

## A NOTE ON BOUNDARY BLOW-UP PROBLEM OF $\Delta u = u^p$

SEICK KIM

ABSTRACT. Assume that  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with  $n \geq 2$ . We study positive solutions to the problem,  $\Delta u = u^p$  in  $\Omega$ ,  $u(x) \rightarrow \infty$  as  $x \rightarrow \partial\Omega$ , where  $p > 1$ . Such solutions are called boundary blow-up solutions of  $\Delta u = u^p$ . We show that a boundary blow-up solution exists in any bounded domain if  $1 < p < \frac{n}{n-2}$ . In particular, when  $n = 2$ , there exists a boundary blow-up solution to  $\Delta u = u^p$  for all  $p \in (1, \infty)$ . We also prove the uniqueness under the additional assumption that the domain satisfies the condition  $\partial\Omega = \partial\bar{\Omega}$ .

### 1. Introduction

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^n$  with  $n \geq 2$  and let  $\partial\Omega$  denote its boundary. In this article we study the problem

- (1)  $\Delta u(x) = f(u(x))$  for  $x \in \Omega$ ,  
(2)  $u(x) \rightarrow +\infty$  as  $d(x) := \text{dist}(x, \partial\Omega) \rightarrow 0$ ,

where  $f(t) = t_+^p := \{\max(t, 0)\}^p$  with  $p > 1$ . Solutions to the problem (1), (2) are called *boundary blow-up solutions*.

In 1957, Keller [5] and Osserman [11] proved existence of solutions to problem (1), (2) for a rather general class of functions  $f$ ; i.e.,  $f : \mathbb{R} \rightarrow [0, \infty)$  is a locally Lipschitz continuous function which is increasing and satisfies the following growth condition called Keller-Osserman condition:

$$(3) \quad \int_{t_0}^{\infty} \left\{ \int_0^t f(s) ds \right\}^{-1/2} dt < +\infty \quad \text{for all } t_0 > 0.$$

It is easy to check that  $f(t) = t_+^p$  with  $p > 1$  satisfies (3). They showed that (3) is a necessary condition for the existence of blow-up solutions. Indeed, if the domain  $\Omega$  is regular enough, say Lipschitz, then the existence of a classical solution to the problem (1), (2) is established by the method of supersolutions and subsolutions together with the uniform estimates of Keller [5]. We will briefly review existence results in the next section.

---

Received March 13, 2018; Accepted May 29, 2018.

2010 *Mathematics Subject Classification*. Primary 35J65; Secondary 35B05.

*Key words and phrases*. blow-up, semi-linear equation, existence, uniqueness.

The author was partially supported by NRF Grant No. NRF-20151009350.

The case  $f(t) = t_+^p$  with  $p > 1$  is of special interest, and in this article only this case will be treated. Loewner and Nirenberg [7] studied the case when  $p = \frac{n+2}{n-2}$  with  $n > 2$ , which is related to a problem in differential geometry. The problem (1), (2) is also related to probability theory. The equation  $\Delta u = u_+^p$ ,  $1 < p \leq 2$ , appears in the analytical theory of a Markov process called superdiffusion; see [2]. By means of a probabilistic representation, a uniqueness result in domains with non-smooth boundary was established by Le Gall [6] in the case when  $p = 2$ . Later, Marcus and Véron [8,9] extended the uniqueness in very general domains for all  $p > 1$ , using purely analytical method; they proved uniqueness in a domain whose boundary is locally represented as a graph of a continuous function. However, it is not clear whether a boundary blow-up solution exists or not in such a general domain. In [10], Matero constructed a boundary blow-up solution of  $\Delta u = u_+^p$  with  $1 < p < \infty$ , in a two-dimensional domain with fractal boundary called the von Koch snowflake domain. His approach is based on the comparison with boundary blow-up solutions in a cut-off open cone.

