Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b180188
pISSN: 1015-8634 / eISSN: 2234-3016

SOME INTEGRATIONS ON NULL HYPERSURFACES IN LORENTZIAN MANIFOLDS

Fortuné Massamba and Samuel Ssekajja

Abstract. We use the so-called pseudoinversion of degenerate metrics technique on foliated compact null hypersurface, M^{n+1}, in Lorentzian manifold $\overline{M}^{n+2}$, to derive an integral formula involving the r-th order mean curvatures of its foliations, $\mathcal{F}^n$. We apply our formula to minimal foliations, showing that, under certain geometric conditions, they are isomorphic to n-dimensional spheres. We also use the formula to deduce expressions for total mean curvatures of such foliations.

1. Introduction

The natural metric induced on a null hypersurface of a semi-Riemannian manifold is generally *degenerate*, which renders the known definitions of some operators, like gradient, divergence, Laplacian, etc., null and void. Moreover, one cannot define a volume element with such a metric. In [3], the authors introduced a new non-degenerate metric, with a pseudo inverse, on a null hypersurface which has helped in the re-defining of such operators on null hypersurfaces. Using such a metric and the concept of Newton transformations [1, 4, 8] on a compact null hypersurface of a Lorentzian manifold, we derive a new integral formula for such null hypersurface, admitting a codimension one foliation (see Theorem 3.5). Consequently, we apply the above theorem to minimal foliations, showing that at some point they are isomorphic to n-dimensional spheres (see Theorem 4.3) as well as deducing a formula for the total mean curvatures of such foliations (see Theorem 4.6). Integral formulae are fundamentally important as they provide obstructions to the existence of foliations whose leaves enjoy some special geometric properties-totally geodesic (or totally umbilic), minimal, constant mean curvature and many more.

The theory of null submanifolds was introduced independently by Duggal and Bejancu [9] and Kuperli [15] and later studied by many other researchers [3, 4, 7, 8, 10, 11, 13]. null hypersurfaces are fundamental to general relativity and electromagnetism. In fact, null hypersurfaces represent different types of

Received March 3, 2018; Accepted July 20, 2018.
2010 *Mathematics Subject Classification.* Primary 53C50; Secondary 53C12, 53C40.
Key words and phrases. null hypersurface, Newton transformation, foliation.

black hole horizons (see details in [9] and [14]). The first order mean curvature is useful in describing the behavior of such horizons, for instance see the Raychaudhuris equation and its applications in [10]. Currently, there is little work on the study of higher order mean curvatures of null hyperfaces and submanifolds in general. Some of the existing pieces of work are due to [4] and [8]. In [4], the authors derive some generalized differential equations (involving higher order mean curvatures) of null hypersurfaces in Lorentzian manifolds. They also present a set of integral formulae, known as Minkowski integral formulae, for such hypersurfaces, admitting some conformal vector fields. For the case of [8], some inequalities involving higher order mean curvatures are proved, leading to some interesting information about totally umbilic null hypersurfaces.

The purpose of this paper is to derive a new integral formulae for compact null hypersurface in Lorentzian manifolds, admitting a parallel foliation. Some of its applications are also given in case of minimal foliations by such hypersurfaces. The paper is arranged as follows; In Section 2 we quote the basic notions on null hypersurfaces and Newton transformations necessary for the rest of the sections. Section 3 presents a new integral formula of foliations by null hypersurfaces in time-orientable Lorentzian manifolds. Section 4 focuses on minimal foliations.

2. Preliminaries

Let $(\overline{M}, \overline{g})$ be a $(n+2)$-dimensional semi-Riemannian manifold [14] and M be a hypersurface of $\overline{M}$. Denote by g the induced tensor field on M and suppose that rank $g = n$ on M. Then we say that M is a null hypersurface of $\overline{M}$ [9]. Moreover, it is easy to see that M is a null hypersurface of $\overline{M}$ if and only if the vector bundle

$$TM^{\perp} = \bigcup_{p\in M} T_pM^{\perp}; \;\; T_pM^{\perp} = \{X_p \in T_p\overline{M} : \overline{g}_g(X_p, Y_p) = 0, \;\; \forall\, Y_p \in T_pM\},$$

becomes a distribution of rank 1 on M. A complementary distribution $S(TM)$ to $TM^{\perp}$ in TM is called a *screen distribution*. Thus, we have the following decomposition

$$TM = S(TM) \oplus_{orth} TM^{\perp},$$

where $\oplus_{orth}$ denotes the orthogonal direct sum of bundles. From now on, we denote by $\Gamma(\Xi)$ the set of smooth sections of a vector bundle Ξ. Let M be a null hypersurface of $(\overline{M}, \overline{g})$ and $S(TM)$ be a screen distribution on M. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that, for any non-zero section of E of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique local section N of $tr(TM)$ satisfying

$$\overline{g}(E, N) = 1, \;\; \overline{g}(N, N) = 0, \;\; \overline{g}(X, N) = 0, \;\; \forall\, X \in \Gamma(S(TM)). \tag{2.1}$$

Throughout the paper, all manifolds are supposed to be paracompact and smooth. By (2.1) it follows that $tr(TM)$ is a null vector bundle which enables us to write down the decomposition

$$T\overline{M} = S(TM) \oplus_{orth} \{TM^{\perp} \oplus tr(TM)\} = TM \oplus tr(TM),$$

where $\oplus$ denotes a non-orthogonal direct sum. We call $tr(TM)$ a *null transversal bundle.*

Next, let ∇ and ∇^* denote the induced connections on M and $S(TM)$, respectively, P be the projection of TM onto $S(TM)$, then the local Gauss-Weingarten equations of M and $S(TM)$ are the following [9]

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.2}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.3}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)E, \tag{2.4}$$

$$\nabla_X E = -A^*_E X - \tau(X)E, \quad A^*_E E = 0, \tag{2.5}$$

for all $X, Y \in \Gamma(TM)$, $E \in \Gamma(TM^{\perp})$ and $N \in \Gamma(tr(TM))$. In the above setting, B is the local second fundamental form of M and C is the local second fundamental form on $S(TM)$. A_N and A^*_E are the shape operators on TM and $S(TM)$ respectively, while τ is a 1-form on TM. The above shape operators are related to their local fundamental forms by

$$g(A^*_E X, Y) = B(X,Y), \ \ g(A_N X, PY) = C(X, PY), \tag{2.6}$$

$$\overline{g}(A^*_E X, N) = 0, \ \ \overline{g}(A_N X, N) = 0, \ \ \forall\, X, Y \in \Gamma(TM). \tag{2.7}$$

From (2.7) we notice that A^*_E and A_N are both screen-valued operators. Let $\theta = \overline{g}(N, \cdot)$ be a 1-form metrically equivalent to N defined on $\overline{M}$. Take

$$\lambda = i^*\theta \tag{2.8}$$

to be its restriction on M, where $i : M \to \overline{M}$ is the inclusion map.

