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A CHARACTERIZATION OF PRIME SUBMODULES OF AN INJECTIVE MODULE OVER A NOETHERIAN RING

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Abstract. In this paper, we give a characterization of prime submodules of an injective module over a Noetherian ring.

0. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Let M be an R-module. We denote a (proper) submodule N of M, by ($N \lneqq M$) $N \leq M$. A proper submodule P of an R-module M is called prime, if $rm \in P$ for some $r \in R$ and $m \in M$ implies $m \in P$ or $r \in (P : M)$, where $(P : M) = \{r \in R \mid rM \subseteq P\}$. If P is a prime submodule of an R-module M, then $(P : M)$ is a prime ideal of R. The set of all prime submodules of an R-module M is denoted by $\mathrm{Spec}(M)$. An R-module M is injective if for every R-module monomorphism $f : N \longrightarrow N'$ and for every R-module homomorphism $g : N \longrightarrow M$, there exists an R-module homomorphism $h : N' \longrightarrow M$ such that $hf = g$. Let $N \subseteq M$ be R-modules. We say that M is an essential extension of N, if for any nonzero R-submodue U of M one has $U \cap N \neq 0$. Let M be an R-module. An injective module E is called an injective envelope of M, if E is an essential extension of M and denoted by $E(M)$. We know that any module M can be embedded into an injective module; and injective envelope of M is the minimal embedding. In this case, the corresponding injective module is unique up to isomorphism. An element x of an R-module M is called torsion, if it has a nonzero annihilator in R. Let M_t be the set of all torsion elements of M. It is clear that if R is an integral domain, then M_t is a submodule of M. We say that M_t is the torsion submodule of M. An R-module M is divisible if for every $0 \neq r \in R$, $rM = M$. It is easy to see that every injective module over an integral domain R is divisible. If M is a divisible R-module, then for every proper submodule N of M, $(N : M) = 0$.

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Prime submodules of a module over a commutative ring have been studied by many authors, see [4, 7, 11]. Also prime submodules of a finitely generated free module over a PID were studied in [2, 3]. The authors in [2], described prime submodules of a finitely generated free module over a UFD and characterized the prime submodules of a free module of finite rank over a PID. The authors in [8,9], extended some results obtained in [2] to a Dedekind and valuation domain. In [10], we have characterized prime submodules of an injective module over a Noetherian domain. In this paper, we extend our results to Noetherian ring.

1. Prime submodules of $E(\frac{R}{\mathfrak{p}})$

In this section, we give some results about prime submodules of $E(\frac{R}{\mathfrak{p}})$, when R is a Noetherian ring and $\mathfrak{p} \in \mathrm{Spec}(R)$. Then we characterize all prime submodules of $E(\frac{R}{\mathfrak{p}})$.

Lemma 1.1. *Let R be a Noetherian ring, $\mathfrak{p} \in Spec(R)$ and $E = E(\frac{R}{\mathfrak{p}})$. We have the following*:
i) $ann_R(E) \subseteq \mathfrak{p}$.
ii) *If $P \in Spec(E)$, then $\mathfrak{p} \subseteq (P : E)$.*
iii) *If $0 \neq P \in Spec(E)$ and $\mathfrak{q} = (P : E)$, then $\frac{R}{\mathfrak{p}} \subseteq P$ or $P \cap \frac{R}{\mathfrak{p}} = \frac{\mathfrak{q}}{\mathfrak{p}}$.*
iv) *If $0 \neq P \in Spec(E)$ and $(P : E) = \mathfrak{p}$, then $\frac{R}{\mathfrak{p}} \subseteq P$.*
v) *If $\mathfrak{p} \in Max(R)$, then $Spec(E) = \{P \subsetneqq E \mid \mathfrak{p}E \subseteq P\}$ and in this case for every $P \in Spec(E)$, we have $(P : E) = \mathfrak{p}$.*