We treat a special case when  $p \in (1, \frac{n}{n-2})$  for  $n \geq 3$  and  $p \in (1, \infty)$  for  $n = 2$ . Some interesting results are obtained in that case. We will prove that a boundary blow-up solution exists in every bounded domain. As a consequence, it will imply a result of Matero [10] mentioned above. We will also show the uniqueness if the domain satisfies an additional assumption,  $\partial\Omega = \partial\bar{\Omega}$ . For example, if  $\partial\Omega$  can be locally represented as a graph of a continuous function, then it satisfies the above condition. In this case, uniqueness was earlier proved by Marcus and Véron [8,9].

## 2. Preliminaries

In this section, we briefly discuss the existence results of Keller [5], Loewner and Nirenberg [7]. We also introduce some terminology which will be used in the later parts of the paper. We begin with a simple lemma.

**Lemma 2.1** (Comparison principle). *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain. Assume that  $f$  is increasing. Let  $u, v \in C^2(\Omega)$  be solutions of  $\Delta u \geq f(u)$  and  $\Delta v \leq f(v)$  respectively. If  $\liminf_{x \rightarrow \partial\Omega} (v - u)(x) \geq 0$ , then  $v \geq u$  in  $\Omega$ .*

*Proof.* Suppose, to the contrary, that there exists  $x_0 \in \Omega$  such that  $u(x_0) > v(x_0)$ . Then for sufficiently small  $\epsilon > 0$ ,  $\Omega_\epsilon := \{u - v > \epsilon\} \neq \emptyset$  and  $\bar{\Omega}_\epsilon \subset \Omega$ . Let  $w := u - v - \epsilon$ . Then  $w = 0$  on  $\partial\Omega_\epsilon$ . Since  $f$  is increasing,

$$Lw \geq f(u) - f(v) \geq f(u) - f(v + \epsilon) \geq 0 \quad \text{in } \Omega_\epsilon.$$

Then the maximum principle implies  $w \leq 0$  in  $\Omega_\epsilon$ . This contradiction proves the lemma.  $\square$

*Remark 2.2.* Let  $\Omega_1, \Omega_2 \subset \mathbb{R}^n$  be bounded domains such that  $\Omega_1 \Subset \Omega_2$ , i.e.,  $\bar{\Omega}_1 \subset \Omega_2$ . Suppose  $u_i$  ( $i = 1, 2$ ) are solutions to (1), (2) in  $\Omega_i$ . Then, it follows from Lemma 2.1 that  $u_1 \geq u_2$  in  $\Omega_1$ .

The next theorem is quoted from [5]; see also [11].

**Theorem 2.3** (Keller [5, pp. 505–507]). *Let  $u$  be a solution of (1) in a bounded domain  $\Omega$ . There exist a continuous, decreasing function  $g : (0, \infty) \rightarrow \mathbb{R}$  determined by  $f$  such that  $\lim_{t \rightarrow 0} g(t) = +\infty$  and*

$$(4) \quad u(x) \leq g(d(x)), \quad \text{where } d(x) := \text{dist}(x, \partial\Omega).$$

Using the above estimate (4), Keller proved the existence of a boundary blow-up solution. Although he claimed the existence in arbitrary domains, his argument seems to require certain smoothness assumption on  $\Omega$ . Let  $\Omega$  be a regular domain, say a Lipschitz domain. By the method of supersolutions and subsolutions (see e.g. [3, pp. 507–511]), one can show that, for each  $m \geq 1$ , there exists a unique solution  $u_m \in C^0(\bar{\Omega}) \cap C^2(\Omega)$  of (1) such that  $u_m = \alpha_m$  on  $\partial\Omega$ , where  $\alpha_m < \alpha_{m+1}$  and  $\alpha_m \rightarrow \infty$  as  $m \rightarrow \infty$ . Then by the comparison principle,  $\{u_m\}_{m=1}^\infty$  is an increasing sequence of functions. By (4),  $u_m(x) \leq g(d(x))$  uniformly for  $m \geq 1$ . Denote by  $u(x)$  the pointwise limit of  $\{u_m(x)\}_{m=1}^\infty$ . Then by the standard elliptic theory (see e.g. [4]),  $u \in C^2(\Omega)$  and  $u$  is a solution of (1). As  $x$  approaches  $\partial\Omega$ ,  $u(x)$  increases indefinitely since  $u_m = \alpha_m$  becomes infinite on  $\partial\Omega$ ; thus  $u$  is a solution of the problem (1), (2).

