

## RICCI CURVATURE, CIRCULANTS, AND EXTENDED MATCHING CONDITIONS

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**ABSTRACT.** Ricci curvature for locally finite graphs, as proposed by Lin, Lu and Yau, provides a useful isomorphism invariant. A Matching Condition was introduced as a key tool for computation of this Ricci curvature. The scope of the Matching Condition is quite broad, but it does not cover all cases. Thus the current paper introduces extended versions of the Matching Condition, and applies them to the computation of the Ricci curvature of a class of circulants determined by certain number-theoretic data. The classical Matching Condition is also applied to determine the Ricci curvature for other families of circulants, along with Cayley graphs of abelian groups that are generated by the complements of (unions of) subgroups.

### 1. Introduction

This paper is devoted to the study of the Lin-Lu-Yau version of Ricci curvature, as an isomorphism invariant for locally finite graphs. In [7], the Matching Condition for an edge of a graph (Definition 2.1) was introduced as a tool for the computation of these Ricci curvatures. However, there are various cases where the Matching Condition cannot be applied, for example the 5-cycle  $C_5$ . Thus the current paper introduces new Extended Matching Conditions (§2), more specifically of types 2 (Definition 2.5) and 3 (Definition 2.9). For example, (each edge in)  $C_5$  satisfies the Extended Matching Condition of Type 2. The respective Theorems 2.7 and 2.10 then provide formulas for the Ricci curvature of an edge satisfying the Extended Matching Condition of Type 2 or 3.

The applications of Ricci curvature discussed here are mainly focused on the class of circulant graphs. We employ various conventions for the specification of these graphs. The basic convention  $C_n(J)$  denotes the circulant graph on the vertex set  $\mathbb{Z}/n$  of residues modulo a positive integer  $n$ , with edges  $\{r, s\}$  precisely when  $r - s$  or  $s - r$  lies in the *jump set*  $J$ . In this notation, the set  $J$

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may be replaced by a list of its elements. The  $n$ -element cycle, corresponding to a jump set  $\{u\}$  with  $u$  coprime to  $n$ , is denoted as usual by  $C_n$ .

Section 3 uses the Extended Matching Conditions of Types 2 and 3 to compute the Ricci curvature of circulants  $C_n(u, v)$  with prime  $n \equiv 1 \pmod{4}$  and  $u^2 + v^2 = 0$  in  $\mathbb{Z}/n$ . The main result of that section, Theorem 3.4, shows that these circulants are Ricci-flat for  $n > 5$ , with the exceptions of  $C_{13}(2, 3)$  and  $C_{17}(1, 4)$ .

Section 4 studies the Cayley graph  $\Gamma(G, S)$  of the complement  $S = G \setminus H$  of a subgroup  $H$  of a finite abelian group  $G$ . This class of graphs includes many circulants (§4.3). Theorem 4.3 computes

$$\frac{2 + |G| - 2|H|}{|G| - |H|}$$

as the constant Ricci curvature of  $\Gamma(G, S)$ , using the classical Matching Condition.

The remaining part of the paper, Section 5, computes the constant Ricci curvature of a circulant  $C_n(J)$ , where  $n$  is a product of mutually coprime factors  $p_i > 3$ , and the jump set  $J$  is the complement  $\mathbb{Z}/n \setminus S$  of the union  $S = \bigcup_{i=1}^m p_i \mathbb{Z}/n$  of subgroups of the cyclic group  $\mathbb{Z}/n$ . Relevant counting lemmas are assembled in §5.2. A matching derived in §5.3 then enables the classical Matching Condition to be applied, producing the formula

$$\frac{2 + \prod_{i=1}^m (p_i - 2)}{\prod_{i=1}^m (p_i - 1)}$$

for the Ricci curvature of  $C_n(J)$  that is exhibited in Theorem 5.13. The concluding Remark 5.14 discusses potential extensions of the theorem, but these remain open in general.

### 1.1. Notational conventions

We regard a simple graph  $(V, E)$  as a relational structure, with  $E$  as an irreflexive symmetric binary relation on  $V$ . Thus for  $x \in V$ , we have  $x^E$  as the set of neighbors of  $x$ . Elements of  $E$  are written as doubletons from  $V$ .

The symbol  $\uplus$  denotes the disjoint union of sets. In this paper, the symbol may normally be regarded as the usual union, coming with a certificate that the uniands are actually disjoint.

### 1.2. Ricci curvatures and Ricci flatness

Since inequivalent but similar notions of Ricci curvature for a simple graph  $(V, E)$  are now current in the literature, we feel that it may be helpful to make some comments. Recall that for a vertex  $x$  in  $V$ , the  $\varepsilon$ -ball  $b_x^\varepsilon$  centered at  $x$  is

the probability distribution on  $V$  defined by

$$b_x^\varepsilon(v) = \begin{cases} 1 - \varepsilon, & v = x; \\ \varepsilon/|x^E|, & v \in x^E; \\ 0, & \text{otherwise.} \end{cases}$$

For an edge  $\{x, y\}$  in  $E$ , the *Wasserstein distance*  $W(b_x^\varepsilon, b_y^\varepsilon)$  is the cost of an optimal transport (minimizing total distance  $\times$  probability weight) from  $b_x^\varepsilon$  to  $b_y^\varepsilon$ . Then  $\kappa_\varepsilon(x, y) = 1 - W(b_x^\varepsilon, b_y^\varepsilon)$ , and the Lin-Lu-Yau version of Ricci curvature is  $\kappa(x, y) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \kappa_\varepsilon(x, y)$ . See [7] for the details, including the dual role played by 1-Lipschitz functions  $f: V \rightarrow \mathbb{R}$ . The Ricci curvature for graphs proposed by Lin, Lu and Yau in [4] is a variant of that proposed by Ollivier [6], who considered the case  $\varepsilon = 1$  and  $\varepsilon = \frac{1}{2}$  in the probability distribution. Here, the corresponding quantity  $1 - \varepsilon$  is often described as the *idleness* [2].