In [3], the authors introduced a non-degenerate metric on a null hypersurface (M, g), in terms of the degenerate metric g and the 1-form λ of (3.4) as follows; Let $X \in \Gamma(TM)$, then $X = PX + \lambda(X)E$ and $\lambda(X) = 0$ if and only if $X \in \Gamma(S(TM))$. Define $\flat$ by

$$\begin{aligned} \flat : \Gamma(TM) &\longrightarrow \Gamma(T^*M) \\ X &\longmapsto X^{\flat} = g(X, \cdot) + \lambda(X)\lambda(\cdot). \end{aligned} \tag{2.9}$$

Then, $\flat$ is an isomorphism of TM onto T^*M which can be used to generalize the usual non-degenerate theory. In the latter case, $S(TM)$ coincides with TM. Consequently, λ vanish identically and the projection morphism P becomes the identity on TM. Let $\sharp$ be the inverse of the isomorphism $\flat$ in (2.9). For some $X \in \Gamma(TM)$ (resp. $\omega \in T^*M$), $X^{\flat}$ (resp. $\omega^{\sharp}$) is called the dual 1-form of X (resp. the dual vector field of ω) with respect to the degenerate metric g. If ω is a 1-form on M, then (2.9) gives $\omega(X) = g(\omega^{\sharp}, X) + \omega(E)\lambda(X)$ for any $X \in \Gamma(TM)$. Define a $(0,2)$-tensor $\widehat{g}$ by $\widehat{g}(X,Y) = X^{\flat}(Y)$. It is obvious that

$\widehat{g}$ is symmetric in X and Y. With respect to a quasi-orthonormal local frame field $\{X_0 = E, X_1, \dots, X_n, N\}$ adapted to $T\overline{M}$, we have

$$\widehat{g}(X,Y) = g(X,Y) + \lambda(X)\lambda(Y), \ \ \forall\, X,Y \in \Gamma(TM). \tag{2.10}$$

The metric $\widehat{g}$ is invertible and its inverse, $g^{[\cdot,\cdot]}$, is called the *pseudo-inverse* of g [3]. Also, observe that $\widehat{g}$ coincides with g if the latter is non-degenerate. In the sequel, we shall make use of the following convention on the range of indices:

$$0 \leq a,b,c \leq n, \quad 1 \leq i,j,k \leq n.$$

The metric $\widehat{g}$ has been used to define the usual divergence operator for a null hypersurface as follows: Let X be a smooth vector field on M and let $\operatorname{div}^M(X)$ be the divergence of X with respect to the non-degenerate metric $\widehat{g}$, then

$$\operatorname{div}^M(X) = -\sum_{a=0}^{n} \epsilon_a \widehat{g}(\nabla_{X_a} X, X_a), \ \ \epsilon_0 = 1, \tag{2.11}$$

where $\{\epsilon_a\}$ is the signature of the basis $\{X_a\}$. It is well known that the induced degenerate metric g is not compatible with the induced connection ∇ in general, and this compatibility arises if and only if the null hypersurface M is totally geodesic in $\overline{M}$ [9]. Also, $\widehat{g}$ is not compatible with ∇ and, in fact, C. Atindogbé et al. [3, pp. 3489–3490] showed that

$$\begin{aligned}(\nabla_X \widehat{g})(Y,Z) = \lambda(Y)\{B(X,PZ) - C(X,PZ)\} + \lambda(Z)\{B(X,PY),\\ - C(X,PY)\} + 2\tau(X)\lambda(Y)\lambda(Z), \ \ X,Y,Z \in \Gamma(TM).\end{aligned} \tag{2.12}$$

Consider an orientable Lorentzian manifold $(\overline{M}, \overline{g})$ and (M,g) be its $(n+1)$-dimensional null hypersurface. Moreover, M is also orientable (cf. [14]) and one can choose a globally defined unit normal vector field (with respect to the non-degenerate metric $\widehat{g}$) $E \in \Gamma(TM^{\perp})$ on M having the same time-orientation of M. The ambient manifold being Lorentzian, the induced metric g on M has signature $(0,n)$. It follows that the hypersurface M equipped with the associated metric $\widehat{g}$ is a Riemannian manifold and thus, $\epsilon_a = \widehat{g}(X_a, X_a) = 1$ for all $a \in \{0, \dots, n\}$. Suppose that M admits an integrable screen $S(TM)$ and let $\mathcal{F}$ be a codimension one foliation of M. Then, the leaves of $\mathcal{F}$ are n-dimensional submanifolds, L, of M. Given a point $p \in M$, we may always choose an orthonormal frame field $\{e_0 = E, e_1, \dots, e_n\}$ defined in a neighborhood of p such that the vectors $e_1, \dots, e_n$ are tangent to the leaves of $\mathcal{F}$ and e_0 is normal to them. Such a frame is called an adapted frame field. Equation (2.6) shows that A^*_E is a self-adjoint operator on $S(TM)$. This is not the case with A_N. However, when $S(TM)$ is integrable [9, Theorem 2.5] showed that A_N is self-adjoint on $S(TM)$. As an example, we have the following

Example 2.1. Consider a Minkowski spacetime manifold $(\mathbb{R}^4_1, \overline{g})$, where $\overline{g}(x,y) = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3$ for any $x, y \in \mathbb{R}^4$. Let Ω be an open set of $\mathbb{R}^4$ and consider a smooth function $G : \Omega \longrightarrow \mathbb{R}^4$. Then

$$M = \{(x^0, \dots, x^3) \in \mathbb{R}^4_1 : x^0 = G(x^1, \dots, x^3)\}$$

is called a Monge hypersurface [9]. Consider a parameterization on M as $x^0 = f(v^0, \ldots, v^3)$; $x^{d+1} = v^d$, $d \in \{0, \ldots, 3\}$. In this case, a natural frame field on M is given by $\partial_{v^d} = G'_{x^{d+1}} \partial_{x^0} + \partial_{x^{d+1}}$, for all $d \in \{0, \ldots, 3\}$. Then it follows that $TM^\perp$ is spanned by $E = \partial_{x^0} + \sum_{i=1}^3 G'_{x^i} \partial_{x^i}$. It is known [9] that M is a null hypersurface if $TM^\perp = \operatorname{Rad} TM$ (the radical distribution), which means that E must be a null vector field with respect to g. Hence, M is null Monge hypersurface if G satisfy the differential equation $\sum_{i=1}^3 G'^2_{x^i} = 1$. The corresponding null transversal vector N is given by $N = \frac{1}{2}\{-\partial_{x^0} + \sum_{i=1}^3 G'_{x^i} \partial_{x^i}\}$. Then, $S(TM) = \operatorname{span}\{X, Y\}$ where $X = G'_{x^3} \partial_{x^1} - G'_{x^1} \partial_{x^3}$ and $Y = G'_{x^3} \partial_{x^2} - G'_{x^2} \partial_{x^3}$, in which we have considered $G'_{x^3} \neq 0$ locally on M. By simple calculations we have $\overline{g}([X, Y], N) = 0$. Hence, $S(TM)$ is integrable. Now, using the fact that $\overline{g}([X, Y], N) = 0$, we have $\overline{g}(Y, \overline{\nabla}_X N) - \overline{g}(X, \overline{\nabla}_Y N) = 0$. Hence, from this last equation we have $g(A_N X, Y) = g(X, A_N Y)$, which shows that A_N is self-adjoint on $S(TM)$.