The solution  $u$  constructed above is called a *minimal boundary blow-up solution*. Indeed, if  $v$  is a boundary blow-up solution, then by the comparison principle,  $u_m \leq v$  in  $\Omega$  for all  $m \geq 1$  and thus,  $u = \lim_{m \rightarrow \infty} u_m \leq v$  follows.

Loewner and Nirenberg [7] introduced another important solution of (1) called a *maximal solution* which is not necessarily a blow-up solution but can be constructed in any bounded domain  $\Omega$ . Let  $\{\Omega_m\}_{m=1}^\infty$  be an exhausting sequence of smooth subdomains of  $\Omega$ ; i.e.,  $\Omega_m \Subset \Omega_{m+1} \Subset \Omega$  and  $\bigcup_{m=1}^\infty \Omega_m = \Omega$ . Let  $u_m$  be the minimal blow-up solution in  $\Omega_m$  for each  $m \geq 1$ , and let  $v$  be the minimal blow-up solution in a ball containing  $\bar{\Omega}$ . By Remark 2.2,  $\{u_m\}_{m=1}^\infty$  is decreasing and bounded below by  $v$ . Hence, the limit function  $u$  exists and by the standard elliptic theory, it is a solution to (1). This solution  $u$  is maximal since if  $v$  is a solution of (1) in  $\Omega$ , then by the comparison principle, we see  $u_m \geq v$  for all  $m \geq 1$ . In next section, we will provide an example of maximal solution which is not a boundary blow-up solution; see Remark 3.3 below.

### 3. Main results

We consider the problem (1), (2) with  $f(t) = t_+^p$ . Note that in this case, a solution to the problem (1), (2) must be positive, which is a simple consequence of the maximum principle. Indeed, more generally, let  $t_0 := \sup\{t : f(t) = 0\}$ . If  $t_0 \neq -\infty$ , then by continuity,  $f(t_0) = 0$  and thus,  $u \equiv t_0$  is a solution to (1). By Lemma 2.1, we find that any blow-up solution of (1) is bounded below by  $t_0$ .

Hereafter, we always assume  $p \in (1, \infty)$  when  $n = 2$ , and  $p \in (1, \frac{n}{n-2})$  when  $n \geq 3$ . We will show that in that case, a boundary blow-up solution exists in any bounded domain, which obviously include the domain considered by

Matero in [10]. Also, by using Safonov's iteration technique in [12], we prove uniqueness provided that  $\Omega$  satisfies the condition  $\partial\Omega = \partial\bar{\Omega}$ . For example, if  $\partial\Omega$  can be locally represented as a graph of a continuous function, then it satisfies the above condition.

### 3.1. Construction of a barrier in $\mathbb{R}^n \setminus \{0\}$

We will construct a solution of  $\Delta u = u_+^p$  in  $\mathbb{R}^n \setminus \{0\}$  which blows up at the origin. We look for a solution of the form  $v(x) = c_p |x|^{-\gamma}$ , where  $c_p, \gamma > 0$ . Since  $v$  is positive and radially symmetric,  $v(r) = c_p r^{-\gamma}$ , where  $r = |x|$ , must solve the following ODE:

$$(5) \quad v''(r) + \frac{n-1}{r} v'(r) = v^p(r) \quad \text{in } (0, \infty).$$