Proof. i) Let $r \in \mathrm{ann}_R(E)$. So $rE = 0$ and hence $r(\frac{R}{\mathfrak{p}}) = 0$. Thus $r + \mathfrak{p} = r(1 + \mathfrak{p}) = \mathfrak{p}$ and so $r \in \mathfrak{p}$. Therefore $\mathrm{ann}_R(E) \subseteq \mathfrak{p}$.

ii) Let $\mathfrak{q} = (P : E)$ and $\mathfrak{p} \nsubseteq \mathfrak{q}$. We show that for every $x \in E$, $\mathrm{ann}_R(x) \nsubseteq \mathfrak{q}$. Let $y \in E$ and $\mathrm{ann}_R(y) \subseteq \mathfrak{q}$. Since R is Noetherian, by [6, Theorem 3.4(1)], $E = \bigcup_{m=1}^{\infty} A_m$, where $A_m = \{x \in E \mid \mathfrak{p}^m x = 0\}$. So there exists $m \in \mathbb{N}$ such that $\mathfrak{p}^m y = 0$ and hence $\mathfrak{p}^m \subseteq \mathfrak{q}$. Then $\mathfrak{p} \subseteq \mathfrak{q}$, which is a contradiction. Therefore for every $x \in E$, $\mathrm{ann}_R(x) \nsubseteq \mathfrak{q}$. Now Let $x \in E$. So there exists $r \in R \setminus \mathfrak{q}$ such that $rx = 0$ and hence $x \in P$. Now we have $P = E$, which is a contradiction. Therefore $\mathfrak{p} \subseteq \mathfrak{q}$.

iii) Let $\frac{R}{\mathfrak{p}} \nsubseteq P$. We show that $P \cap \frac{R}{\mathfrak{p}} = \frac{\mathfrak{q}}{\mathfrak{p}}$. Since $\mathfrak{q}E \subseteq P$, $\mathfrak{q}(\frac{R}{\mathfrak{p}}) \subseteq P$ and hence $\frac{\mathfrak{q}}{\mathfrak{p}} \subseteq P \cap \frac{R}{\mathfrak{p}}$. Now let $P \cap \frac{R}{\mathfrak{p}} = \frac{\mathfrak{a}}{\mathfrak{p}}$ for some ideal $\mathfrak{a}$ of R. If $\mathfrak{a} = \mathfrak{p}$, then $P \cap \frac{R}{\mathfrak{p}} = \{0\}$ and since $E(\frac{R}{\mathfrak{p}})$ is an essential extension of $\frac{R}{\mathfrak{p}}$, $P = 0$, which is a contradiction. Thus $\mathfrak{a} \neq \mathfrak{p}$. Let $r \in \mathfrak{a} \setminus \mathfrak{p}$. So $r + \mathfrak{p} = r(1 + \mathfrak{p}) \in P$ and since $1 + \mathfrak{p} \notin P$, we have $r \in \mathfrak{q}$. Therefore $P \cap \frac{R}{\mathfrak{p}} = \frac{\mathfrak{q}}{\mathfrak{p}}$.

iv) It follows by part (iii).

v) Let $P \in \mathrm{Spec}(E)$. By part (ii), $\mathfrak{p} \subseteq (P : E)$ and hence $\mathfrak{p}E \subseteq P$. Conversely, let $P \subsetneqq E$ and $\mathfrak{p}E \subseteq P$. Then $\mathfrak{p} \subseteq (P : E) \neq R$. Since $\mathfrak{p} \in \mathrm{Max}(R)$, we have $(P : E) = \mathfrak{p}$. Therefore $P \in \mathrm{Spec}(E)$. □

Let R be a ring, $\mathfrak{p} \in \operatorname{Spec}(R)$, M be an R-module and $N \leq M$. Lu in [5], defined the saturation of N with respect to $\mathfrak{p}$ by $S_{\mathfrak{p}}(N) = \{x \in M \mid sx \in N$ for some $s \in R \setminus \mathfrak{p}\}$.