We will consider codimension one foliations $\mathcal{F}^n$ of M furnished with the operator A^*_E. The operator A^*_E, restricted to $S(TM)$, is diagonalizable with n real-valued eigenvalues $\kappa^*_1, \ldots, \kappa^*_n$ with respect to the eigenvector fields $e_1, \ldots, e_n$ tangent to the leaves of $\mathcal{F}$. For $1 \leq k \leq n$, let $\mathcal{S}^*_r$ denote the r-th elementary symmetric function on the eigenvalues $\kappa^*_1, \ldots, \kappa^*_n$; this way, one gets n smooth functions $\mathcal{S}^*_r : \mathcal{F}^n \longrightarrow \mathbb{R}$, such that $\det(t\mathbb{I} - A^*_E) = \sum_{k=0}^n (-1)^k \mathcal{S}^*_k t^{n-k}$, where $\mathcal{S}^*_0 = 1$ by definition and $\mathbb{I}$ is the identity on $\mathcal{F}$. One immediately sees that $\mathcal{S}^*_r = \sigma_r(\kappa^*_1, \ldots, \kappa^*_n)$, where $\sigma_r = \mathbb{R}[\kappa^*_1, \ldots, \kappa^*_n]$, is the r-th elementary symmetric polynomial on the indeterminates $\kappa^*_1, \ldots, \kappa^*_n$. For $1 \leq k \leq n$, one defines the r-th mean curvature H^* of $\mathcal{F}$ by $\binom{n}{r}\mathcal{H}^*_r = \mathcal{S}^*_r = \sigma_r(\kappa^*_1, \ldots, \kappa^*_n)$. Sometimes, $\mathcal{S}^*_r$ instead of $\mathcal{H}^*$ is referred to as the r-th mean curvature. In that regard, we will adopt the latter for mean curvature in this paper. The r-th Newton transformation T^*_r, for $0 \leq r \leq n$, on $\mathcal{F}$ is defined by setting $T^*_0 = \mathbb{I}$ and, for $1 \leq r \leq n$, by the recurrence relation

$$T^*_r = \mathcal{S}^*_r \mathbb{I} - A^*_E \circ T^*_{r-1}. \tag{2.13}$$

By Caylay-Hamiliton theorem $T^*_n = 0$. Since T^*_r is a polynomial in A^*_E for every r, it is also self-adjoint and commutes with A^*_E. Therefore, the basis $\{e_1, \ldots, e_n\}$ diagonalizes T^*_r. Let $\operatorname{tr}^s(\cdot)$ denote the trace with respect to $T\mathcal{F}$ and $\operatorname{tr}(\cdot)$, the trace with respect to TM. Then, the Newton transformation T^*_r satisfy the following relations, for any $X \in \Gamma(M)$, (see [1, 4] for details)

$$\operatorname{tr}^s(T^*_r) = (n - r)\mathcal{S}^*_r, \quad \operatorname{tr}^s(A^*_E \circ T^*_r) = (r + 1)\mathcal{S}^*_{r+1}, \tag{2.14}$$

$$\operatorname{tr}^s(A^{*^2}_E \circ T^*_r) = \mathcal{S}^*_1 \mathcal{S}^*_{r+1} - (r + 2)\mathcal{S}^*_{r+2}, \ \operatorname{tr}^s(T^*_r \circ \nabla^*_X A^*_E) = X(\mathcal{S}_{r+1}). \tag{2.15}$$

The results of the present paper are based on the computation of the divergence of some smooth vector fields globally defined on M. In order to do this, we will need to extent the definition of T^*_r to TM. If we denote by T_r this new Newton transformation on M, then $T_r = \operatorname{diag}(T^*_r, \binom{n}{r}\mathcal{H}^*_r)$.

The divergence of the operator $T_r : \Gamma(TM) \longrightarrow \Gamma(TM)$ is defined as the trace of the $\mathrm{End}(TM)$-valued operator ∇T_r and given by

$$\mathrm{div}^M(T_r) = \mathrm{tr}(\nabla T_r) = \sum_{a,b=0}^{n} g^{[ab]}(\nabla T_r)(e_a, e_b) = \sum_{a=0}^{n}(\nabla_{e_a} T_r)e_a. \tag{2.16}$$

Denote by R^*, R and $\overline{R}$, the curvature tensors of $S(TM)$, M and $\overline{M}$ respectively. Then, by (cf. [9, p. 94]), we have

$$\overline{g}(R(X,Y)Z,N) = \overline{g}(\overline{R}(X,Y)Z,N), \ \ \forall\, X,Y,Z \in \Gamma(TM), \tag{2.17}$$

where $N \in \Gamma(tr(TM))$ and by [10, p. 66] we have

$$\begin{aligned} g(R(X,Y)PZ,PW) &= g(R^*(X,Y)PZ,PW) + C(X,PZ)B(Y,PW) \\ &\quad - C(Y,PZ)B(X,PW), \quad \forall\, X,Y,Z,W \in \Gamma(TM). \end{aligned} \tag{2.18}$$

3. An integral formula

In this section, we present a special integral formula for a foliation, $\mathcal{F}^n$, of a compact null hypersurface $(M^{n+1}, \widehat{g})$ in a Lorentzian manifold $(\overline{M}^{n+2}, \overline{g})$. Integral formulae are important in differential geometry since they provide obstructions to the existence of foliations whose leaves enjoy some special geometric properties, as being totally geodesic (or totally umbilic), minimal, constant mean curvature and many more. Most of the interesting and useful integral formulae in both Riemannian and semi-Riemannian geometry are obtained by computing the divergence of certain vector fields and applying Stoke's theorem (see some examples in [1]).

Observe that the normal null vector field E, with respect to g, is unitary with respect to the nondegenerate metric $\widehat{g}$. That is $\widehat{g}(E,E) = 1$, and hence E becomes a unit normal vector field on M with respect to the associated metric $\widehat{g}$. In that line, we can apply the method used in [1] to compute the divergence of the vector fields $T_r\overline{\nabla}_E E$ $(= T_r\nabla_E E$ as $B(E,\cdot) = 0)$ and $r\mathcal{S}_{r+1}E$ and then apply Stoke's theorem to obtain analogous integration formulae for foliations by null hypersurfaces in Lorentzian manifolds.

Definition 3.1. Let (M,g) be a screen integrable null hypersurface of $(\overline{M}, \overline{g})$. A foliation $\mathcal{F}$ on M is said to be parallel if $\nabla^*_X Y = 0$ for all $X \in \Gamma(TM)$ and $Y \in \Gamma(T\mathcal{F})$.

Furthermore, we will suppose that $C(E,Y) = 0$ for any $Y \in \Gamma(T\mathcal{F})$. With the above definition, we state the following.

Proposition 3.2. *Let (M,g) be a screen integrable null hypersurface of a Lorentzian manifold $(\overline{M}, \overline{g})$ such that $C(E,\cdot) = 0$. Let $\mathcal{F}$ be a parallel foliation of M and let $\{e_i\}$, for $i \in \{1,\ldots,n\}$, be a local field tangent to $\mathcal{F}$. Then*

$$\begin{aligned} \widehat{g}(\nabla_{e_i}\nabla_E E, e_j) &= \widehat{g}(A^*_E e_i, A_N e_j) - \widehat{g}((\nabla^*_E A^*_E)e_i, e_j) - \overline{g}(R(e_i,E)e_j, N) \\ &\quad + \tau(E)\{B(e_i,e_j) + C(e_i,e_j)\} + E\cdot B(e_i,e_j) - E\cdot C(e_i,e_j), \end{aligned}$$

where R is the curvature tensor of M.

