

## NULLITY OF THE LEVI-FORM AND THE ASSOCIATED SUBVARIETIES FOR PSEUDO-CONVEX CR STRUCTURES OF HYPERSURFACE TYPE

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ABSTRACT. Let  $M^{2n+1}$ ,  $n \geq 1$ , be a smooth manifold with a pseudo-convex integrable CR structure of hypersurface type. We consider a sequence of CR invariant subsets  $M = \mathcal{S}_0 \supset \mathcal{S}_1 \supset \cdots \supset \mathcal{S}_n$ , where  $\mathcal{S}_q$  is the set of points where the Levi-form has nullity  $\geq q$ . We prove that  $\mathcal{S}_q$ 's are locally given as common zero sets of the coefficients  $A_j$ ,  $j = 0, 1, \dots, q-1$ , of the characteristic polynomial of the Levi-form. Some sufficient conditions for local existence of complex submanifolds are presented in terms of the coefficients  $A_j$ .

### Introduction

Let  $M$ , or to specify the dimension  $M^{2n+1}$ ,  $n \geq 1$ , be a smooth ( $C^\infty$ ) manifold equipped with a pseudo-convex integrable CR structure  $(H(M), J)$  of hypersurface type. We consider a sequence of CR-invariant subsets

$$M = \mathcal{S}_0 \supset \mathcal{S}_1 \supset \cdots \supset \mathcal{S}_n,$$

where  $\mathcal{S}_q$  is the set of those points where the Levi-form has nullity greater than or equal to  $q$ . In §2 we prove that  $\mathcal{S}_q$  is locally given as a common zero set of the coefficients of the characteristic polynomial of the matrix representation of the Levi-form. A complex submanifold of  $M$  of complex dimension  $q$ ,  $1 \leq q \leq n$ , is a local embedding  $f$  of an open subset  $\mathcal{O} \subset \mathbb{C}^q$  that satisfies

$$(0.1) \quad \begin{aligned} df(T_x(\mathcal{O})) &\subset H_{f(x)}(M), \quad \forall x \in \mathcal{O} \quad \text{and} \\ J \circ df &= df \circ J_{st}, \end{aligned}$$

where  $J_{st}$  is the standard complex structure tensor of  $\mathbb{C}^q$ . Solving (0.1) for  $f$  is an over-determined problem and there is no solution generically. If such  $f$

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exists, then its image must be contained in  $\mathcal{S}_q$ . Hence a necessary condition for the existence of a complex manifold of dimension  $q$  is that  $\mathcal{S}_q$  has real dimension  $\geq 2q$ . In §3 we present some sufficient conditions for the existence of real defining functions for a complex submanifold of dimension  $q$ . These are conditions on the coefficients of the characteristic polynomial of the Levi-form that define  $\mathcal{S}_q$ . In several complex variables the function theory of a domain often depends on the geometry of the boundary, for which we refer the readers to [3] and [8]. In particular, at a boundary point  $x$  of a pseudoconvex domain with smooth real-analytic boundary the subelliptic estimate for the  $\bar{\partial}$ -Neumann problem holds for  $(p, q)$ -forms if there is no germ at  $x$  of a complex variety of dimension greater than or equal to  $q$  in the boundary (see [5] and [9]). For real hypersurfaces in general, without assuming pseudoconvexity, the existence of a germ of complex hypersurface is an obstruction to the extension of holomorphic functions of one side of  $M$  to the other side (see [11]). The existence of complex submanifolds in real hypersurfaces has been studied in [1], [6] and [7] without assuming the pseudoconvexity. The present paper makes essential use of the fact that the eigen-values of the Levi-form are non-negative for pseudoconvex CR manifolds. The coefficients  $A_j$  of the characteristic polynomial of the Levi-form depends on the choice of local basis of  $(1, 0)$  vectors, they are invariant only under unitary change of basis. However,  $\mathcal{S}_q$  the common zero set of  $A_j$ ,  $j = 0, 1, \dots, q - 1$ , is a CR invariant, which is interesting from the viewpoint of the invariant theory. We work in  $C^\infty$  category. All the manifolds in this paper shall be assumed to be connected and oriented, or to be a connected submanifold sitting in a small neighborhood  $U \subset M$ . The authors thank Dmitri Zaitsev for the discussions we had while he was visiting KIAS. We thank Sungyeon Kim for the valuable comments and suggestions that she gave us.