Although both definitions have some similar properties, they vary in several aspects. The difference can be seen, for example, by considering Ricci-flat graphs, i.e., graphs with constant curvature 0. Recently, Lin and Lu classified all Ricci-flat connected graphs with girth at least five [5]. They showed that if  $G$  is a Ricci-flat graph with girth  $g(G) \geq 5$ , then  $G$  is one of the following graphs: the infinite path, a cycle  $C_n$  with  $n \geq 6$ , the dodecahedral graph, the Petersen graph, or the half-dodecahedral graph. They also constructed several Ricci-flat graphs with girth 3 or 4. An analogue of this study with Ollivier's definition of Ricci curvature was conducted by Bhattacharya and Mukherjee [1]. They proved that a connected graph  $G$  with girth  $g(G) \geq 5$  is Ricci-flat in the sense of Ollivier if and only if  $G$  is one of the following: a path  $P_n$  with  $n \geq 2$ , the infinite ray, the infinite path, a cycle  $C_n$  with  $n \geq 5$ , or the star graph  $T_n$  with  $n \geq 3$ . Except for the infinite path and cycles  $C_n$  with  $n \geq 6$ , the respective Ricci-flat graphs according to Lin-Lu-Yau and Ollivier are different.

## 2. Matching Conditions

Throughout this section, we will consider a locally finite simple graph  $G = (V, E)$ .

### 2.1. The classical Matching Condition

**Definition 2.1** ([7, Definition 6.1]). An edge  $\{x, y\}$  of the graph  $G$  is said to satisfy the *Matching Condition* if there is a perfect matching (in the graph-theoretical sense) between the two sets  $x^E \setminus (\{y\} \cup y^E)$  and  $y^E \setminus (\{x\} \cup x^E)$ .

If an edge  $\{x, y\}$  of a graph  $G$  satisfies the Matching Condition, then  $x$  and  $y$  have the same degree in  $G$  [7, Lemma 6.2(a)].

**Theorem 2.2** ([7, Theorem 6.3]). *In a locally finite graph  $G = (V, E)$ , suppose that  $\{x, y\}$  is an edge satisfying the Matching Condition. Then*

$$\kappa(x, y) = (2 + |x^E \cap y^E|) / \delta,$$

where  $\delta$  is the common degree of  $x$  and  $y$ .

## 2.2. The Extended Matching Condition

For a vertex  $p$  of  $G$  and subset  $S$  of  $V$ , we will adopt the usual metric space convention

$$d(p, S) = \inf\{d(p, s) \mid s \in S\}.$$

Suppose that  $\{x, y\}$  is an edge of  $G$ . For  $p$  in  $x^E \setminus (\{y\} \cup y^E)$  and  $S = y^E \setminus (\{x\} \cup x^E)$ , note  $d(p, S) \leq 3$ . Similarly, note  $d(q, T) \leq 3$  for  $q$  in  $y^E \setminus (\{x\} \cup x^E)$  and  $T = x^E \setminus (\{y\} \cup y^E)$ .

**Definition 2.3.** Let  $\{x, y\}$  be an edge of  $G$ . Suppose that there are disjoint partitions

$$x^E \setminus (\{y\} \cup y^E) = P_1 \uplus P_2 \uplus P_3$$

and

$$y^E \setminus (\{x\} \cup x^E) = Q_1 \uplus Q_2 \uplus Q_3$$

such that for  $1 \leq i \leq 3$ , each vertex in  $P_i$  is matched to a unique vertex in  $Q_i$  by a path of length  $i$ , and *vice versa*. Then the edge  $\{x, y\}$  is said to satisfy the *Extended Matching Condition*.

If  $P_2$  and  $P_3$  are empty, the Extended Matching Condition reduces to the original Matching Condition of Definition 2.1. The following observation extends [7, Lemma 6.2(a)].

**Lemma 2.4.** *Suppose that an edge  $\{x, y\}$  of  $G$  satisfies the Extended Matching Condition. Then the vertices  $x$  and  $y$  have the same degree in  $G$ .*

*Proof.* Note that

$$\begin{aligned} |x^E| &= |x^E \setminus (\{y\} \cup y^E)| + |x^E \cap y^E| + 1 \\ &= |P_1| + |P_2| + |P_3| + |x^E \cap y^E| + 1 \\ &= |Q_1| + |Q_2| + |Q_3| + |x^E \cap y^E| + 1 \\ &= |y^E \setminus (\{x\} \cup x^E)| + |x^E \cap y^E| + 1 = |y^E|. \end{aligned}$$

Thus the degrees of  $x$  and  $y$  coincide.  $\square$

## 2.3. The Extended Matching Condition of Type 2

**Definition 2.5.** The Extended Matching Condition of Definition 2.3 is said to be of *Type 2* if the following hold:

- (a)  $P_2$  is a singleton  $\{p\}$ ;
- (b)  $Q_2$  is a singleton  $\{q\}$ ;
- (c)  $d(p, Q_1) \geq 2$ ;
- (d)  $d(q, P_1) \geq 2$ ; and
- (e)  $P_3 = Q_3 = \emptyset$ .

**Example 2.6.** Each edge within  $C_5$  satisfies the Extended Matching Condition of Type 2. For example, with  $x = 0$  and  $y = 1$ , one has  $p = 4$  and  $q = 2$ , while  $P_1$  and  $Q_1$  are empty.

**Theorem 2.7.** *In a locally finite graph  $G = (V, E)$ , suppose that  $\{x, y\}$  is an edge satisfying the Extended Matching Condition of Type 2. Then*

$$\kappa(x, y) = (1 + |x^E \cap y^E|) / \delta,$$

where  $\delta$  is the common degree of  $x$  and  $y$ .

*Proof.* Let  $\varepsilon$  be a small positive number. The measure  $b_x^\varepsilon$  assigns weight  $1 - \varepsilon$  to the vertex  $x$ , and weight  $\varepsilon/\delta$  to each vertex in  $x^E$ . The measure  $b_y^\varepsilon$  assigns weight  $1 - \varepsilon$  to the vertex  $y$ , and weight  $\varepsilon/\delta$  to each vertex in  $y^E$ . Both measures assign weight 0 to the remaining vertices. Set  $c = |x^E \cap y^E|$ .

