_e)^j \mathbb{1}_{\{V_T > K_e\}} \middle| \mathcal{F}_t \right] \\ &= S_t e^{(r_f - r_d - q - \rho\sigma_S\sigma_V)(T-t)} \\ &\quad \times \sum_{j=0}^{n} \binom{n}{j} (-K_e)^j \mathbb{E}_{\mathbb{Q}^d}\left[e^{-\frac{\sigma_S^2}{2}(T-t) + \sigma_S\left(B_T^{\mathbb{Q}^d} - B_t^{\mathbb{Q}^d}\right)} V_T^{n-j} \mathbb{1}_{\{V_T > K_e\}} \middle| \mathcal{F}_t \right]. \end{aligned} \tag{3.30}$$

For a new risk-neutral probability measure $\tilde{\mathbb{Q}}^d$, the Radon-Nykodým derivative of $\tilde{\mathbb{Q}}^d$ with respect to $\mathbb{Q}^d$ is defined by

$$\left.\frac{d\tilde{\mathbb{Q}}^d}{d\mathbb{Q}^d}\right|_{\mathcal{F}_t} = e^{-\frac{\sigma_S^2}{2}t + \sigma_S B_t^{\mathbb{Q}^d}}.$$

Then Girsanov's theorem implies that

$$B_t^{\tilde{\mathbb{Q}}^d} = B_t^{\mathbb{Q}^d} - \sigma_S t \tag{3.31}$$

and

$$W_t^{\tilde{\mathbb{Q}}^d} = W_t^{\mathbb{Q}^d} - \rho\sigma_S t \tag{3.32}$$

are again two correlated $\tilde{\mathbb{Q}}^d$-standard Brownian motions. Note then that the $\tilde{\mathbb{Q}}^d$-dynamic of S_t is given by

$$dV_t = (r_d - r_f + \rho\sigma_S\sigma_V) V_t dt + \sigma_V V_t dW_t^{\tilde{\mathbb{Q}}^d} \tag{3.33}$$

from (2.2) and (3.32). Thus, (3.30) becomes

$$C_4^{(n)}(t, S_t, V_t) = S_t e^{(r_f - r_d - q - \rho\sigma_S\sigma_V)(T-t)} \sum_{j=0}^{n} \binom{n}{j} (-K_e)^j \mathbb{E}_{\tilde{\mathbb{Q}}^d}\left[V_T^{n-j} \mathbb{1}_{\{V_T > K_e\}} \middle| \mathcal{F}_t \right]. \tag{3.34}$$

Likewise, for a new risk-neutral probability measure $\hat{\mathbb{Q}}^d$, the Radon-Nykodým derivative of $\hat{\mathbb{Q}}^d$ with respect to $\tilde{\mathbb{Q}}^d$ is defined by

$$\frac{d\hat{\mathbb{Q}}^d}{d\tilde{\mathbb{Q}}^d} = \frac{V_T^{n-j}}{\mathbb{E}_{\tilde{\mathbb{Q}}^d}\left[V_T^{n-j} \middle| \mathcal{F}_t \right]} \tag{3.35}$$

on $\mathcal{F}_T$. On the other hand, the dynamics of V_t^{n-j} under the measure $\tilde{\mathbb{Q}}^d$ is given by

$$\begin{aligned} dV_t^{n-j} = &\ (n-j)\left\{ r_d - r_f + \rho\sigma_S\sigma_V + \frac{(n-j-1)\,\sigma_V^2}{2} \right\} V_t^{n-j} dt \\ &+ (n-j)\,\sigma_V V_t^{n-j} dW_t^{\tilde{\mathbb{Q}}^d} \end{aligned} \tag{3.36}$$

from (3.33). Then Girsanov's theorem implies from (3.31) that

$$\begin{aligned} W_t^{\hat{\mathbb{Q}}^d} &= W_t^{\tilde{\mathbb{Q}}^d} - (n-j)\,\sigma_V t \\ &= W_t^{\mathbb{Q}^d} - \{\rho\sigma_S + (n-j)\,\sigma_V\}\, t \end{aligned} \tag{3.37}$$