Proof. Considering Definition 3.1, we see that $\nabla^*_X e_i = 0$ for any $X \in \Gamma(TM)$, and $C(E, e_i) = 0$. From these conditions we deduce that $\nabla_E e_i = 0$. In fact, from (2.4), we have $\nabla_E e_i = \nabla^*_E e_i + C(E, e_i)E = 0$. Thus, using (2.12) and Definition 3.1, we derive

$$\begin{aligned} \nabla_{e_i}\widehat{g}(\nabla_E E, e_j) = & -\tau(E)\{B(e_i, e_j) - C(e_i, e_j)\} \\ & + \widehat{g}(\nabla_{e_i}\nabla_E E, e_j) + \widehat{g}(\nabla_E E, \nabla_{e_i} e_j). \end{aligned} \tag{3.1}$$

By (2.5) we have $\widehat{g}(\nabla_E E, e_j) = -g(A^*_E E, e_j) + \tau(E)g(E, e_j) = 0$, and hence equation (3.1) gives

$$-\widehat{g}(\nabla_E E, \nabla_{e_i} e_j) = \widehat{g}(\nabla_{e_i}\nabla_E E, e_j) - \tau(E)\{B(e_i, e_j) - C(e_i, e_j)\}. \tag{3.2}$$

Also, from (2.5), (2.12) and Definition 3.1, we derive

$$\begin{aligned} \widehat{g}((\nabla^*_E A^*_E)e_i, e_j) = & \widehat{g}(\nabla_E E, \nabla_{e_i} e_j) + \widehat{g}(E, \nabla_E \nabla_{e_i} e_j) \\ & + E \cdot B(e_i, e_j) - E \cdot C(e_i, e_j) + 2\tau(E)C(e_i, e_j). \end{aligned} \tag{3.3}$$

Now, using (3.3), we derive

$$\begin{aligned} & \widehat{g}(A^*_E e_i, A_N e_j) - \widehat{g}(R(e_i, E)e_j, E) - \widehat{g}((\nabla^*_E A^*_E)e_i, e_j) \\ = & \widehat{g}(A^*_E e_i, A_N e_j) - \widehat{g}(\nabla_E E, \nabla_{e_i} e_j) + \widehat{g}(\nabla_{[e_i,E]} e_j, E) \\ & - E \cdot B(e_i, e_j) + E \cdot C(e_i, e_j) - 2\tau(E)C(e_i, e_j). \end{aligned} \tag{3.4}$$

Decomposing the screen component of $\nabla_{e_i} E$ in the basis $\{e_k\}$ for $k \in \{1, \ldots, n\}$, we have $P(\nabla_{e_i} E) = \sum_{k=1}^n \widehat{g}(\nabla_{e_i} E, e_k)e_k$. Since $\nabla_E e_j = 0$ (see Definition 3.1), we have

$$\widehat{g}(\nabla_{[e_i,E]} e_j, E) = \widehat{g}(\nabla_{\nabla_{e_i} E} e_j, E) = \sum_{k=1}^n \widehat{g}(\nabla_{e_i} E, e_k)\widehat{g}(E, \nabla_{e_k} e_j),$$

which on using the compatibility relation (2.12) and the fact that $\widehat{g}(X, Y) = \sum_{a=0}^n \widehat{g}(X, e_a)\widehat{g}(e_a, Y)$, we deduce that

$$\widehat{g}(\nabla_{[e_i,E]} e_j, E) = -\sum_{k=1}^n B(e_i, e_k)C(e_k, e_j) = -\widehat{g}(A^*_E e_i, A_N e_j). \tag{3.5}$$

Then replacing (3.5) in (3.4), we have

$$\begin{aligned} & \widehat{g}(A^*_E e_i, A_N e_j) - \widehat{g}(R(e_i, E)e_j, E) - \widehat{g}((\nabla^*_E A^*_E)e_i, e_j) \\ = & -\widehat{g}(\nabla_E E, \nabla_{e_i} e_j) - \widehat{g}(E, \nabla_{e_i}\nabla_E e_j) - E \cdot B(e_i, e_j) \\ & + E \cdot C(e_i, e_j) - 2\tau(E)C(e_i, e_j). \end{aligned} \tag{3.6}$$

Putting (3.2) in (3.6), we have

$$\begin{aligned} & \widehat{g}(A^*_E e_i, A_N e_j) - \widehat{g}(R(e_i, E)e_j, E) - \widehat{g}((\nabla^*_E A^*_E)e_i, e_j) \\ = & \widehat{g}(\nabla_{e_i}\nabla_E E, e_j) - E \cdot B(e_i, e_j) + E \cdot C(e_i, e_j) \\ & - \tau(E)\{B(e_i, e_j) + C(e_i, e_j)\}, \end{aligned} \tag{3.7}$$

from which the result follows by considering the fact that $\widehat{g}(R(e_i,E)e_j,E) = \overline{g}(R(e_i,E)e_j,N)$, which ends the proof. □

Proposition 3.3. *Under the assumptions of Proposition 3.2, we have*

$$\begin{aligned}\operatorname{div}^M(T_r\nabla_E E) = &-\widehat{g}(\operatorname{div}^M(T_r),\nabla_E E) - \operatorname{tr}^s(A_E^*\circ A_N\circ T_r)\\ &+\operatorname{tr}^s(T_r\circ(\nabla_E^* A_E^*)) - E\cdot\operatorname{tr}^s(A_E^*\circ T_r) + E\cdot\operatorname{tr}^s(A_N\circ T_r)\\ &-2\tau(E)\operatorname{tr}^s(A_N\circ T_r) - \tau(E)\{\mathcal{S}_1^* - \operatorname{tr}^s(A_N)\}\mathcal{S}_r^*\\ &-\widehat{g}(\nabla_E\nabla_E E, T_r E) - \operatorname{tr}^s(\overline{R}(E)T_r) - \sum_{i=1}^n \overline{g}(\overline{R}(T_r e_i,N)E,e_i)\end{aligned}$$

for all $r\in\{0,\dots,n-1\}$.