Proposition 1.2. *Let R be a Noetherian ring, $\mathfrak{p} \in Spec(R)$ and $E = E(\frac{R}{\mathfrak{p}})$. Then*
i) *$S_{\mathfrak{p}}(0) = \{0\}$, where $\{0\}$ is the zero submodule of E.*
ii) *$ann_R(E) = \mathfrak{p}$ if and only if $\{0\} \in Spec(E)$.*

Proof. i) Let $S_{\mathfrak{p}}(0) \cap \frac{R}{\mathfrak{p}} = \frac{\mathfrak{a}}{\mathfrak{p}}$, where $\mathfrak{a}$ is an ideal of R. Suppose that $\mathfrak{a} \neq \mathfrak{p}$ and choose $r \in \mathfrak{a} \setminus \mathfrak{p}$. So $r + \mathfrak{p} \in S_{\mathfrak{p}}(0)$ and hence there exists $s \in R \setminus \mathfrak{p}$ such that $sr + \mathfrak{p} = s(r + \mathfrak{p}) = \mathfrak{p}$. Then $sr \in \mathfrak{p}$ and hence $r \in \mathfrak{p}$ or $s \in \mathfrak{p}$, which is a contradiction. Therefore $\mathfrak{a} = \mathfrak{p}$. Thus $S_{\mathfrak{p}}(0) \cap \frac{R}{\mathfrak{p}} = \{0\}$ and since $E(\frac{R}{\mathfrak{p}})$ is an essential extension of $\frac{R}{\mathfrak{p}}$, $S_{\mathfrak{p}}(0) = \{0\}$.

ii) Let $\operatorname{ann}_R(E) = \mathfrak{p}$. Suppose that $0 \neq x \in E$, $r \in R$ such that $rx = 0$. If $r \in R \setminus \mathfrak{p}$, by part (i), we have $x \in S_{\mathfrak{p}}(0) = \{0\}$, which is a contradiction. So $r \in \mathfrak{p}$ and hence $\{0\} \in \operatorname{Spec}(E)$. Conversely, let $\{0\} \in \operatorname{Spec}(E)$. By Lemma 1.1, parts (i) and (ii), we have $\mathfrak{p} \subseteq (0 : E) = \operatorname{ann}_R(E) \subseteq \mathfrak{p}$ and hence $\operatorname{ann}_R(E) = \mathfrak{p}$. □

In [10, Theorem 2.6], the authors prove that, if R is a Noetherian domain with quotient filed K and M is an injective R-module, then
i) $M = M_t \oplus N$, where $N \simeq \oplus_{i \in I} K$ for some index set I.
ii) $\operatorname{Spec}(M) = \emptyset$ or $\operatorname{Spec}(M) = \{M_t \oplus D \mid D \subsetneqq N,\ D \simeq \oplus_{j \in J} K$ for some index set $J\}$.

Proposition 1.3. *Let R be a Noetherian ring, $\mathfrak{p} \in Spec(R)$, $E = E(\frac{R}{\mathfrak{p}})$ and $ann_R(E) = \mathfrak{p}$. Let K be the quotient field of $\frac{R}{\mathfrak{p}}$. We have:*
i) *$E \simeq \oplus_{i \in I} K$ for some index set I.*
ii) *$Spec(E) = \{P \subsetneqq E \mid P \simeq \oplus_{j \in J} K$ for some index set $J\}$.*
iii) *If $P \in Spec(E)$, then $(P : E) = \mathfrak{p}$.*

Proof. i) If $\operatorname{ann}_R(E) = \mathfrak{p}$, then E is an $\frac{R}{\mathfrak{p}}$-module. Since E is an injective R-module, by the Baer's Criterion it is easy to show that E is an injective $\frac{R}{\mathfrak{p}}$-module. Since $E_t = S_{\mathfrak{p}}(0)$ as $\frac{R}{\mathfrak{p}}$-module, then by Proposition 1.2(i), $E_t = 0$. Now by [10, Theorem 2.6(i)], $E \simeq \oplus_{i \in I} K$, for some index set I.

ii) It follows by part (i) and [10, Theorem 2.6(ii)].

iii) Since E is an injective $\frac{R}{\mathfrak{p}}$-module, E is a divisible $\frac{R}{\mathfrak{p}}$-module and hence $(P :_{R/\mathfrak{p}} E) = 0$. So $(P :_R E) = \mathfrak{p}$. □

For the characterization of prime submodules of $E = E(\frac{R}{\mathfrak{p}})$, we need the following lemma.