Hence, the unknown constants  $c_p, \gamma$  should satisfy

$$(6) \quad c_p \gamma(\gamma + 2 - n) r^{-\gamma-2} = c_p^p r^{-\gamma p}.$$

Set  $\gamma = 2/(p-1)$  so that  $\gamma + 2 = \gamma p$ . The assumption  $c_p > 0$  requires a restriction on  $p$ , namely  $2/(p-1) > n-2$ . It is satisfied for all  $p > 1$  when  $n = 2$  and for  $p \in (1, \frac{n}{n-2})$  when  $n \geq 3$ . If we choose

$$(7) \quad c_p = \{\gamma(\gamma + 2 - n)\}^{\gamma/2} = \left\{ \frac{2n - 2(n-2)p}{(p-1)^2} \right\}^{1/(p-1)},$$

it follows  $c_p^p = c_p \gamma(\gamma + 2 - n)$ .

Then,  $v(x) = c_p |x|^{-\gamma}$  is a solution of  $\Delta v = v_+^p$  on  $\mathbb{R}^n \setminus \{0\}$  such that  $v(x) \rightarrow +\infty$  as  $|x| \rightarrow 0$ . We summarize the above result as a lemma.

**Lemma 3.1.** *Let  $p > 1$  when  $n = 2$ , and let  $p \in (1, \frac{n}{n-2})$  when  $n \geq 3$ . Then,  $v(x) := c_p |x|^{-\gamma}$  is a solution of  $\Delta v = v_+^p$  in  $\mathbb{R}^n \setminus \{0\}$  such that  $v(x) \rightarrow +\infty$  as  $|x| \rightarrow 0$ . Here,  $\gamma = 2/(p-1)$  and  $c_p = \left\{ \frac{2n-2(n-2)p}{(p-1)^2} \right\}^{1/(p-1)}$ .*

### 3.2. Existence and uniqueness of boundary blow-up solution

**Theorem 3.2.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain. Then, there exists a solution  $u$  to the problem (1), (2).*

*Proof.* Let  $\{\Omega_m\}_{m=1}^\infty$  be an exhausting sequence of smooth subdomains of  $\Omega$ , and let  $u_m$  be the minimal blow-up solution of (1) in  $\Omega_m$ . Then, the limit  $u := \lim_{m \rightarrow \infty} u_m$  is a maximal solution; see Section 2.

We need to show that  $u$  is indeed a boundary blow-up solution. For any  $y \in \Omega$ , choose a point  $y_0 \in \partial\Omega$  such that  $d(y) = |y - y_0|$ . Let  $v(x) := c_p |x - y_0|^{-\gamma}$  with  $c_p, \gamma$  defined as in Lemma 3.1. Since  $y_0 \notin \Omega_m$  for each  $m \geq 1$ , we find  $v(x) < +\infty$  for all  $x \in \Omega_m$ . Hence, we conclude by Lemma 2.1 that

$$(8) \quad u_m(y) \geq v(y) = c_p d^{-\gamma}(y)$$

provided  $m$  is large enough so that  $y \in \Omega_m$ .

Therefore, by passing to the limit, we find  $u(y) \geq c_p d^{-\gamma}(y)$  for any  $y \in \Omega$ . Clearly,  $u(y) \rightarrow +\infty$  as  $d(y) \rightarrow 0$ , and thus,  $u$  is a desired solution.  $\square$

*Remark 3.3.* In Theorem 3.2, the restriction that  $p < \frac{n}{n-2}$  when  $n > 2$  is essential. Let  $\Omega := \{x \in \mathbb{R}^n : 0 < |x| < 1\}$ , where  $n > 2$ . Brézis and Véron [1] showed that if  $p \geq \frac{n}{n-2}$ , then any positive solution  $u$  of  $\Delta u = u^p$  in  $\Omega$  satisfies  $\overline{\lim}_{x \rightarrow 0} u(x) < +\infty$ . Consequently, there is no solution of the problem (1), (2) in  $\Omega$ . This also shows that in general, a maximal solution is not necessarily a boundary blow-up solution.