Proof. Using (2.12) and the symmetry of T_r, we get

$$\begin{aligned}\widehat{g}((\nabla_{e_i}T_r)e_i,\nabla_E E) = &\,\widehat{g}(e_i,(\nabla_{e_i}T_r)\nabla_E E) - \tau(E)\lambda(T_r E)\{B(e_i,e_i)\\ &-C(e_i,e_i)\} + \tau(E)\{B(e_i,T_r e_i) - C(e_i,T_r e_i)\}.\end{aligned}\tag{3.8}$$

Now, by (2.16) and (3.8) we have

$$\begin{aligned}\widehat{g}(\operatorname{div}^M(T_r),\nabla_E E) = &\sum_{a=0}^n \widehat{g}(\nabla_{e_a}T_r\nabla_E E, e_a) - \sum_{a=0}^n \widehat{g}(\nabla_{e_a}\nabla_E E, T_r e_a)\\ &-\tau(E)\{\mathcal{S}_1^* - \operatorname{tr}^s(A_N)\}\mathcal{S}_r^*\\ &+\tau(E)\{\operatorname{tr}^s(A_E^*\circ T_r) - \operatorname{tr}^s(A_N\circ T_r)\}.\end{aligned}\tag{3.9}$$

Finally, letting $e_j = T_r e_i$ in Proposition 3.2 and then substituting the resulting equation in (3.9), while considering (2.11) and (2.17), we get the desired result. □

Proposition 3.4. *Let* (M,g) *be a screen integrable null hypersurface of a Lorentzian manifold* $(\overline{M},\overline{g})$ *and* $\mathcal{F}$ *be a parallel foliation on* M*. Then*

$$\begin{aligned}\operatorname{div}^M(T_r\nabla_E E + r\mathcal{S}_{r+1}^* E) = &-\widehat{g}(\operatorname{div}^M(T_r),\nabla_E E) - \operatorname{tr}^s(A_E^*\circ A_N\circ T_r)\\ &-\tau(E)\{\mathcal{S}_1^* - \operatorname{tr}^s(A_N)\}\mathcal{S}_r^* - r[\mathcal{S}_1^* + \tau(E)]\mathcal{S}_{r+1}^*\\ &+E\cdot\operatorname{tr}^s(A_N\circ T_r) - 2\tau(E)\operatorname{tr}^s(A_N\circ T_r)\\ &-\widehat{g}(\nabla_E\nabla_E E, T_r E) - \sum_{i=1}^n \overline{g}(\overline{R}(T_r e_i,N)E,e_i)\end{aligned}$$

for all $r\in\{0,\dots,n-1\}$.

Proof. A proof follows immediately from Proposition 3.3, (2.14), (2.15) and the fact that $\operatorname{div}^M(\mathcal{S}_{r+1}^* E) = \mathcal{S}_{r+1}^*\operatorname{div}^M(E) + E(\mathcal{S}_{r+1}^*) = -[\mathcal{S}_1^* + \tau(E)]\mathcal{S}_{r+1}^* + E(\mathcal{S}_{r+1}^*)$. □

Let $(\overline{M},\overline{g})$ be $(n+2)$-dimensional orientable Lorentzian manifold and $(M,\widehat{g})$ be its $(n+1)$-dimensional null hypersurface. Moreover, M is also orientable (cf. [14]). Therefore, by Lemma 20 of [14, p. 195], M admits a (global) $(n+1)$-dimensional volume element, which we will denote by $d\mathrm{vol}_{\widehat{g}}(M)$. In terms of the metric $\widehat{g}$ and differential forms dx^a for $a \in \{0, \dots, n\}$ on M, we have $d\mathrm{vol}_{\widehat{g}}(M) = |\det(\widehat{g}_{ab})|^{\frac{1}{2}} dx^0 \wedge \cdots \wedge dx^n$. Now, from Proposition 3.4 we state the following.

Theorem 3.5. *Let (M, g) be a compact orientable screen integrable null hypersurface of a Lorentzian manifold $(\overline{M}, \overline{g})$ and $\mathcal{F}$ be a parallel foliation on M such that $C(E, \cdot) = 0$. Then*

$$\begin{aligned}\int_M \{&\widehat{g}(\mathrm{div}^M(T_r), \nabla_E E) + \mathrm{tr}^s(A^*_E \circ A_N \circ T_r) - E \cdot \mathrm{tr}^s(A_N \circ T_r)\\ &+ 2\tau(E)\mathrm{tr}^s(A_N \circ T_r) + \tau(E)\{\mathcal{S}^*_1 - \mathrm{tr}^s(A_N)\}\mathcal{S}^*_r + r[\mathcal{S}^*_1 + \tau(E)]\mathcal{S}^*_{r+1}\\ &+ \widehat{g}(\nabla_E \nabla_E E, T_r E) + \sum_{i=1}^{n} \overline{g}(\overline{R}(T_r e_i, N)E, e_i)\}\mathrm{dvol}_{\widehat{g}}(M) = 0\end{aligned}$$

for all $r \in \{0, \dots, n-1\}$.

Proof. Integrating the equation of Proposition 3.4 over M and using Stoke's theorem, we have $\int_M \mathrm{div}^M(T_r \nabla_E E + r\mathcal{S}^*_{r+1} E)\mathrm{dvol}_{\widehat{g}}(M) = 0$ and the result follows immediately, which ends the proof. □

Remark 3.6. Notice that Theorem 3.5 is the null version of Corollary 3.6 of [1]. The two results shows remarkable differences by the terms involved and the mixture of shape operators A^*_E and A_N. However, under conformality of shape operators (this is covered in the next section), our formula gives, among other new results, analogous results as those seen in [1].

4. Application of Theorem 3.5 to minimal foliations

In this section, we apply Theorem 3.5 to minimal codimension one foliation $\mathcal{F}$ of M. A foliation $\mathcal{F}$ on M is said to be minimal if its first order mean curvature $\mathcal{S}^*_1$ vanishes, i.e., $\mathcal{S}^*_1 = 0$. It is important to stress that the 1-form τ defined in (2.3) and (2.5) depends on the section E of $TM^\perp$. It has been proved in [9, p. 99] that there exists a pair (E, N) on an open neighborhood $\mathcal{U} \subset M$ such that the corresponding 1-form τ vanishes identically. We will consider such a section in this section.

Example 4.1. The following example appeared in [4]. Consider the 6-dimensional space $\overline{M} = \mathbb{R}^6$ endowed with a Lorentzian metric $\overline{g} = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + e^{2x^0}\{(\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2\} + e^{2x^1}\{(\mathrm{d}x^4)^2 + (\mathrm{d}x^5)^2\}$, where $(x^0, \dots, x^5)$ are the usual rectangular coordinates on $\overline{M}$. The non-zero Christoffel coefficients of the Levi-Civita connection of $\overline{g}$ are $\Gamma^2_{02} = \Gamma^3_{03} = \Gamma^4_{14} = \Gamma^5_{15} = 1$, $\Gamma^0_{22} = \Gamma^0_{33} = -e^{2x^0}$ and $\Gamma^1_{44} = \Gamma^1_{55} = e^{2x^1}$. Consider a hypersurface M of $\overline{M}$ given by $M = \{(x^0, \dots, x^5) \in \mathbb{R}^6 : x^0 + x^1 = 0\}$. Then, M is a null hypersurface with $N = -\frac{1}{2}(\partial x^0 + \partial x^1)$ and $E = \partial x^0 - \partial x^1$. Also, $S(TM) = \mathrm{span}\{e_1, e_2, e_3, e_4\}$, where $e_1 = e^{-2x^0}\partial x^2$, $e_2 = e^{-2x^0}\partial x^3$, $e_3 = e^{-2x^1}\partial x^4$ and $e_1 = e^{-2x^1}\partial x^5$. Notice that $[e_i, e_j] = 0$ for all $i, j \in \{1, 2, 3, 4\}$. Hence, $S(TM)$ is integrable. By a straightforward calculation, we have $\nabla_{e_1} E = e_1$, $\nabla_{e_2} E = e_2$, $\nabla_{e_3} E = -e_3$ and $\nabla_{e_4} E = -e_4$. Thus, $\kappa_1^* = \kappa_2^* = -1$ and $\kappa_3^* = \kappa_4^* = 1$, from which $\mathcal{S}_1^* = \kappa_1^* + \kappa_2^* + \kappa_3^* + \kappa_4^* = 0$. Hence, any codimension one foliation $\mathcal{F}$ of M in this example is minimal.