A coupling from  $b_x^\varepsilon$  to  $b_y^\varepsilon$  is obtained as follows. First, transfer the weight  $1 - \varepsilon - \varepsilon/\delta$  from the vertex  $x$  to the vertex  $y$ . Then, for each vertex in  $x^E \cap y^E$ , leave the weight  $\varepsilon/\delta$  there alone. Next, transfer the weights  $\varepsilon/\delta$  from each vertex in  $x^E \setminus (\{p, y\} \cup y^E)$  to the corresponding vertex in  $y^E \setminus (\{q, x\} \cup x^E)$  along the matching edge. Finally, transfer the weight  $\varepsilon/\delta$  from the vertex  $p$  to the vertex  $q$  along a path of distance 2. Then the transportation cost is

$$(2.1) \quad W(b_x^\varepsilon, b_y^\varepsilon) \leq 1 - \varepsilon - \frac{\varepsilon}{\delta} + \frac{\varepsilon}{\delta}(\delta - c - 2) + \frac{\varepsilon}{\delta} \cdot 2 = 1 - (c + 1) \frac{\varepsilon}{\delta}.$$

Next, consider the function  $f: V \rightarrow \mathbb{R}$  defined by

$$f(v) = \begin{cases} 1, & v \in P_1 \cup P_2 \cup \{x\}; \\ -1, & v \in Q_2; \\ 0, & \text{otherwise.} \end{cases}$$

Note that the function  $f$  is 1-Lipschitz, since  $d(x, q) = d(p, q) = 2$  and  $d(q, P_1) \geq 2$ . Then

$$W(b_x^\varepsilon, b_y^\varepsilon) \geq \left(1 - \varepsilon - \frac{\varepsilon}{\delta}\right) \cdot 1 + \frac{\varepsilon}{\delta}(\delta - c - 1) \cdot 1 - \frac{\varepsilon}{\delta}(-1) = 1 - (c + 1) \frac{\varepsilon}{\delta}.$$

It follows that  $\kappa(x, y) = (1 + c)/\delta = (1 + |x^E \cap y^E|) / \delta$ .  $\square$

**Example 2.8.** The circulant  $C_{10}(1, 4)$  is connected and edge-transitive, with an automorphism group that includes the transpositions

$$(0 \ 5), (1 \ 6), (2 \ 7), (3 \ 8), (4 \ 9).$$

Each edge  $\{x, y\}$  of  $C_{10}(1, 4)$  satisfies the Extended Matching Condition of Type 2, with  $|x^E \cap y^E| = 0$  and  $\delta = 4$ . For example, taking  $x = 0$  and  $y = 1$ , one has  $P_1 = \{4, 6\}$ ,  $Q_1 = \{5, 7\}$ ,  $p = 9$  and  $q = 2$ . Thus  $\kappa(x, y) = 1/4$ .

## 2.4. The Extended Matching Condition of Type 3

**Definition 2.9.** The Extended Matching Condition of Definition 2.3 is said to be of *Type 3* if the following hold:

- (a)  $P_3$  is a singleton  $\{p\}$ ;
- (b)  $Q_3$  is a singleton  $\{q\}$ ;

- (c)  $d(p, Q_1) = 3$ ;
- (d)  $d(q, P_1) = 3$ ; and
- (e)  $P_2 = Q_2 = \emptyset$ .

**Theorem 2.10.** *In a locally finite graph  $G = (V, E)$ , suppose that  $\{x, y\}$  is an edge satisfying the Extended Matching Condition of Type 3. Then*

$$\kappa(x, y) = |x^E \cap y^E| / \delta,$$

where  $\delta$  is the common degree of  $x$  and  $y$ .

*Proof.* The proof is similar to the proof of Theorem 2.7, but with

$$f(v) = \begin{cases} 1, & v \in P_1 \cup \{x\} \cup p^E, \\ -1, & v \in Q_3, \\ 2, & v \in P_3, \\ 0, & \text{otherwise.} \end{cases}$$

as the 1-Lipschitz function  $f: V \rightarrow \mathbb{R}$ . □

### 3. Sums of two squares

The Extended Matching Conditions of Types 2 and 3 will now be used to compute the Ricci curvature of circulants  $C_n(u, v)$  with prime  $n \equiv 1 \pmod{4}$  and  $u^2 + v^2 = 0$  in  $\mathbb{Z}/n$ . Since the latter condition, taking  $0 < u \neq v \leq n/2$ , implies that  $n$  itself is the largest power of  $n$  that divides the integer  $u^2 + v^2 < n^2/4$ , the ‘‘Sum of Two Squares Theorem’’ [3, Th. 366] forces the congruence restriction on  $n$ .

#### 3.1. Preliminary lemmas

**Lemma 3.1.** *Consider a circulant  $C_n(u, v)$  with prime  $n \equiv 1 \pmod{4}$ , and nonzero residues  $u, v$  in  $\mathbb{Z}/n$  with  $u^2 + v^2 = 0$ . Then  $C_n(u, v)$  is isomorphic to a circulant  $C_n(a, b)$  with  $a, b$  coprime and  $a^2 + b^2 = 0$ .*

*Proof.* Suppose that the integers  $u$  and  $v$  have greatest common divisor  $d$ , say  $u = da$  and  $v = db$  with  $a$  and  $b$  coprime. Then the prime number  $n$  divides the integer  $u^2 + v^2 = d^2(a^2 + b^2)$ . Since the residues  $u$  and  $v$  are nonzero,  $n$  does not divide  $d$ . Thus  $a^2 + b^2 = 0$  in  $\mathbb{Z}/n$ , and

$$\mathbb{Z}/n \rightarrow \mathbb{Z}/n; x \mapsto dx$$

yields an isomorphism from  $C_n(a, b)$  to  $C_n(u, v)$ . □

**Lemma 3.2.** *For a modulus  $n$ , consider non-zero coprime residues  $a, b$  in  $\mathbb{Z}/n$ .*

- (a) *The circulant  $C_n(a, b)$  is connected.*
- (b) *If  $n \equiv 1 \pmod{4}$  is a prime with  $a^2 + b^2 = 0$ , the circulant  $C_n(a, b)$  is edge-transitive.*

*Proof.* (a) Because the integers  $a$  and  $b$  are coprime, there are integers  $l$  and  $m$  with  $la + mb = 1$ . It follows that any two residues in  $\mathbb{Z}/n$  are connected by a series of steps forward or backward by  $a$  or  $b$ .