### 1. Pseudo-convexity and the nullity of the Levi-form

Let  $M$  be a smooth ( $C^\infty$ ) manifold of dimension  $2n + 1$ ,  $n \geq 1$ . A CR structure of hypersurface type, or simply a CR structure, on  $M$  is a pair  $(H(M), J)$ , where  $H(M)$  is a smooth subbundle of codimension 1 of the tangent bundle  $T(M)$  and  $J$  is an almost complex structure on  $H(M)$ , that is,  $J : H(M) \rightarrow H(M)$  is a smooth bundle isomorphism for which

$$\begin{array}{ccc} H(M) & \xrightarrow{J} & H(M) \\ & \searrow & \swarrow \\ & M & \end{array}$$

commutes and

$$J^2 = -I_{2n}.$$

Then the complexification  $\mathbb{C} \otimes H(M)$  has a decomposition

$$\mathbb{C} \otimes H(M) = H^{1,0}(M) \oplus H^{0,1}(M),$$

where  $H^{1,0}(M)$  (resp.  $H^{0,1}(M)$ ) is the set of eigen-vectors of  $J$  corresponding to the eigen-value  $\sqrt{-1}$  (resp.  $-\sqrt{-1}$ ). Locally, there exist real vector fields  $X_1, \dots, X_n$  so that  $X_j, JX_j, j = 1, \dots, n$ , span  $H(M)$ . Then

$$(1.1) \quad L_j := X_j - \sqrt{-1}JX_j, \quad j = 1, \dots, n$$

span  $H^{1,0}(M)$  and

$$(1.2) \quad \bar{L}_j := X_j + \sqrt{-1}JX_j, \quad j = 1, \dots, n$$

span  $H^{0,1}(M)$ , respectively. A diffeomorphism  $F$  of  $M$  onto a CR manifold of the same dimension  $\widetilde{M}$  is called a CR mapping if  $F$  preserves the CR structure, namely,  $dF$  maps  $H(M)$  onto  $H(\widetilde{M})$  and

$$(1.3) \quad dF \circ J = J \circ dF$$

on  $H(M)$ . The CR structure bundle  $H(M)$  is locally given by a non-vanishing real 1-form that annihilates  $H(M)$ : any point  $x \in M$  has a neighborhood  $U$  on which there is a smooth non-vanishing real 1-form  $\theta$  such that

$$(1.4) \quad H(U) = \theta^\perp := \{V \in T(U) : \theta(V) = 0\}.$$

We fix  $U$  for our local arguments assuming the local bases (1.1) and (1.2) are defined on  $U$ . By  $\Omega^0(U) = C^\infty(U)$  we denote the ring of smooth complex valued functions on  $U$  and by  $\Omega^p(U)$  the module over  $\Omega^0(U)$  of smooth  $p$ -forms and by

$$\Omega^*(U) = \bigoplus_{p=0}^{2n+1} \Omega^p(U)$$

the exterior algebra of smooth differential forms on  $U$  with complex coefficients. A subalgebra  $\mathcal{I} \subset \Omega^*(U)$  is called an ideal if the following conditions hold:

- i)  $\mathcal{I} \wedge \Omega^*(U) \subset \mathcal{I}$ ,
- ii) if  $\phi = \sum_{p=0}^{2n+1} \phi_p \in \mathcal{I}$ ,  $\phi_p \in \Omega^p(U)$ , then each  $\phi_p \in \mathcal{I}$  (homogeneity).

Because of the homogeneity condition  $\mathcal{I}$  is two-sided, that is,

$$\Omega^*(U) \wedge \mathcal{I} \subset \mathcal{I}.$$

We consider in this paper only those ideals that are generated by 0-forms (functions) and 1-forms: for a finite set of smooth functions  $\mathbf{r} = (r^1, \dots, r^d)$  and a finite set of smooth 1-forms  $\Theta = (\theta^1, \dots, \theta^s)$  let  $\mathcal{I}(\mathbf{r}, \Theta)$  be the ideal generated by  $\mathbf{r}$  and  $\Theta$ . For two elements  $\phi, \psi \in \Omega^*(U)$  we write

$$\phi \equiv \psi, \quad \text{mod } (\mathbf{r}, \Theta)$$

or

$$\phi \stackrel{(\mathbf{r}, \Theta)}{\equiv} \psi,$$

if and only if

$$\phi - \psi \in \mathcal{I}(\mathbf{r}, \Theta).$$

The system  $\mathbf{r}$  is said to be *non-degenerate* if  $dr^1 \wedge \dots \wedge dr^d \neq 0$ . For other definitions and notations concerning the exterior algebra we refer the readers

to [2]. The CR structure bundle  $H(M)$  is *integrable* in the sense of Frobenius if

$$(1.5) \quad d\theta \equiv 0, \quad \text{mod } (\theta),$$

where  $\theta$  is as in (1.4). By the Frobenius theorem if (1.5) holds  $M$  is locally foliated by integral manifolds of  $H(M)$ . If this is the case  $J$  is an almost complex structure on each leaf.