A semi-Riemannian manifold $(\overline{M}, \overline{g})$ of constant sectional curvature c is called a semi-Riemannian space form and denoted by $\overline{M}(c)$. By the same convention of the curvature we have used in Proposition 3.2, the curvature tensor field $\overline{R}$ of $\overline{M}(c)$ is given by (cf. O'Neill [14, p. 80, p. 89])

$$\overline{R}(\overline{X}, \overline{Y})\overline{Z} = c\{\overline{g}(\overline{Y}, \overline{Z})\overline{X} - \overline{g}(\overline{X}, \overline{Z})\overline{Y}\}, \;\; \forall \overline{X}, \overline{Y}, \overline{Z} \in \Gamma(T\overline{M}). \tag{4.1}$$

The scalar curvature $\mathcal{R}$ is one of the most important concepts in (semi-)Riemannian geometry (and particularly in General Relativity). It is the weakest curvature invariant one can attach (point-wise) to a semi-Riemannian manifold. Its value at any point can be described as the trace of the Ricci tensor, evaluated at that point. On a null hypersurface (M, g), (a) the induced connection is not a Levi-Civita connection (unless M is totally geodesic), (b) the induced Ricci tensor is not symmetric in general and (c) the induced metric g is degenerate. which means g has no inverse. Hence, one cannot contract the Ricci tensor of such a hypersurface to obtain a scalar curvature. To overcome theses difficulties, Duggal-Sahin considered, in [10], a class of null hypersurfaces in ambient Lorentzian signature, called null hypersurfaces of genus zero. Elements of such class are subject to the following constraints: admission of canonical screen distribution that induces a canonical transversal vector bundle and induced symmetric Ricci tensor. Such restrictions were overcame by a pseudo inversion technique of g by C. Atindogbe *et al.* [3]. He proved the following result about the extrinsic scalar curvature $\mathcal{R}$ of M, which we quote for further use.

Theorem 4.2 ([2])**.** *Let (M, g) be a null hypersurface of a $(n+2)$-dimensional space form $\overline{M}(c)$. Then*

$$\mathcal{R} = cn^2 + \mathrm{tr}^s(A_E^*)\mathrm{tr}^s(A_N) - \mathrm{tr}^s(A_E^* \circ A_N),$$

where $\mathcal{R}$ is the extrinsic scalar curvature of M.

From the above result and Theorem 3.5, we state the following.

Theorem 4.3. *Let $\mathcal{F}^n$, $n \geq 2$, be a minimal parallel foliation of a compact null hypersurface (M, g) satisfying $\tau = 0$ in a Lorentzian space form $\overline{M}(c \geq 0)$. Then the extrinsic scalar curvature, $\mathcal{R}$, of M satisfy the following inequality*

$$\int_M \mathcal{R} d\mathrm{vol}_{\widehat{g}}(M) \geq 0,$$

with equality if $\overline{M}$ is flat, that is, $c = 0$. Moreover, if $c > 0$, then M is isomorphic to n-dimensional spheres.

Proof. Setting $r = 0$ in Theorem 3.5 and the fact $\tau = 0$, we have

$$\begin{aligned}&\int_M \{\mathrm{tr}^s(A_E^* \circ A_N) - E \cdot \mathrm{tr}^s(A_N)\\ &+ \sum_{i=1}^{n} \overline{g}(\overline{R}(T_r e_i, N)E, e_i)\} d\mathrm{vol}_{\widehat{g}}(M) = 0.\end{aligned} \tag{4.2}$$

Thus, applying (2.10) and the fact $\overline{M}$ is a space of constant sectional curvature c, we have $\sum_{i=1}^{n} \overline{g}(\overline{R}(e_i, N)E, e_i) = cn$. Then, (4.2) leads to

$$\int_M \{\mathrm{tr}^s(A_E^* \circ A_N) - E \cdot \mathrm{tr}^s(A_N) + cn\} d\mathrm{vol}_{\widehat{g}}(M) = 0. \tag{4.3}$$

As $\mathcal{F}$ is minimal, then $\mathrm{tr}^s(A_E^*) = \mathcal{S}_1^* = 0$ and $E \cdot \mathrm{tr}^s(A_N) = \mathrm{div}^M(\mathrm{tr}^s(A_N)E)$. Hence, (4.3), Theorem 4.2 and the compactness of M leads to

$$\int_M \{\mathcal{R} - c(n^2 + n)\} d\mathrm{vol}_{\widehat{g}}(M) = 0,$$

from which the inequality of our theorem follows by observing that $c(n^2 + n) \geq 0$. The last assertion follows immediately by the well known Gauss-Bonnet Theorem [12], and the proof is complete. $\square$

Let Ric^L denote the Ricci tensor of a leaf L of $\mathcal{F}$. Then putting $X = PW = e_i$, where $i \in \{1, \dots, n\}$, in (2.18) we derive

$$\begin{aligned}Ric^L(Y, PZ) = &\sum_{i=1}^{n} g(R(e_i, Y)PZ, e_i) + \sum_{i=1}^{n} C(Y, PZ)B(e_i, e_i)\\ &- \sum_{i=1}^{n} C(e_i, PZ)B(Y, e_i)\\ = &\sum_{i=1}^{n} g(R(e_i, Y)PZ, e_i) - g(A_N PZ, A_E^* Y) + C(Y, PZ)\mathrm{tr}^s(A_E^*).\end{aligned} \tag{4.4}$$

By [10, p. 138], we have

$$\sum_{i=1}^{n} g(R(e_i, Y)PZ, e_i) = Ric^M(Y, PZ) - \overline{g}(R(E, Y)PZ, N), \tag{4.5}$$

where Ric^M is the Ricci tensor of M. Putting (4.5) in (4.4) we get

$$\begin{aligned}Ric^L(Y, PZ) = &Ric^M(Y, PZ) - g(A_N PZ, A_E^* Y)\\ &+ C(Y, PZ)\mathrm{tr}^s(A_E^*) - \overline{g}(R(E, Y)PZ, N).\end{aligned} \tag{4.6}$$

From [10, p. 138] we have

$$\begin{aligned}Ric^M(Y, PZ) = &\overline{R}ic(Y, PZ) + B(Y, PZ)(A_N)\\ &- g(A_N Y, A_E^* PZ) - \overline{g}(\overline{R}(E, PZ)Y, N),\end{aligned} \tag{4.7}$$

where $\overline{Ric}$ is the Ricci tensor of $\overline{M}$. Then using (4.7), (2.17) and (4.6) we deduce

$$\begin{aligned} Ric^L(Y, PZ) &= \overline{Ric}(Y, PZ) + B(Y, PZ)\mathrm{tr}^s(A_N) + C(Y, PZ)\mathrm{tr}^s(A_E^*) \\ &\quad - g(A_N Y, A_E^* PZ) - g(A_N PZ, A_E^* Y) \\ &\quad - \overline{g}(\overline{R}(E, PZ)Y, N) - \overline{g}(R(E, Y)PZ, N). \end{aligned} \tag{4.8}$$