(b) For any residue  $c$  in  $\mathbb{Z}/n$ , the maps

$$x \mapsto c - x, \quad x \mapsto c + x, \quad \text{and} \quad x \mapsto a^{-1}bx$$

are automorphisms of  $C_n(a, b)$ . The edges of  $C_n(a, b)$  are either of the form  $\{i, i + a\}$  or  $\{j, j + b\}$  for residues  $i, j$ . Now  $x \mapsto k - i + x$  maps the edge  $\{i, i + a\}$  to  $\{k, k + a\}$ , while  $x \mapsto l - j + x$  maps  $\{j, j + b\}$  to  $\{l, l + b\}$ . Finally,  $x \mapsto j - a^{-1}bi + a^{-1}bx$  maps  $\{i, i + a\}$  to  $\{j, j + b\}$ . Thus  $C_n(a, b)$  is edge-transitive.  $\square$

**Example 3.3.** The circulant  $C_5(1, 2)$  is the complete graph  $K_5$ . As such, it has constant Ricci curvature  $5/4$  [4, 7].

### 3.2. The main result

**Theorem 3.4.** *Let  $n \equiv 1 \pmod{4}$  be a prime number greater than 5. Consider non-zero coprime residues  $a, b$  in  $\mathbb{Z}/n$  with  $a^2 + b^2 = 0$ .*

- (a) *In  $C_n(a, b)$ , the neighbor sets  $0^E$  and  $a^E$  are disjoint.*
- (b) *The circulants  $C_{13}(2, 3)$  and  $C_{17}(1, 4)$  have Ricci curvature  $1/4$ . Otherwise,  $C_n(a, b)$  is flat.*

*Proof.* (a) Note  $0^E = \{a, b, -a, -b\}$  and  $a^E = \{0, a + b, a - b, 2a\}$ . Since the residues  $a$  and  $b$  are nonzero, the only potential for a nontrivial intersection would occur if  $2a = \pm b$  or  $2b = \pm a$ . Assume, without loss of generality, that  $4a^2 = b^2 = -a^2$ . Then  $n$  divides  $5a^2$ , so  $n = 5$ . This case (treated in Example 3.3) is now excluded by the hypothesis.

(b) By Lemma 3.2(b), it will suffice to compute  $\kappa(0, a)$ . With reference to the notations of Definitions 2.3, 2.5 and 2.9, write:

$$\begin{cases} P_1 = \{a - b, a + b\}, & p = 2a, \\ Q_1 = \{-b, b\}, & q = -a. \end{cases}$$

There is a matching

$$\begin{array}{cc} a - b & a + b \\ | & | \\ -b & b \end{array}$$

from  $P_1$  to  $Q_1$ . Now  $d(p, Q_1) \geq 2$ , since  $d(p, Q_1) = 1$  would imply the contradictions  $a \in \{0, b\}$  or  $9a^2 = b^2$ , whence  $n = 5$ . Similarly,  $d(q, P_1) \geq 2$  and  $d(-a, 2a) \geq 2$ . However,  $d(-a, 2a) \leq 3$ . Thus two cases remain.

**Case I:**  $d(-a, 2a) = 2$ . Here, the neighbor sets

$$(-a)^E = \{0, -a - b, -a + b, -2a\} \quad \text{and} \quad (2a)^E = \{a, 3a, 2a - b, 2a + b\}$$

intersect, and it follows from Theorem 2.7 that  $\kappa = 1/4$ . This situation arises in two subcases.

**Case I(a):**  $3a = -a \pm b$ , so  $b^2 = 16a^2$ , yielding the circulant  $C_{17}(1, 4)$ .

**Case I(b):**  $-a \pm b = 2a \mp b$ , so  $4b^2 = 9a^2$ , for the circulant  $C_{13}(2, 3)$ .

**Case II:**  $d(-a, 2a) = 3$ . Here, the neighbor sets  $(-a)^E$  and  $(2a)^E$  are disjoint, so Theorem 2.10 shows that  $C_n(a, b)$  is Ricci flat.  $\square$

## 4. Subgroup complements

### 4.1. Cayley graphs

Let  $S$  be a subset of an additive group  $G$ , with  $0 \notin S = -S$ . The *Cayley graph*  $\Gamma(G, S)$  of  $G$  with respect to  $S$  is the simple graph with vertex set  $G$ , in which two vertices  $g$  and  $g'$  are adjacent if and only if  $g - g' \in S$ . This section will use the classical Matching Condition to obtain a formula for the Ricci curvature of certain Cayley graphs defined on finite abelian groups, and in particular for certain circulants implemented by these Cayley graphs.

**Lemma 4.1.** *Suppose that  $S$  is a subset of an additive group  $G$ , with  $0 \notin S = -S$ . Then for each element  $l$  of  $G$ , the invertible translation*

$$R_+(l): G \rightarrow G; x \mapsto x + l$$

*is an automorphism of  $\Gamma(G, S)$ .*

*Proof.* Suppose that  $\{g, g'\}$  is an edge of  $\Gamma(G, S)$ , say  $g = s + g'$  with  $s \in S$ . Then  $g + l = s + g' + l$ , so that  $\{g + l, g' + l\}$  is also an edge of  $\Gamma(G, S)$ .  $\square$

**Lemma 4.2.** *Let  $S$  be the complement  $G \setminus H$  of a proper subgroup  $H$  of a finite additive group  $G$ .*

- (a) *The set  $S$  is closed under negation.*
- (b) *The subset  $S$  generates  $G$ .*
- (c) *The Cayley graph  $\Gamma(G, S)$  is connected.*

*Proof.* (a) Negation is a permutation of the set  $G$ . Then since the subset  $H$  of  $G$  is closed under negation, so is its complement  $S$ .

(b) Consider an element  $s$  of the complement of  $H$ . Then each element  $h$  of  $H$  is the difference  $(h + s) - s$  of elements of  $S$ .