**Definition 1.1.** The *torsion tensor* of the CR structure is

$$(1.6) \quad d\theta, \quad \text{mod } (\theta).$$

We regard (1.6) as an element of the quotient module  $\Omega^2(U)/(\theta)$ , where  $(\theta)$  is the set of all 2-forms

$$\{\theta \wedge \omega : \omega \in \Omega^1(U)\}.$$

From the viewpoint of (1.5), the torsion tensor is the obstruction to the local integrability of  $H(M)$ . The torsion tensor of the CR structure defines a hermitian form  $\mathcal{L}$ , which is called the *Levi-form*, on  $H^{1,0}(U)$  by

$$(1.7) \quad \mathcal{L}(L, L') := \frac{1}{\sqrt{-1}} d\theta(L \wedge \bar{L}').$$

If  $\mathcal{L}$  is semi-definite at  $x$ , then the CR structure is said to be *pseudo-convex* at  $x$ . If  $\mathcal{L}$  is semi-definite at every point of  $M$ , then the CR structure is said to be pseudo-convex.

**Definition 1.2.** At a point  $x \in M$  the *null space* of  $\mathcal{L}$  is

$$(1.8) \quad \mathcal{N}_x := \{L \in H_x^{1,0}(M) : \mathcal{L}(L, L) = 0\}.$$

Notice that if  $M$  is pseudo-convex, then  $\mathcal{N}_x$  is a subspace. The complex dimension of  $\mathcal{N}_x$  is called the *nullity* of the Levi-form at  $x$ .

If  $F$  is a CR mapping of  $M$  onto a CR manifold of same dimension  $\widetilde{M}$ , then by (1.3)  $dF$  preserves the type of vectors, that is,  $dF$  extends to the isomorphism

$$\begin{aligned} H^{1,0}(M) &\xrightarrow{dF} H^{1,0}(\widetilde{M}) \\ H^{0,1}(M) &\xrightarrow{dF} H^{0,1}(\widetilde{M}), \end{aligned}$$

Thus we have:

**Proposition 1.3.** *Let  $(H(M), J)$  be a smooth pseudo-convex CR structure on a smooth manifold  $M^{2n+1}$ ,  $n \geq 1$ . Then the notions of the null space and the nullity of the Levi-form defined locally by (1.7)-(1.8) do not depend on the choice of  $\theta$ , and therefore, well defined globally. The nullity of the Levi-form is invariant under CR mappings.*

Now for each  $q = 0, 1, \dots, n$ , let  $\mathcal{S}_q$  be the set of points where the nullity of the Levi-form is greater than or equal to  $q$ . Then we have a sequence of CR-invariant subsets

$$M = \mathcal{S}_0 \supset \mathcal{S}_1 \supset \dots \supset \mathcal{S}_n.$$

## 2. Invariant subvarieties of pseudo-convex CR manifolds

Consider the matrix representation of the Levi-form  $\mathcal{L}$  with respect to the basis (1.1): let

$$(2.1) \quad T := [T_{jk}]_{n \times n}, \text{ where } T_{jk} := \mathcal{L}(L_j, L_k).$$

$T$  is a hermitian matrix and assumed to be positive semi-definite.

**Theorem 2.1.** *Suppose that a smooth real manifold  $M^{2n+1}$  admits a pseudo-convex CR structure  $(H(M), J)$ . Let  $T$  be a matrix representation of the Levi-form defined locally as in (2.1) and*

$$\det(T - \lambda I) := \sum_{k=0}^{n-1} A_k (-\lambda)^k + (-\lambda)^n$$

be the characteristic polynomial of  $T$ . Then for each  $q = 1, \dots, n$ ,  $\mathcal{S}_q$  is given as a common zero set of  $A_j$  for all  $j$  with  $0 \leq j \leq q - 1$ .