is again a $\hat{\mathbb{Q}}^d$-standard Brownian motion. Moreover, the $\hat{\mathbb{Q}}^d$-dynamics of S_t is given by

$$dV_t = \left\{ r_d - r_f + \rho\sigma_S\sigma_V + (n-j)\,\sigma_V^2 \right\} V_t dt + \sigma_V V_t dW_t^{\hat{\mathbb{Q}}^d} \tag{3.38}$$

from (2.2) and (3.37). Finally, (3.34) becomes

$$\begin{aligned} &C_4^{(n)}(t, S_t, V_t) \\ &= S_t e^{(r_f - r_d - q - \rho\sigma_S\sigma_V)(T-t)} \sum_{j=0}^{n} \binom{n}{j} (-K_e)^j\, \mathbb{E}_{\tilde{\mathbb{Q}}^d}\left[V_T^{n-j} \middle| \mathcal{F}_t \right] \hat{\mathbb{Q}}\,(V_T > K_e) \\ &= S_t e^{(r_f - r_d - q - \rho\sigma_S\sigma_V)(T-t)} \\ &\quad \times \sum_{j=0}^{n} \binom{n}{j} (-K_e)^j\, V_t^{n-j} e^{(n-j)\left\{ r_d - r_f + \rho\sigma_S\sigma_V + \frac{(n-j-1)\sigma_V^2}{2} \right\}(T-t)} N\left(d^{(n-j)} \right) \end{aligned}$$

from (3.36) and (3.38), where

$$d^{(n-j)} = \frac{\ln \frac{V_t}{K_e} + \left\{ r_d - r_f + \rho\sigma_S\sigma_V + + \left(n - j - \frac{1}{2} \right) \sigma_V^2 \right\} (T-t)}{\sigma_V \sqrt{T-t}}. \qquad \square$$

Once more again, we consider another symmetric type power quanto option in a similar type with the maturity payoff given by

$$S_T \max\left\{ (V_T - K_e)^n, 0 \right\}. \tag{3.39}$$

This power quanto option's maturity payoff coincides with the above defined powered option for odd exponents. For even exponents, i.e., $n = 2L$ for any

$L \in \mathbf{N}$, (3.39) can be rewritten as

$$S_T\left(V_T - K_e\right)^{2L} = S_T \sum_{j=0}^{2L} \binom{2L}{j} V_T^{2L-j}\left(-K_e\right)^j. \tag{3.40}$$

Thus, we have the following option pricing formula:

$$\begin{aligned}
&C_4^{(n)}\left(t, S_t, V_t\right) \\
&= e^{-r_d(T-t)} \mathbb{E}_{\mathbb{Q}^d}\left[S_T \sum_{j=0}^{2L} \binom{2L}{j} V_T^{2L-j}\left(-K_e\right)^j \middle| \mathcal{F}_t \right] \\
&= S_t e^{(r_f - r_d - q - \rho\sigma_S\sigma_V)(T-t)} \sum_{j=0}^{2L} \binom{2L}{j} \left(-K_e\right)^j \mathbb{E}_{\tilde{\mathbb{Q}}^d}\left[V_T^{2L-j} \middle| \mathcal{F}_t \right] \\
&= S_t e^{(r_f - r_d - q - \rho\sigma_S\sigma_V)(T-t)} \\
&\quad \times \sum_{j=0}^{2L} \binom{2L}{j} \left(-K_e\right)^j V_t^{2L-j} e^{(2L-j)\left\{ r_d - r_f + \rho\sigma_S\sigma_V + \frac{(2L-j-1)\sigma_V^2}{2} \right\}(T-t)}
\end{aligned}$$

by substituting $2L$ for n from (3.36).

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