**Theorem 3.4.** *In addition, assume that  $\Omega$  satisfies  $\partial\Omega = \partial\overline{\Omega}$ . Then, the solution of the problem (1), (2) is unique.*

*Proof.* Let  $u_1, u_2$  be two boundary blow-up solutions in  $\Omega$ . We claim that the following estimate holds:

$$(9) \quad N_1 d^{-\gamma}(x) \leq u_i(x) \leq N_2 d^{-\gamma}(x) \quad \text{for all } x \in \Omega; \quad i = 1, 2,$$

where  $N_1, N_2 > 0$  are constants depending only on  $n$  and  $p$ .

Fix  $x_0 \in \Omega$  and denote  $r := d(x_0)$ . Choose  $z_0 \in \partial\Omega$  such that  $|x_0 - z_0| = r$ . From the assumption that  $\partial\Omega = \partial\overline{\Omega}$ , there exists a point  $y_0 \in B_r(z_0) \setminus \overline{\Omega}$ . Note that  $r \leq |x_0 - y_0| \leq 2r$ . Let  $v(x) := c_p |x - y_0|^{-\gamma}$ . Since  $\Omega$  is bounded and  $y_0 \notin \overline{\Omega}$ , we find, by see Lemma 2.1, that  $u_i(x) \geq v(x)$ , where  $i = 1, 2$ . In particular,

$$(10) \quad u_i(x_0) \geq c_p |x_0 - y_0|^{-\gamma} \geq c_p 2^{-\gamma} d^{-\gamma}(x_0); \quad i = 1, 2.$$

Also, by considering a ball  $B_r(x_0)$  and the minimal boundary blow-up solution in that ball as a comparison function, it is not hard to see  $u_i(x_0) \leq N_2 d(x_0)^{-\gamma}$ ,  $i = 1, 2$ , for some constant  $N_2 > 0$  depending only on  $n$  and  $p$ ; see e.g. [5].

Therefore, we conclude that the estimate (9) holds. Once we obtain the estimate (9),  $u_1 \equiv u_2$  will follow from the iteration technique of Safonov in [12]. For the reader's convenience, we will reproduce his technique here.

Assume, to the contrary, that  $u_2(x_1)/u_1(x_1) > k > 1$  for some  $x_1 \in \Omega$ . Let  $\Omega_0 := \{u_2 > ku_1\} \cap B_r(x_1)$ , where  $r = \frac{1}{2} d(x_1)$ . Then, we find

$$\Delta(u_2 - ku_1) = u_2^p - ku_1^p > (k^p - k)u_1^p \geq c_1 kr^{-\gamma p},$$

where  $c_1 = 2^{-\gamma p} N_1^p (k^{p-1} - 1)$ . Therefore,  $\Delta(u_2 - ku_1 + w) \geq 0$  in  $\Omega_0$ , where  $w = \frac{c_1}{2n} kr^{-\gamma p} (r^2 - |x - x_1|^2)$ . By the maximum principle

$$w(x_1) < (u_2 - ku_1 + w)(x_1) \leq \sup_{\partial\Omega_0} (u_2 - ku_1 + w).$$

Note that the maximum must be achieved on  $\partial B_r(x_1) \cap \overline{\Omega}_0 \subset \partial\Omega_0$ ; otherwise, it is achieved on  $\{u_2 = ku_1\} \cap \overline{B_r(x_1)}$ , where we have  $u_2 - ku_1 + w \leq w(x_1)$ . Hence,  $w(x_1) < (u_2 - ku_1)(x_2)$ , where  $x_2 \in \partial B_r(x_1) \cap \partial\Omega_0 \subset \Omega$ . On the other hand, by (9), we find (recall  $x_2 \in \partial B_r(x_1)$  so that  $d(x_2) \geq r$ )

$$w(x_1) = \frac{c_1}{2n} kr^{-\gamma p} r^2 = \frac{c_1}{2n} kr^{-\gamma} \geq c_2 ku_1(x_2),$$