(c) Consider elements  $g$  and  $g'$  of  $G$ . By (b), the group element  $g - g'$  is a sum of elements of  $S$ . Thus  $g$  and  $g'$  are connected by a path in  $\Gamma(G, S)$ , along edges corresponding to the summands of  $g - g'$ .  $\square$

### 4.2. Ricci curvature of Cayley graphs

**Theorem 4.3.** *Let  $H$  be a proper subgroup of a finite abelian group  $G$ . Take  $S = G \setminus H$ . Then the Cayley graph  $\Gamma(G, S)$  has*

$$\kappa = \frac{2 + |G| - 2|H|}{|G| - |H|}$$

*as its constant Ricci curvature.*



*Proof.* Let  $E$  be the edge set of  $\Gamma(G, S)$ , so that

$$\{g, g'\} \in E \Leftrightarrow g - g' \notin H.$$

We will compute the Ricci curvature  $\kappa(0, s)$  for an arbitrary element  $s$  of  $S$ . Note that  $0^E = G \setminus H$ , whence

$$s^E = 0^E + s = (G \setminus H) + s = G \setminus (H + s)$$

by Lemma 4.1. Thus

$$\begin{aligned} 0^E &= (0^E \cap s^E) \uplus (0^E \cap \overline{s^E}) \\ &= (0^E \cap s^E) \uplus (0^E \cap (H + s)) \\ &= (0^E \cap s^E) \uplus (H + s) \end{aligned}$$

and

$$\begin{aligned} s^E &= (s^E \cap 0^E) \uplus (s^E \cap \overline{0^E}) \\ &= (s^E \cap 0^E) \uplus (s^E \cap H) \\ &= (0^E \cap s^E) \uplus H. \end{aligned}$$

In particular,

$$(4.1) \quad 0^E \cap s^E = G \setminus (H \uplus (H + s)).$$

The existence of a matching between

$$0^E \setminus (\{s\} \cup s^E) = (H + s) \setminus \{s\}$$

and

$$s^E \setminus (\{0\} \cup 0^E) = H \setminus \{0\}$$

follows since  $G$  is abelian: There is an edge between the two vertices  $h + s$  and  $h$  for all  $h \in H \setminus \{0\}$ . Thus  $\Gamma(G, S)$  satisfies the Matching Condition.

Using (4.1), Theorem 2.2 shows that

$$\kappa(0, s) = \frac{2 + |0^E \cap s^E|}{|G| - |H|} = \frac{2 + |G \setminus (H \uplus (H + s))|}{|G| - |H|} = \frac{2 + |G| - 2|H|}{|G| - |H|}.$$

Now an arbitrary edge of  $\Gamma(G, S)$  has the form  $\{g, s + g\} = \{0, s\} + g$  for some  $g \in G$  and  $s \in S$ . By Lemma 4.1, the translation  $R_+(g)$  is an automorphism of  $\Gamma(G, S)$ . Thus

$$\kappa(g, s + g) = \kappa(0, s) = \frac{2 + |G| - 2|H|}{|G| - |H|},$$

as required to complete the proof of the theorem.  $\square$

### 4.3. Circulants from subgroup complements

To conclude the current section, we exhibit a direct application of Theorem 4.3 to the determination of Ricci curvatures of circulants. Consider the product  $n = pq$  of positive integers  $p$  and  $q$ , with  $p > 1$ . Consider the circulant  $C_n(J)$ , with  $J = \{s \in \mathbb{Z}/n \setminus p\mathbb{Z}/n \mid s \leq \lfloor n/2 \rfloor\}$ .

**Corollary 4.4.** *The circulant graph  $C_n(J)$  has*

$$\kappa = \frac{2 + q(p - 2)}{q(p - 1)}$$

*as its constant Ricci curvature.*

*Proof.* Take  $G = \mathbb{Z}/n$  and  $H = p\mathbb{Z}/n$  in Theorem 4.3, so  $|H| = q$ .  $\square$

**Example 4.5.** If  $q = 1$ , then Corollary 4.4 offers one more derivation of the well-known formula  $n/(n - 1)$  [4, 7] for the constant Ricci curvature of the complete graph  $K_n = C_n(1, 2, \dots, \lfloor n/2 \rfloor)$ .

## 5. Subgroup union complements

Consider a product

$$(5.1) \quad n = \prod_{i=1}^m p_i$$

of mutually coprime integer factors  $p_i > 3$ . Within the cyclic group  $\mathbb{Z}/n$ , consider the union

$$S = \bigcup_{i=1}^m p_i \mathbb{Z}/n$$

of subgroups, and the jump set

$$(5.2) \quad J_+ = \{s \in \mathbb{Z}/n \setminus S \mid s \leq \lfloor n/2 \rfloor\}.$$

The goal of this section is to compute the (constant) Ricci curvature of the circulant  $C_n(J_+)$ . Note that Example 4.5 represents the case  $m = 1$ .

### 5.1. Notational conventions

Although the jump set (5.2) follows the standard convention for specifying a circulant structure on  $\mathbb{Z}/n$ , in listing jumps of up to  $\lfloor n/2 \rfloor$  that may be forward or backward, it will actually be more convenient for the purposes of this section to take the equivalent full jump set  $J = \mathbb{Z}/n \setminus S$ , and refer to the circulant  $C_n(J)$  rather than  $C_n(J_+)$ .

There is an isomorphism

$$(5.3) \quad \mathbb{Z}/n \rightarrow \bigoplus_{i=1}^m \mathbb{Z}/p_i; n\mathbb{Z} + x \mapsto (p_1\mathbb{Z} + x, \dots, p_m\mathbb{Z} + x)$$

given by the Chinese Remainder Theorem. Since it is easier to work in  $\mathbb{Z}/p_1 \oplus \cdots \oplus \mathbb{Z}/p_m$  than in  $\mathbb{Z}/n$ , we will use the following notational conventions for isomorphic images under (5.3):

The image of the jump set  $J$  is

$$(5.4) \quad J' = \{(j_1, \dots, j_m) \mid \forall 1 \leq k \leq m, j_k \neq 0\},$$

and the image of  $C_n(J)$  is written as  $C_n(J')$ .