*Proof.* We fix a point  $x \in M$  and a neighborhood  $U$  of  $x$ . By a unitary change of basis  $\mathcal{L}$  can be diagonalized. The pseudo-convexity implies that the diagonal elements are non-negative, and the number of zeros in the diagonal is the nullity of the Levi-form. To be precise, define a hermitian metric on  $H_x^{1,0}(M)$  by declaring  $L_j$ 's as in (1.1) are orthonormal. By the spectral theorem, there exists an orthonormal basis  $Q_1, \dots, Q_n$  that are eigen-vectors of  $T$ . Let  $d_j$  be the eigen-value corresponding to  $Q_j$ . Setting  $Q_j = \sum_{k=1}^n Q_j^k L_k$  and  $Q = [Q_j^k]$  we have

$$QT(x)Q^* = QT(x)Q^{-1} = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}, \quad d_j \geq 0,$$

so that

$$\mathcal{L}(Q_j, Q_k) = \begin{cases} d_j, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

Suppose  $x \in \mathcal{S}_1$ . Then the Levi-form  $\mathcal{L}(x)$  has a non-trivial null vector. Setting it by

$$V = \sum_{j=1}^n b^j Q_j,$$

we have

$$(2.2) \quad \begin{aligned} 0 &= \mathcal{L}(V, V) \\ &= \sum_{j=1}^n |b^j|^2 d_j. \end{aligned}$$

Since  $d_j \geq 0$ , (2.2) implies that at least one of  $d_j$ 's is zero. Therefore,

$$\mathcal{A}_0(x) = d_1 \cdots d_n = 0.$$

Now suppose  $x \in \mathcal{S}_q$ . Then the Levi-form  $\mathcal{L}(x)$  has at least  $q$  independent null vectors. Setting them by

$$V_\lambda = \sum_{j=1}^n b_\lambda^j Q_j,$$

we have

$$(2.3) \quad \begin{aligned} 0 &= \mathcal{L}(V_\lambda, V_\lambda) \\ &= \sum_{j=1}^n |b_\lambda^j|^2 d_j, \quad \lambda = 1, \dots, q. \end{aligned}$$

We restate (2.3) in matrices as

$$(2.4) \quad \underbrace{\begin{bmatrix} |b_1^1|^2 & \cdots & |b_1^n|^2 \\ \vdots & & \vdots \\ |b_q^1|^2 & \cdots & |b_q^n|^2 \end{bmatrix}}_B \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since  $V_\lambda$ 's are independent

$$(2.5) \quad \beta := \begin{bmatrix} b_1^1 & \cdots & b_1^n \\ \vdots & & \vdots \\ b_q^1 & \cdots & b_q^n \end{bmatrix}$$

has rank  $q$ , so that  $\beta$  has  $q$  independent columns, which implies that  $q$  columns of  $B$  are non-zeros. Since  $d_j \geq 0$ , for all  $j = 1, \dots, n$ , (2.4) implies that at least  $q$  of  $d_j$ 's are zeros. For  $k = 0, 1, \dots, q-1$ ,  $A_k(x)$  is the symmetric polynomial of degree  $n-k$  in  $d_1, \dots, d_n$ , therefore,  $\mathcal{A}_k(x) = 0$ .

Conversely, if  $A_0(x) = \cdots = A_{q-1}(x) = 0$ , then there are at least  $q$  zeros in  $\{d_1, \dots, d_n\}$ , say

$$d_1 = \cdots = d_q = 0.$$

Then  $Q_1, \dots, Q_q$  are null vectors of  $\mathcal{L}(x)$ . □

### 3. Integrable CR structures

The CR structure  $(H(M), J)$  is said to be *integrable* if the module of smooth sections of  $H^{1,0}(M)$ , denoted by  $\Gamma(H^{1,0}(M))$ , is closed under the Lie bracket:

$$(3.1) \quad [L, L'] \in \Gamma(H^{1,0}(M)), \quad \forall L, L' \in \Gamma(H^{1,0}(M)).$$

(3.1) is equivalent to

$$(3.2) \quad [JX, JY] = [X, Y] + J[JX, Y] + J[X, JY], \quad \forall X, Y \in \Gamma(H(M)).$$