Lemma 1.4. *Let R be a Noetherian ring, $\mathfrak{p} \in Spec(R)$ and $E = E(\frac{R}{\mathfrak{p}})$. If $s \in R \setminus \mathfrak{p}$, then the R-homomorphism $f_s : E \longrightarrow E$ defined by $x \mapsto sx$ is an automorphism of E.*

Proof. [6, Lemma 3.2(2)]. □

Theorem 1.5. *Let R be a Noetherian ring, $\mathfrak{p} \in Spec(R)$ and $E = E(\frac{R}{\mathfrak{p}})$. Then $Spec(E) = \{P \subsetneqq E \mid \mathfrak{p} \subseteq (P : E) = \mathfrak{q} \in Spec(R)$ and $\frac{E}{P}$ is a K-module, where K is the quotient field of $\frac{R}{\mathfrak{q}}\}$.*

Proof. Let $\Sigma = \{P \subsetneqq E \mid \mathfrak{p} \subseteq (P : E) = \mathfrak{q} \in \mathrm{Spec}(R)$ and $\frac{E}{P}$ is a K-module, where K is the quotient field of $\frac{R}{\mathfrak{q}}\}$. We show that $\mathrm{Spec}(E) = \Sigma$. Let $P \in \Sigma$. Since every proper submodule of a vector space is $\{0\}$-prime, $\{P\}$ is a $\{0\}$-prime submodule of K-vector space $\frac{E}{P}$. So P is a $\{0\}$-prime submodule of $\frac{R}{\mathfrak{q}}$-module $\frac{E}{P}$ and hence P is a $\mathfrak{q}$-prime submodule of R-module E. Thus $\Sigma \subseteq \mathrm{Spec}(E)$. Conversely, let $P \in \mathrm{Spec}(E)$. By Lemma 1.1(ii), $\mathfrak{p} \subseteq (P : E) = \mathfrak{q}$. Since $\mathfrak{q}E \subseteq P$, $\frac{E}{P}$ is an $\frac{R}{\mathfrak{q}}$-module. Let K be the quotient field of $\frac{R}{\mathfrak{q}}$. For every $r \in R$, $s \in R \setminus \mathfrak{q}$ and $x \in E$, we put $\overline{r} = r + \mathfrak{q}$, $\overline{s} = s + \mathfrak{q}$ and $\overline{x} = x + P$. By Lemma 1.4, for every $s \in R \setminus \mathfrak{q}$ and $x \in E$ there exists a unique $y \in E$ such that $sy = x$. Now we define the map $K \times \frac{E}{P} \longrightarrow \frac{E}{P}$ by $\frac{\overline{r}}{\overline{s}} \, . \, \overline{x} = \overline{ry}$, where $sy = x$. We show that this map is well-defined. Let $\frac{\overline{r}}{\overline{s}} = \frac{\overline{r'}}{\overline{s'}}$, $\overline{x} = \overline{x'}$, where $sy = x$ and $s'y' = x'$. So $rs' - sr' \in q$, $x - x' \in P$ and hence $sy - s'y' \in P$. We prove that $ry - r'y' \in P$. Since $rr'(sy - s'y') \in P$, hence $rr'sy - rr's'y' = rr'sy - {r'}^2 sy' + {r'}^2 sy' - rr's'y' = r's(ry - r'y') + (r's - rs')r'y' \in P$. But $r's - rs' \in \mathfrak{q}$ and $\mathfrak{q}E \subseteq P$, hence $(r's - rs')r'y' \in P$. Thus $r's(ry - r'y') \in P$. If $r' \in \mathfrak{q}$, then $r \in \mathfrak{q}$ and we have $ry - r'y' \in P$. Let $r' \notin \mathfrak{q}$. Since $r's \notin \mathfrak{q}$ and P is a $\mathfrak{q}$-prime submodule, $ry - r'y' \in P$. So $\frac{E}{P}$ is a K-module and hence $P \in \Sigma$. Therefore $\mathrm{Spec}(E) = \Sigma$. □