Let  $j \in J$  and  $j_k = p_k \mathbb{Z} + j \in \mathbb{Z}/p_k$  for each  $1 \leq k \leq m$ . For given  $1 \leq k \leq m$ , the isomorphic images of the subgroup  $p_k \mathbb{Z}/n$  of  $\mathbb{Z}/n$  and its coset  $j + p_k \mathbb{Z}/n$  are respectively given by

$$(5.5) \quad S'_k = \left\{ (s_1, \dots, s_m) \in \bigoplus_{i=1}^m \mathbb{Z}/p_i \mid s_k = 0 \right\}$$

and

$$(5.6) \quad T'_k = \left\{ (t_1, \dots, t_m) \in \bigoplus_{i=1}^m \mathbb{Z}/p_i \mid t_k = j_k \right\}.$$

Therefore, the unions

$$(5.7) \quad S' = \bigcup_{k=1}^m S'_k \quad \text{and} \quad T' = \bigcup_{k=1}^m T'_k$$

are the isomorphic images of  $S$  and  $j + S$ , respectively.

## 5.2. Counting lemmas

This section prepares four technical lemmas for subsequent use. The first of the four is a standard identity for binomial coefficients.

**Lemma 5.1.** *The identity*

$$(5.8) \quad \sum_{k=0}^l \binom{m}{k} \binom{m-k}{l-k} = \binom{m}{l} 2^l$$

holds for positive integers  $l$  and  $m$  with  $0 \leq l \leq m$ .

*Proof.* Consider a set  $M$  with  $m$  elements. The right hand side of (5.8) counts the number of subset chains

$$(5.9) \quad K \subseteq L \subseteq M$$

with  $|L| = l$ , by first making any one of  $\binom{m}{l}$  choices for the  $l$ -element subset  $L$  of  $N$ , and then choosing  $K$  from any of the  $2^l$  subsets of  $L$ .

The left hand side of (5.8) counts the same number of subset chains (5.9) by first selecting an integer  $k$  from  $\{0, \dots, l\}$ , next making any one of  $\binom{m}{k}$  choices for a  $k$ -element subset  $K$  of  $M$ , and finally, from among the  $m - k$  elements of  $M \setminus K$ , choosing any one of the  $\binom{m-k}{l-k}$  subsets of size  $l - k$  to make up the complement of  $K$  in  $L$ .  $\square$

We now resume the notation of §5.1.

**Lemma 5.2.** *Consider the circulant  $C_n(J)$  on the vertex set  $\mathbb{Z}/n$ , with jump set  $J = \mathbb{Z}/n \setminus S$ , edge set  $E$ , and  $j \in J$ .*

- (a)  $|0^E| = \prod_{i=1}^m (p_i - 1)$ .
- (b)  $|j + S| = n - \prod_{i=1}^m (p_i - 1)$ .

*Proof.* (a) With the notation of §5.1, we take subset complements as appropriate in the isomorphic sets  $\mathbb{Z}/n$  or  $\bigoplus_{i=1}^m \mathbb{Z}/p_i$ . Using (5.7), we have

$$(5.10) \quad |0^E| = |J| = |\bar{S}| = |\bar{S}'| = \left| \bigcup_{k=1}^m S_k \right| = \left| \bigcap_{k=1}^m \bar{S}_k \right| = \prod_{i=1}^m (p_i - 1)$$

as required.

(b) By (5.10), we have

$$|j + S| = |S| = |\mathbb{Z}/n| - |\bar{S}| = n - \prod_{i=1}^m (p_i - 1)$$

as required.  $\square$

**Lemma 5.3.** *The cardinality of  $S \cap (j + S)$  is given by*

$$(5.11) \quad |S \cap (j + S)| = \sum_{l=2}^m (-1)^l (2^l - 2) \sum_{1 \leq i_1 < \dots < i_l \leq m} \frac{n}{p_{i_1} \cdots p_{i_l}}.$$

*Proof.* We compute the cardinality of  $S' \cap T'$  instead of  $S \cap (j + S)$ . Note that  $S'_k \cap T'_k = \emptyset$  for each  $k$ , and  $S'_k \cap T'_l = \{(s_1, \dots, s_m) \mid s_k = 0, s_l = j\}$  for  $k \neq l$ . Thus

$$(5.12) \quad S' \cap T' = \{(s_1, \dots, s_m) \mid \exists 1 \leq k, l \leq m. s_k = 0 \text{ and } s_l = j\}.$$

We use the inclusion-exclusion principle, step by step, to compute the cardinality of  $S' \cap T'$ .

We start by counting the number of  $m$ -tuples containing at least one 0-component and one  $j_l$ -component (“ $j$ -component” for short) in  $S' \cap T'$ . One location for 0 and one location for  $j$  from an  $m$ -tuple can be selected in  $\binom{m}{1} \binom{m-1}{1}$  ways. Thus the number of such elements in  $S' \cap T'$  is given by

$$\binom{m}{1} \binom{m-1}{1} \sum_{1 \leq i_1, i_2 \leq m} \frac{n}{p_{i_1} p_{i_2}} = \frac{\binom{m}{1} \binom{m-1}{1}}{\binom{m}{2}} \sum_{1 \leq i_1 < i_2 \leq m} \frac{n}{p_{i_1} p_{i_2}}.$$

However, this number exceeds the cardinality of  $S' \cap T'$ , since each  $m$ -tuple containing one 0-component and two  $j$ -components, or two 0-components and one  $j$ -component, is counted twice. In fact, two locations for 0 and one location for  $j$ , or one location for 0 and two locations for  $j$ , within an  $m$ -tuple, can be selected in  $\binom{n}{2} \binom{n-2}{1} + \binom{n}{1} \binom{n-1}{2}$  ways. Therefore, we subtract the number

$$\frac{\binom{m}{2} \binom{m-2}{1} + \binom{m}{1} \binom{m-1}{2}}{\binom{m}{3}} \sum_{1 \leq i_1 < i_2 < i_3 \leq m} \frac{n}{p_{i_1} p_{i_2} p_{i_3}}$$

from our initial count. But then, we obtain a count which is smaller than the actual cardinality of  $S' \cap T'$ , since each  $m$ -tuple containing three 0 components and one  $j$ -component, two 0-components and two  $j$ -components, or one 0-component and three  $j$ -components, is then subtracted twice. To correct, we add their number to the count. In fact, whenever we select a total of  $l$  locations for 0 and  $j$ , we add the number

$$(5.13) \quad \frac{(-1)^l \sum_{k=1}^{l-1} \binom{m}{k} \binom{m-k}{l-k}}{\binom{m}{l}} \sum_{1 \leq i_1 < \dots < i_l \leq m} \frac{n}{p_{i_1} \cdots p_{i_l}}$$

to the count. Since  $m$  is finite, the recursive procedure stops after  $m - 1$  steps. By Lemma 5.1, we have

$$(5.14) \quad (-1)^l \sum_{k=1}^{l-1} \binom{m}{l} \binom{m-k}{l-k} / \binom{m}{l} = (-1)^l (2^l - 2).$$