In this section  $M$  is assumed to be a CR manifold with an integrable CR structure. Then if a submanifold  $\mathcal{S}$  of  $M$  is  $J$ -invariant, namely, if  $JT(\mathcal{S}) \subset T(\mathcal{S})$ , then  $\mathcal{S}$  is a complex manifold with the complex structure  $J$  by the following

**Theorem 3.1** (Newlander-Nirenberg [10]). *Let  $\mathcal{S}$  be a smooth real manifold of dimension  $2m$  with a smooth almost complex structure  $J$  that is integrable in the sense that any vector fields  $X$  and  $Y$  defined locally on an open neighborhood  $U \subset \mathcal{S}$  satisfy (3.2). Then  $\mathcal{S}$  is covered by coordinate patches with complex coordinates in which the coordinates in overlapping patches are related by holomorphic transformations, so that for a local coordinate system  $(z_1, \dots, z_m)$ ,  $z_j = x_j + \sqrt{-1}y_j$ ,*

$$\begin{aligned} J\left(\frac{\partial}{\partial x_j}\right) &= \frac{\partial}{\partial y_j}, \\ J\left(\frac{\partial}{\partial y_j}\right) &= -\frac{\partial}{\partial x_j}, \quad j = 1, \dots, m. \end{aligned}$$

For the definition of the tangential Cauchy-Riemann operator  $\bar{\partial}_b$  we refer the readers to [4]. For a function  $f$ ,  $\bar{\partial}_b f$  is a section of  $H^{*0,1}(M) := (H^{0,1}(M))^*$  defined by

$$(\bar{\partial}_b f)(V) = df(V) \quad \text{for any } V \in \Gamma(H^{0,1}(M)).$$

Let  $\pi_{1,0}$ ,  $\pi_{0,1}$  and  $\pi_t$  be the projections of  $\mathbb{C} \otimes T^*(M)$  onto each component of

$$\mathbb{C} \otimes T^*(M) = H^{*1,0}(M) \oplus H^{*0,1}(M) \oplus \langle \theta \rangle.$$

Then

$$\begin{aligned} \bar{\partial}_b f &:= \pi_{0,1} df, \\ \partial_b f &:= \pi_{1,0} df. \end{aligned}$$

**Theorem 3.2.** *Suppose that  $M^{2n+1}$  is a smooth manifold with integrable CR structure  $(H(M), J)$ . For an integer  $k$ ,  $0 \leq k \leq n$ , let  $\rho_1, \dots, \rho_{2k+1}$  be a non-degenerate system of real-valued functions defined locally on an open subset  $U \subset M$ . Let  $\mathcal{S}$  be the common zero set of  $\rho_j$ ,  $j = 1, \dots, 2k+1$ . Then  $\mathcal{S}$  is a complex manifold of complex dimension  $n - k$  with  $J$  as its complex structure if and only if the linear span of  $\{\bar{\partial}_b \rho_1, \dots, \bar{\partial}_b \rho_{2k+1}\}$  has constant rank  $k$  on  $\mathcal{S}$ .*

*Proof.* Suppose that  $\mathcal{S}$  is a complex manifold of complex dimension  $n - k$ . Then

$$(3.3) \quad \mathbb{C} \otimes T(\mathcal{S}) = T^{1,0}(\mathcal{S}) \oplus T^{0,1}(\mathcal{S}),$$

where  $T^{1,0}(\mathcal{S}) := H^{1,0}(M) \cap \mathbb{C} \otimes T(\mathcal{S})$  and so forth. Each of the direct summand of the right-hand side of (3.3) has rank  $n - k$ . For a local section  $V$  of  $H^{0,1}(M)$  it is obvious that  $V \in T^{0,1}(\mathcal{S})$  if and only if

$$(3.4) \quad d\rho_j(V) = 0, \quad j = 1, \dots, 2k+1.$$

Since  $V \in H^{0,1}(M)$ , (3.4) is equivalent to

$$\bar{\partial}_b \rho_j(V) = 0, \quad j = 1, \dots, 2k+1.$$

We note that there are  $n - k$  independent vectors  $V_1, \dots, V_{n-k}$  in  $T^{0,1}(\mathcal{S})$ . Therefore, the linear span

$$(3.5) \quad \langle \bar{\partial}_b \rho_1, \dots, \bar{\partial}_b \rho_{2k+1} \rangle(x) \subset H_x^{*0,1}(M), \quad \forall x \in \mathcal{S}$$

has rank  $n - (n - k) = k$ .