Relation (4.8) gives the Ricci tensor of a leaf L of $\mathcal{F}$. Denote by $\mathcal{R}^L$ and $\overline{\mathcal{R}}$ the intrinsic scalar curvatures of L and $\overline{M}$ respectively. Then (4.8) gives

$$\begin{aligned} \mathcal{R}^L &= \overline{\mathcal{R}} - \overline{Ric}(E, E) - \overline{Ric}(N, N) + 2\mathrm{tr}^s(A_E^*)\mathrm{tr}^s(A_N) \\ &\quad - 2\mathrm{tr}^s(A_E^* \circ A_N) - 2\sum_{i=1}^{n} \overline{g}(\overline{R}(E, e_i)e_i, N). \end{aligned} \tag{4.9}$$

Now by (3.5.26) of [10] and (4.9), we deduce that

$$\begin{aligned} \mathcal{R}^L &= \mathcal{R} - \overline{Ric}(E, E) + \mathrm{tr}^s(A_E^*)\mathrm{tr}^s(A_N) \\ &\quad - \mathrm{tr}^s(A_E^* \circ A_N) - \sum_{i=1}^{n} \overline{g}(\overline{R}(E, e_i)e_i, N). \end{aligned}$$

When $\overline{M}$ is a space of constant sectional curvature c, the above relation reduces to

$$\mathcal{R}^L = \mathcal{R} + \mathrm{tr}^s(A_E^*)\mathrm{tr}^s(A_N) - \mathrm{tr}^s(A_E^* \circ A_N) - cn. \tag{4.10}$$

Using (4.10) and Theorem 4.3 we state the following.

Corollary 4.4. *Under the hypothesis of Theorem 4.3, the leaves of $\mathcal{F}$ are isomorphic to n-dimensional spheres if* $\int_M (\mathrm{tr}^s(A_E^*)\mathrm{tr}^s(A_N) - \mathrm{tr}^s(A_E^* \circ A_N) - cn) d\mathrm{vol}_{\widehat{g}}(M) \geq 0$.

The integral in Theorem 3.5 depends on both shape operators A_E^* and A_N. As the shape operator is an information tool in studying geometry of submanifolds, we can consider a foliation $\mathcal{F}$ whose shape operator A_E^* is conformal to that of M (i.e., A_N).

A null hypersurface (M, g) of a semi-Riemannian manifold is called screen locally conformal [9] if the shape operators A_N and A_E^* of M and $S(TM)$, respectively, are related by

$$A_N = \varphi A_E^*, \tag{4.11}$$

where φ is a non-vanishing smooth function on a neighborhood $\mathcal{U} \subset M$. In particular, M is screen homothetic if φ is non-zero constant.

As an example of a minimal screen conformal null hypersurface, we have the following.

Example 4.5 (Null Monge hypersurface)**.** Let M be the null Monge hyperface of Example 2.1 such that the function G satisfies $G(x_1, \ldots, x_3) =$

$\alpha_0 + \sum_{k=1}^{3} \alpha_k x_k$, where $\sum_{k=1}^{3} \alpha_k^2 = 1$. By virtue of Theorem 2 (or Theorem 3) of [6, p. 102], M is minimal. Consider along M the time-like section $\overline{V} = \partial_{x^0} \in \Gamma(T\mathbb{R}_1^4)$. Then $\overline{g}(\overline{V}, E) = -1$ which implies that $\overline{V}$ is not tangent to M. Therefore, the vector bundle $H = \text{span}\{\overline{V}, E\}$ is non-degenerate on M. The complementary orthogonal vector bundle $S(TM)$ to H in $T\mathbb{R}_1^4$ is a non-degenerate distribution on M and is complementary to $TM^{\perp}$. Hence, $S(TM)$ is a screen distribution on M. The corresponding null transversal bundle $\text{tr}(TM)$ is spanned by $N = \overline{V} + \frac{1}{2}E$ and $\tau = 0$. Indeed, $\tau(X) = \overline{g}(\overline{\nabla}_X N, E) = \frac{1}{2}\overline{g}(\overline{\nabla}_X E, E) = 0$ for any $X \in \Gamma(TM)$. The Gauss-Weingarten equations simplifies to $\overline{\nabla}_X N = -A_N X$ and $\overline{\nabla}_X E = -A_E^* X$. On the other hand, following simple calculations we get $\overline{\nabla}_X E = 0$ and $\overline{\nabla}_X N = 0$. Thus M is screen globally conformal with a conformal factor $\varphi = 1/2$ [9].

Theorem 4.6. *Let $(M(c), g)$ be a compact screen conformal null hypersurface in a Lorentzian space form $(\overline{M}(c), \overline{g})$. Let $\mathcal{F}$ be a minimal parallel foliation of M. If $\tau = 0$ then, for all $r \in \{0, \ldots, n-1\}$,*

$$\int_M \left\{ \varphi \mathcal{S}_{r+2}^* - \frac{c(n-r)}{r+2} \mathcal{S}_r^* \right\} d\text{vol}_{\widehat{g}}(M) = 0. \tag{4.12}$$

Proof. Since $\mathcal{F}$ is minimal and $\overline{M}$ is of constant sectional curvature, we respectively have $\mathcal{S}_1^* = 0$ and, by (4.1), $\sum_{i=1}^{n} \overline{g}(\overline{R}(T_r e_i, N)E, e_i) = c(n-r)\mathcal{S}_r^*$. Hence by Theorem 3.5, we have

$$\begin{aligned} &\int_M \{\text{tr}^s(A_E^* \circ A_N \circ T_r) - E \cdot \text{tr}^s(A_N \circ T_r) \\ &+ c(n-r)\mathcal{S}_r^*\} d\text{vol}_{\widehat{g}}(M) = 0. \end{aligned} \tag{4.13}$$

Since $\mathcal{S}_1^* = 0$, we have

$$\text{div}^M(\text{tr}^s(A_N \circ T_r)E) = E \cdot \text{tr}^s(A_N \circ T_r). \tag{4.14}$$

Putting (4.14) in (4.13) and considering $A_N = \varphi A_E^*$ and that M is compact, we have

$$\int_M \{\varphi \text{tr}^s(A_E^{*^2} \circ T_r) + c(n-r)\mathcal{S}_r^*\} d\text{vol}_{\widehat{g}}(M) = 0, \tag{4.15}$$

from which (4.12) follows by applying (2.15). □

In the event that M is screen homothetic, we get the following result which is similar to that in [1, p. 109].