Thus, using (5.14) to simplify the summands (5.13), the formula (5.11) is obtained.  $\square$

**Lemma 5.4.** *Consider the circulant  $C_n(J)$  on the vertex set  $\mathbb{Z}/n$ , with jump set  $J = \mathbb{Z}/n \setminus S$ , edge set  $E$ , and  $j \in J$ .*

(a) *The neighbor set of 0 is*

$$(5.15) \quad 0^E = (0^E \cap j^E) \cup ((j + S) \setminus S).$$

(b) *The neighbor set of  $j$  is*

$$(5.16) \quad j^E = (0^E \cap j^E) \cup (S \setminus (j + S)).$$

(c) *The cardinality of  $0^E \cap j^E$  is given by*

$$|0^E \cap j^E| = \prod_{i=1}^m (p_i - 2).$$

*Proof.* (a) Note that  $j^E = j + J = \mathbb{Z}/n \setminus (j + S)$ . Since

$$\begin{aligned} 0^E \setminus j^E &= (\mathbb{Z}/n \setminus S) \setminus (\mathbb{Z}/n \setminus (j + S)) \\ &= \overline{S} \setminus \overline{j + S} = \overline{S} \cap (j + S) = (j + S) \setminus S, \end{aligned}$$

we have  $0^E = (0^E \cap j^E) \cup (0^E \setminus j^E) = (0^E \cap j^E) \cup ((j + S) \setminus S)$ .

(b) Since

$$\begin{aligned} j^E \setminus 0^E &= (\mathbb{Z}/n \setminus (j + S)) \setminus (\mathbb{Z}/n \setminus S) \\ &= \overline{j + S} \setminus \overline{S} = (\overline{j + S}) \cap S = S \setminus (j + S), \end{aligned}$$

we have  $j^E = (j^E \cap 0^E) \cup (j^E \setminus 0^E) = (0^E \cap j^E) \cup (S \setminus (j + S))$ .

(c) By Lemmas 5.2 and 5.3, it follows that

$$|0^E \cap j^E| = |0^E| - |j + S| + |S \cap (j + S)|$$

$$\begin{aligned}
&= \prod_{i=1}^m (p_i - 1) - \left( n - \prod_{i=1}^m (p_i - 1) \right) \\
&\quad + \sum_{l=2}^m (-1)^l (2^l - 2) \sum_{1 \leq i_1 < \dots < i_l \leq m} \frac{n}{p_{i_1} \cdots p_{i_l}} \\
&= -n + 2 \prod_{i=1}^m (p_i - 1) + \sum_{l=2}^m (-1)^l (2^l - 2) \sum_{1 \leq i_1 < \dots < i_l \leq m} \frac{n}{p_{i_1} \cdots p_{i_l}} \\
&= n + \sum_{l=1}^m (-1)^l 2^l \sum_{1 \leq i_1 < \dots < i_l \leq m} \frac{n}{p_{i_1} \cdots p_{i_l}} \\
&\quad + \sum_{l=2}^m (-1)^l (2^l - 2) \sum_{1 \leq i_1 < \dots < i_l \leq m} \frac{n}{p_{i_1} \cdots p_{i_l}} \\
&= n - 2 \sum_{1 \leq i_1 \leq m} \frac{n}{p_{i_1}} + \sum_{l=2}^m (-1)^l 2^l \sum_{1 \leq i_1 < \dots < i_l \leq m} \frac{n}{p_{i_1} \cdots p_{i_l}} \\
&= n + \sum_{l=1}^m (-1)^l 2^l \sum_{1 \leq i_1 < \dots < i_l \leq m} \frac{n}{p_{i_1} \cdots p_{i_l}} = \prod_{1 \leq i \leq m} (p_i - 2),
\end{aligned}$$

as required.  $\square$

### 5.3. A perfect matching

The goal of this section is to prove the following.

**Lemma 5.5.** *For each element  $j \in J = \mathbb{Z}/n \setminus S$ , there is a perfect matching between the sets*

$$S \setminus (\{0\} \cup (j + S)) \quad \text{and} \quad (j + S) \setminus (\{j\} \cup S)$$

in the circulant  $C_n(J)$ .

The proof of Lemma 5.5 relies on the regularity properties of certain bipartite graphs, induced subgraphs of  $C_n(J')$ , constructed in a chain of definitions. Specifically, for each subset

$$(5.17) \quad \emptyset \subset I \subset \{1, \dots, m\}$$

(with  $0 < |I| < m$ ), a bipartite graph  $G_I$  will be presented.

As before, set  $j_k = \mathbb{Z}/p_k + j \in \mathbb{Z}/p_k \setminus \{0\}$  for  $1 \leq k \leq m$ . Consider the respective isomorphic images  $S'$  and  $T'$  of  $S$  and  $j + S$  given in (5.7). Taking (5.12) into account, one then has

$$S' \setminus T' = \{(s_1, \dots, s_m) \mid \exists 1 \leq k \leq m. s_k = 0 \text{ and } \forall 1 \leq l \leq m, s_l \neq j_l\}$$

and

$$T' \setminus S' = \{(t_1, \dots, t_m) \mid \exists 1 \leq k \leq m. t_k = j_k \text{ and } \forall 1 \leq l \leq m, t_l \neq 0\}$$

as the respective isomorphic images of  $S \setminus (j + S)$  and  $(j + S) \setminus S$ .

**Definition 5.6.** For a subset (5.17), define the subset

$$S_I = \{(s_1, \dots, s_m) \mid \forall i \notin I, s_i \notin \{0, j_i\} \text{ and } \forall i \in I, s_i = 0\}$$

of  $S' \setminus T'$ . Such a subset is described as a *source part*.