Conversely, suppose that (3.5) has rank  $k$  at  $x \in \mathcal{S}$ . Assuming the first  $k$  elements of  $\{\bar{\partial}_b \rho_1, \dots, \bar{\partial}_b \rho_{2k+1}\}$  are independent, we choose 1-forms  $\omega^1, \dots, \omega^{n-k} \in H^{*0,1}(M)$  on a neighborhood  $U \subset M$  of  $x$  so that  $H^{*0,1}(M)$  is spanned by

$$\{\bar{\partial}_b \rho_1, \dots, \bar{\partial}_b \rho_k, \omega^1, \dots, \omega^{n-k}\}.$$

Then

$$(3.6) \quad \underbrace{\partial_b \rho_1, \dots, \partial_b \rho_k, \bar{\omega}^1, \dots, \bar{\omega}^{n-k}}_{H^{*1,0}(M)}, \underbrace{\bar{\partial}_b \rho_1, \dots, \bar{\partial}_b \rho_k, \omega^1, \dots, \omega^{n-k}}_{H^{*0,1}(M)}, \theta$$

is a local basis of the whole cotangent bundle  $\mathbb{C} \otimes T^*(M)$ . Let

$$(3.7) \quad \underbrace{L_1, \dots, L_k, \bar{V}_1, \dots, \bar{V}_{n-k}}_{H^{1,0}(M)}, \underbrace{\bar{L}_1, \dots, \bar{L}_k, V_1, \dots, V_{n-k}}_{H^{0,1}(M)}, T$$

be the dual basis of (3.6) for  $\mathbb{C} \otimes T(M)$ . Now consider the bundle of maximal complex subspaces of  $T(\mathcal{S})$  given by

$$H(\mathcal{S}) := T(\mathcal{S}) \cap J T(\mathcal{S}).$$

Let  $\mathcal{V}(\mathcal{S})$  be a sub-bundle of  $T(\mathcal{S})$ , locally defined on the neighborhood  $U \cap \mathcal{S}$  of  $x$ , so that

$$T(\mathcal{S}) = H(\mathcal{S}) \oplus \mathcal{V}(\mathcal{S}).$$

Then the  $J$ -invariance of  $\mathcal{S}$  comes from the dimension count of

$$(3.8) \quad \mathbb{C} \otimes T(\mathcal{S}) = H^{1,0}(\mathcal{S}) \oplus H^{0,1}(\mathcal{S}) \oplus (\mathbb{C} \otimes \mathcal{V}(\mathcal{S})).$$

To count the dimension of  $H^{0,1}(\mathcal{S})$ , let  $V := X + \sqrt{-1}JX \in H^{0,1}(M)$ ,  $X$  is a real vector, be any one of  $V_1, \dots, V_{n-k}$  in (3.7) and  $\rho$  be any one of  $\rho_1, \dots, \rho_{2k+1}$ . We have

$$\begin{aligned} 0 &= \bar{\partial}_b \rho(V) = d\rho(V) \\ &= d\rho(X + \sqrt{-1}JX) \end{aligned}$$

which yields

$$d\rho(X) = 0 \quad \text{and} \quad d\rho(JX) = 0$$

so that  $X$  and  $JX$  are tangent to  $\mathcal{S}$ . Therefore,  $V_1, \dots, V_{n-k}$  are elements of  $H^{0,1}(\mathcal{S})$ . Hence  $H^{0,1}(\mathcal{S})$  has complex dimension  $\geq n - k$ . But each fibre in the left-hand side of (3.8) has complex dimension  $2(n - k)$ . Therefore, the last component of the right-hand side of (3.8) has rank zero, which implies that  $\mathcal{V}(\mathcal{S}) = 0$ . Thus we have

$$T(\mathcal{S}) = H(\mathcal{S}),$$

which means that  $\mathcal{S}$  is  $J$ -invariant and the conclusion follows from Theorem 3.1.  $\square$



If  $(M, H(M), J)$  is pseudo-convex and  $\mathcal{S}$  has real dimension  $\geq 2q$ , then the defining functions  $\rho_j$ 's of Theorem 3.2 can be found from the coefficients and their radicals of the characteristic polynomial of the Levi-form. We adopt from [9] the following.