where  $c_2 = \frac{c_1}{2nN_2} = \frac{N_1^p}{2^{\gamma p+1}nN_2}(k^{p-1} - 1)$ . Therefore, we conclude  $u_2(x_2)/u_1(x_2) > (1 + c_2)k$ . By iterating the above process (both  $c_1$  and  $c_2$  are monotone increasing in  $k$ ), we obtain a sequence of points  $\{x_j\}_{j=1}^\infty$  in  $\Omega$  satisfying  $u_2(x_j)/u_1(x_j) > (1 + c_2)^j k$ , which tends to infinity as  $j \rightarrow \infty$ . On the other hand, by (9),

$$\frac{u_2(x)}{u_1(x)} < \frac{N_2 d^{-\gamma}(x)}{N_1 d^{-\gamma}(x)} = \frac{N_2}{N_1} \quad \forall x \in \Omega.$$

This contradiction proves the uniqueness.  $\square$

*Remark 3.5.* If  $\Omega = \{x \in \mathbb{R}^2 : 0 < |x| < 1\}$ , then  $\partial\Omega \neq \bar{\partial}\Omega$ . The uniqueness fails in this case; see [13].

**Acknowledgment.** This paper is based on a presentation by the author at the 2001 AMS sectional meeting, Williamstown, MA, and was supported in part by NSF Grant No. DMS-9971052.

**Addendum.** This paper originally appeared as IMA preprint No.1872 but it was never published (<https://www.ima.umn.edu/sites/default/files/1872.pdf>). Later, the essential part of it was included in the paper ‘‘On uniqueness of boundary blow-up solutions of a class of nonlinear elliptic equations’’ by Hongjie Dong, Seick Kim, and Mikhail Safonov. (Communications in Partial Differential Equations **33** (2008), no. 2, 177–188.)

## References

- [1] H. Brézis and L. Véron, *Removable singularities for some nonlinear elliptic equations*, Arch. Rational Mech. Anal. **75** (1980/81), no. 1, 1–6.
- [2] E. B. Dynkin, *Diffusions, superdiffusions and partial differential equations*, American Mathematical Society Colloquium Publications, **50**, American Mathematical Society, Providence, RI, 2002.
- [3] L. C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, **19**, American Mathematical Society, Providence, RI, 1998.
- [4] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, second edition, Grundlehren der Mathematischen Wissenschaften, **224**, Springer-Verlag, Berlin, 1983.
- [5] J. B. Keller, *On solutions of  $\Delta u = f(u)$* , Comm. Pure Appl. Math. **10** (1957), 503–510.
- [6] J.-F. Le Gall, *A path-valued Markov process and its connections with partial differential equations*, in First European Congress of Mathematics, Vol. II (Paris, 1992), 185–212, Progr. Math., **120**, Birkhäuser, Basel, 1994.
- [7] C. Loewner and L. Nirenberg, *Partial differential equations invariant under conformal or projective transformations*, in Contributions to analysis (a collection of papers dedicated to Lipman Bers), 245–272, Academic Press, New York, 1974.
- [8] M. Marcus and L. Véron, *Uniqueness of solutions with blowup at the boundary for a class of nonlinear elliptic equations*, C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), no. 6, 559–563.
- [9] ———, *Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), no. 2, 237–274.

- [10] J. Matero, *Boundary-blow up problems in a fractal domain*, Z. Anal. Anwendungen **15** (1996), no. 2, 419–444.
- [11] R. Osserman, *On the inequality  $\Delta u \geq f(u)$* , Pacific J. Math. **7** (1957), 1641–1647.
- [12] M. V. Safonov, *On the uniqueness of blowup solutions for semilinear elliptic equations*, Preprint.
- [13] L. Véron, *Solutions singulières d'équations elliptiques semilinéaires*, C. R. Acad. Sci. Paris Sér. A-B **288** (1979), no. 18, A867–A869.

SEICK KIM  
DEPARTMENT OF MATHEMATICS  
YONSEI UNIVERSITY  
SEOUL 03722, KOREA  
*Email address:* kimseick@yonsei.ac.kr