**Lemma 5.7.** (a) Consider  $I_1 \neq I_2 \subset \{1, \dots, m\}$ . Then  $S_{I_1} \cap S_{I_2} = \emptyset$ .

(b) The source parts provide a disjoint union decomposition

$$S' \setminus (\{(0, \dots, 0)\} \cup T') = \bigsqcup_{\emptyset \subset I \subset \{1, \dots, m\}} S_I$$

of the isomorphic image of  $S \setminus (\{0\} \cup (j + S))$ .

**Definition 5.8.** For a subset (5.17), define the subset

$$T_I = \{(t_1, \dots, t_m) \mid \forall i \notin I, t_i \notin \{0, j_i\} \text{ and } \forall i \in I, t_i = j_i\}$$

of  $T' \setminus S'$ . Such a subset is described as a *target part*.

**Lemma 5.9.** (a) Consider  $I \subset \{1, \dots, m\}$ . Then  $S_I \cap T_I = \emptyset$ .

(b) Consider  $I_1 \neq I_2 \subset \{1, \dots, m\}$ . Then  $T_{I_1} \cap T_{I_2} = \emptyset$ .

(c) The target parts provide a disjoint union decomposition

$$T' \setminus (\{(j_1, \dots, j_m)\} \cup S') = \bigsqcup_{\emptyset \subset I \subset \{1, \dots, m\}} T_I$$

of the isomorphic image of  $(j + S) \setminus (\{j\} \cup (S))$ .

**Definition 5.10.** Consider a subset (5.17).

(a) Define  $V_I = S_I \sqcup T_I$ .

(b) Define

$$E_I = \left\{ \left\{ (s_1, \dots, s_m), (t_1, \dots, t_m) \right\} \mid \right. \\ \left. (s_1, \dots, s_m) \in S_I, (t_1, \dots, t_m) \in T_I, \forall 1 \leq j \leq m, s_j \neq t_j \right\}.$$

(c) Define  $G_I$  to be the bipartite graph with vertex set  $V_I$  and edge set  $E_I$ .

**Lemma 5.11.** For each non-empty, proper subset  $I$  of  $\{1, \dots, m\}$ , the bipartite graph  $G_I$  is the subgraph of  $C_n(J')$  induced on  $S_I \cup T_I$ .

*Proof.* Consider the form (5.4) of  $J'$ . □

**Lemma 5.12.** Consider a subset (5.17). Write  $d = \prod_{i \notin I} (p_i - 3)$ .

(a) One has

$$|(s_1, \dots, s_m)^{E_I}| = d$$

for  $(s_1, \dots, s_m) \in S_I$ .

(b) One has

$$|(t_1, \dots, t_m)^{E_I}| = d$$

for  $(t_1, \dots, t_m) \in T_I$ .

(c) The bipartite graph  $G_I$  is  $d$ -regular.

(d) *The graph  $G_I$  has a perfect matching.*

*Proof.* (a) For  $(s_1, \dots, s_m) \in S_I$ , consider a neighbor  $(t_1, \dots, t_m) \in T_I$ . For  $i \in I$ , one has  $s_i = 0$  and  $t_i = j_i$ , so there are no choices for these coordinates of  $(t_1, \dots, t_m)$ . On the other hand, for  $i \notin I$ , the coordinate  $t_i$  may be chosen from any one of the  $p_i - 3$  elements of  $\mathbb{Z}/p_i \setminus \{0, j_i, s_i\}$ .

(b) is proved similarly to (a).

(c) follows from (a) and (b).

(d) follows from (c).  $\square$

The proof of Lemma 5.5 now follows from Lemmas 5.7, 5.9 and 5.12: Take the perfect matching between the sets

$$S \setminus (\{0\} \cup (j + S)) \quad \text{and} \quad (j + S) \setminus (\{j\} \cup S)$$

in the circulant  $C_n(J)$  isomorphic to the perfect matching between the sets

$$S' \setminus (\{(0, \dots, 0)\} \cup T') \quad \text{and} \quad T' \setminus (\{(j_1, \dots, j_m)\} \cup S')$$

in the circulant  $C_n(J')$  that is given by the disjoint union of the perfect matchings from Lemma 5.12(d) in the induced subgraphs  $G_I$  for all  $\emptyset \subset I \subset \{1, \dots, m\}$ .

#### 5.4. The Ricci curvature

**Theorem 5.13.** *The circulant graph  $C_n(J)$  with jump set  $J = \mathbb{Z}_n \setminus S$  has constant Ricci curvature*

$$(5.18) \quad \kappa = \frac{2 + \prod_{i=1}^m (p_i - 2)}{\prod_{i=1}^m (p_i - 1)}.$$

*Proof.* Since  $C_n(J)$  has rotational symmetry, it suffices to compute  $\kappa(0, j)$  for an arbitrary element  $j$  of  $J$ . By (5.15), we have

$$0^E = (0^E \cap j^E) \cup ((j + S) \setminus S),$$

and by (5.16), we have

$$j^E = (0^E \cap j^E) \cup (S \setminus (j + S)).$$

By Lemma 5.5, there is a matching between the sets  $(j + S) \setminus (\{j\} \cup S)$  and  $S \setminus (\{0\} \cup (j + S))$ . Thus  $C_n(J)$  satisfies the Matching Condition. Since  $|0^E \cap j^E| = \prod_{i=1}^m (p_i - 2)$  by Lemma 5.4(c), Theorem 2.2 yields

$$\kappa = \frac{2 + |0^E \cap j^E|}{\delta} = \frac{2 + \prod_{i=1}^m (p_i - 2)}{\prod_{i=1}^m (p_i - 1)},$$

as required.  $\square$

*Remark 5.14.* We conjecture that the condition  $p_i > 3$  imposed on the factorization  $n = \prod_{i=1}^m p_i$  may be relaxed to  $p_i > 2$ . For example, a matching can be constructed for  $m = 2$ ,  $p_1 = 3$  and  $3 \nmid p_2$  so that the formula (5.18) remains valid. However, for general multiples  $n$  of 3, a different matching construction is needed, since Lemma 5.12(d) may fail in this case.



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