**Definition 3.3.** Let  $\mathcal{J}$  be a subset of the ring  $C^\infty(x_0)$  of germs at  $x_0 \in M$  of smooth functions. The *real radical* of  $\mathcal{J}$ , denoted by  $\sqrt[\mathbb{R}]{\mathcal{J}}$ , is the set of all  $g \in C^\infty(x_0)$  such that there exist an integer  $m$  and an  $f \in \mathcal{J}$  so that

$$|g|^m \leq |f|$$

on some neighborhood of  $x_0$ .

$\sqrt[\mathbb{R}]{\mathcal{J}}$  is an ideal. We have:

**Corollary 3.4.** Let  $M^{2n+1}$  be a smooth CR manifold with pseudo-convex integrable CR structure  $(H(M), J)$ . For an integer  $q$ ,  $0 \leq q \leq n$ , let  $\mathcal{S}_q$  be the set of points of  $M$  where the Levi-form has nullity  $\geq q$ . Suppose that the real dimension of  $\mathcal{S}_q$  is greater than or equal to  $2q$  and that there is a non-degenerate set of real-valued functions  $\rho_1, \dots, \rho_{2(n-q)+1}$  that generates  $\sqrt[\mathbb{R}]{\{A_0, \dots, A_{q-1}\}}$  and satisfies

$$\text{rank } \langle \bar{\partial}_b \rho_1, \dots, \bar{\partial}_b \rho_{2(n-q)+1} \rangle = n - k$$

on the common zero set  $\mathcal{S}$  of  $\rho_j$ 's. Then  $\mathcal{S}$  is a complex manifold of complex dimension  $q$ .

*Proof.* On  $\mathcal{S}_q$ ,  $A_j = 0$  for all  $j = 0, 1, \dots, q-1$ . Since  $\mathcal{S}$  is the common zero set of  $\rho_j$ 's we have  $\mathcal{S} \subset \mathcal{S}_q$ . The conclusion follows from Theorem 3.2.  $\square$

*Remark 3.5.* In analytic ( $C^\omega$ ) category we check a finite set of non-degenerate functions  $\vec{\rho} := (\rho_1, \rho_2, \dots, \rho_{2(n-q)+1})$  including the generators of

$$\sqrt[\mathbb{R}]{\{A_0, \dots, A_{q-1}\}}$$

and determine the existence of a complex manifold by showing that

$$\begin{aligned} \bar{\partial}_b \rho_1 \wedge \cdots \wedge \bar{\partial}_b \rho_{n-q} &\neq 0, \\ \bar{\partial}_b \rho_1 \wedge \cdots \wedge \bar{\partial}_b \rho_{n-q} \wedge \bar{\partial}_b \rho_\ell &\equiv 0 \pmod{(\vec{\rho})}, \quad \forall \ell. \end{aligned}$$

## References

- [1] H. Ahn and C.-K. Han, *Local geometry of Levi-forms associated with the existence of complex submanifolds and the minimality of generic CR manifolds*, J. Geom. Anal. **22** (2012), no. 2, 561–582.
- [2] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths, *Exterior Differential Systems*, Mathematical Sciences Research Institute Publications, **18**, Springer-Verlag, New York, 1991.
- [3] S. S. Chern and J. K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271.
- [4] S.-C. Chen and M.-C. Shaw, *Partial differential equations in several complex variables*, AMS/IP Studies in Advanced Mathematics, **19**, American Mathematical Society, Providence, RI, 2001.

- [5] K. Diederich and J. E. Fornæss, *Pseudoconvex domains with real-analytic boundary*, Ann. Math. (2) **107** (1978), no. 2, 371–384.
- [6] N. Q. Dieu, *Zero sets of real polynomials containing complex varieties*, Illinois J. Math. **55** (2011), no. 1, 69–76 (2012).
- [7] C.-K. Han and G. Tomassini, *Complex submanifolds in real hypersurfaces*, J. Korean Math. Soc. **47** (2010), no. 5, 1001–1015.
- [8] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, second revised edition, North-Holland Publishing Co., Amsterdam, 1973.
- [9] J. J. Kohn, *Subellipticity of the  $\bar{\partial}$ -Neumann problem on pseudo-convex domains: sufficient conditions*, Acta Math. **142** (1979), no. 1-2, 79–122.
- [10] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. (2) **65** (1957), 391–404.
- [11] J.-M. Trépreau, *Sur le prolongement holomorphe des fonctions  $C$ - $R$  définies sur une hypersurface réelle de classe  $C^2$  dans  $\mathbf{C}^n$* , Invent. Math. **83** (1986), no. 3, 583–592.

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