Corollary 4.7. *Under the assumptions of Theorem 4.6, if M is screen homothetic, then*

$$\int_M \mathcal{S}_r^* d\text{vol}_{\widehat{g}}(M) = \begin{cases} \left(\dfrac{c}{\varphi}\right)^{\frac{r}{2}} \dbinom{\frac{n}{2}}{\frac{r}{2}} \text{vol}_{\widehat{g}}(M), & n, r \text{ even}, \\ 0, & n \text{ or } r \text{ odd}. \end{cases} \tag{4.16}$$

Next, we consider the case $\mathcal{S}_1^* = 0$ and $\mathcal{S}_2^*$ constant.

Theorem 4.8. *Let (M, g) be a compact screen homothetic null hypersurface with $\varphi > 0$ in a Lorentzian space form $(\overline{M}(c), \overline{g})$ and $\mathcal{F}$, a codimension one parallel foliation of M whose leaves are minimal and with constant $\mathcal{S}_2^*$. Then $\mathcal{S}_2^*$ must be constant on M. Furthermore, M is locally a product space of a null curve $\mathcal{C}$ and a leaf L of $\mathcal{F}$.*

Proof. By Proposition 2.31 of [5, p. 101], either $\mathcal{S}_2^*$ is constant on M, and the assertions of the theorem are satisfied, or there exists a closed leaf L of $\mathcal{F}$ with the property that $\mathcal{S}_2^*|_L = \max_M \mathcal{S}_2^*(p)$. Suppose that $\mathcal{S}_2^*$ is non-constant on M. Since $\overline{M}$ has non-negative sectional curvature, we have $\sum_{i=1}^{n} \overline{g}(\overline{R}(e_i, N)E, e_i) = nc$. Thus, by Theorem 3.5, we have $\varphi \int_M \mathcal{S}_2^* d\mathrm{vol}_{\widehat{g}}(M) > 0$. Consequently, $\int_M \mathcal{S}_2^* d\mathrm{vol}_{\widehat{g}}(M) > 0$. As $\mathcal{S}_2^*$ is non-constant on M, then $\mathcal{S}_2^*|_L > 0$ and $\mathcal{S}_2^*$ is positive on L. But $\mathcal{S}_1^{*2} \geq \mathcal{S}_2^* > 0$. As $\mathcal{S}_1^* = 0$, we get $0 \geq \mathcal{S}_2^* > 0$, which is a contradiction. Hence, $\mathcal{S}_2^*$ is a constant on M and the proof is complete. $\square$

It is important to note that the operators T_r^* given by recurrence (2.13) are not unique, mainly due to the fact that A_E^* changes with a change in the screen distribution $S(TM)$. More precisely, let $S(TM)$ and $S(TM)'$ be two screen distributions of M, with respect to the bases $\{e_i\}$ and $\{e_i'\}$ respectively. Denote by N' the section of $\mathrm{tr}(TM)'$ for the same $E \in \Gamma(\mathrm{Rad}\, TM)$. Then, by [9, page 87] we have, for any $X \in \Gamma(TM)$,

$$A_E^{*'} X = A_E^* X + B(X, N - N')E, \ \tau(X)' = \tau(X) + B(X, N' - N)E. \tag{4.17}$$

Notice that the eigenvalues of T_r^* are of the form

$$\mu_{i,r} = \sum_{i_1 < \cdots < i_r, i_j \neq i} \kappa_{i_1}^* \cdots \kappa_{i_r}^*. \tag{4.18}$$

Let $T_r^{*'}$ be the new Newton transformation with respect to $A_E^{*'}$, for a given eigenspace $\mathrm{span}\{e_k\}$. Using the first relation of (4.17), (2.13) and (4.18), we derive

$$T_r^{*'} = T_r^* + (\mathcal{S}_r^{*'} - \mathcal{S}_r^*)\mathbb{I} + (\mu_{i,r-1} - \mu_{i,r-1}')A_E^* - B(T_{r-1}^*, N - N'). \tag{4.19}$$

Hence, from (4.19) and $T_r = \mathrm{diag}(T_r^*, \binom{n}{r}\mathcal{H}_r^*)$ we conclude that T_r also depend on $S(TM)$. From the above discussions we can see that Theorem 3.5 depends on the choice of $S(TM)$.

Acknowledgments. This work is based on the research supported wholly / in part by the National Research Foundation of South Africa (Grant Numbers: 95931 and 106072).

References

[1] K. Andrzejewski and P. G. Walczak, *The Newton transformation and new integral formulae for foliated manifolds*, Ann. Global Anal. Geom. **37** (2010), no. 2, 103–111.
[2] C. Atindogbé, *Scalar curvature on lightlike hypersurfaces*, Appl. Sci. **11** (2009), 9–18.
[3] C. Atindogbé, J.-P. Ezin, and J. Tossa, *Pseudoinversion of degenerate metrics*, Int. J. Math. Math. Sci. **2003** (2003), no. 55, 3479–3501.

[4] C. Atindogbé and H. T. Fotsing, *Newton transformations on null hypersurfaces*, Commun. Math. **23** (2015), no. 1, 57–83.

[5] J. L. M. Barbosa, K. Kenmotsu, and G. Oshikiri, *Foliations by hypersurfaces with constant mean curvature*, Math. Z. **207** (1991), no. 1, 97–107.

[6] I. Burdujan, *On minimal lightlike Monge hypersurfaces of a Lorentz space*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **47** (2001), no. 1, 95–103 (2002).

[7] C. Calin, *Contributions to geometry of CR-submanifold*, Ph.D. Thesis, University of Iasi (Romania), 1998.

[8] J. Dong and X. Liu, *Totally umbilical lightlike hypersurfaces in Robertson-Walker spacetimes*, ISRN Geom. **2014** (2014), Art. ID 974695, 10 pp.

[9] K. L. Duggal and A. Bejancu, *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Mathematics and its Applications, **364**, Kluwer Academic Publishers Group, Dordrecht, 1996.

[10] K. L. Duggal and B. Sahin, *Differential Geometry of Lightlike Submanifolds*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2010.

[11] D. H. Jin, *Ascreen lightlike hypersurfaces of an indefinite Sasakian manifold*, J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math. **20** (2013), no. 1, 25–35.

[12] Y. Li, *The Gauss-Bonnet-Chern theorem on Riemannian manifolds*, arXiv:1111.4972 [math.DG].

[13] F. Massamba and S. Ssekajja, *Quasi generalized CR-lightlike submanifolds of indefinite nearly Sasakian manifolds*, Arab. J. Math. (Springer) **5** (2016), no. 2, 87–101.

[14] B. O'Neill, *Semi-Riemannian Geometry*, Pure and Applied Mathematics, **103**, Academic Press, Inc., New York, 1983.

[15] D. N. Kupeli, *Singular Semi-Riemannian Geometry*, Mathematics and its Applications, **366**, Kluwer Academic Publishers Group, Dordrecht, 1996.

Fortuné Massamba
School of Mathematics
Statistics and Computer Science
University of KwaZulu-Natal
Private Bag X01, Scottsville 3209
South Africa
Email address: massfort@yahoo.fr, Massamba@ukzn.ac.za

Samuel Ssekajja
School of Mathematics
Statistics and Computer Science
University of KwaZulu-Natal
Private Bag X01, Scottsville 3209
South Africa
Email address: ssekajja.samuel.buwaga@aims-senegal.org