Corollary 1.6. *Let R be a Noetherian ring, $\mathfrak{p} \in Spec(R)$, $E = E(\frac{R}{\mathfrak{p}})$. Suppose that $\sqrt{ann_R(E)} = \mathfrak{p}$. Then there exists $m \in \mathbb{N}$ such that $A_m = \{x \in E \mid \mathfrak{p}^m x = 0\} \in Spec(E)$ and $(A_m : E) = \mathfrak{p}$.*

Proof. Since R is a Noetherian ring and $\sqrt{\mathrm{ann}_R(E)} = \mathfrak{p}$, there exists $n \in \mathbb{N}$ such that $\mathfrak{p}^n \subseteq \mathrm{ann}(E)$ and $\mathfrak{p}^{n-1} \nsubseteq \mathrm{ann}(E)$. Put $m = n - 1$. By [6, Theorem 3.4(4)], we have $\frac{E}{A_m}$ is a K-module, where K is the quotient field of $\frac{R}{\mathfrak{p}}$. So by the first part of the proof of Theorem 1.5, we have A_m is a $\mathfrak{p}$-prime submodule of E. □

The following examples show that the assumptions of Corollary 1.6, are satisfied in both cases, that R is an integral domain or it is not.

Example 1.7. Let $R = \mathbb{Z}$ and $\mathfrak{p} = (0)$. We have $E(\frac{R}{\mathfrak{p}}) = \mathbb{Q}$. Then

$$\sqrt{\mathrm{ann}_{\mathbb{Z}}(E(\frac{R}{\mathfrak{p}}))} = \sqrt{(0)} = (0) = \mathfrak{p}.$$

Example 1.8. Let $R = \mathbb{Z}_6$ and $\mathfrak{p} = \langle \overline{2} \rangle$. Clearly $\frac{R}{\mathfrak{p}} \simeq \mathbb{Z}_2$. We show that $E_{\mathbb{Z}_6}(\mathbb{Z}_2) = \mathbb{Z}_2$. We know that $E_{\mathbb{Z}}(\mathbb{Z}_2) \simeq \mathbb{Z}_{2^\infty}$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_6, \mathbb{Z}_{2^\infty})$ is an injective

$\mathbb{Z}_6$-module. It is easy to see that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_6, \mathbb{Z}_{2^\infty}) \simeq \mathbb{Z}_2$. Then $E_{\mathbb{Z}_6}(\mathbb{Z}_2) = \mathbb{Z}_2$. Now we have

$$\begin{aligned}\sqrt{\mathrm{ann}_{\mathbb{Z}_6}(E_{\mathbb{Z}_6}(\frac{R}{\mathfrak{p}}))} &= \sqrt{\mathrm{ann}_{\mathbb{Z}_6}(E_{\mathbb{Z}_6}(\mathbb{Z}_2))} \\ &= \sqrt{\mathrm{ann}_{\mathbb{Z}_6}(\mathbb{Z}_2)} = \sqrt{\langle\overline{2}\rangle} = \langle\overline{2}\rangle = \mathfrak{p}.\end{aligned}$$

2. Prime submodules of an injective module over a Noetherian ring

In this section we characterize the prime submodules of an injective module over a Noetherian ring R.

Proposition 2.1. *Let R be a Noetherian ring, $\mathfrak{p} \in Spec(R)$ and M be an injective R-module such that $\mathfrak{p} \subseteq ann_R(M)$. Let K be the quotient field of $\frac{R}{\mathfrak{p}}$. We have:*

i) *$M = S_{\mathfrak{p}}(0) \oplus N$ such that $N \simeq \bigoplus_{i\in I} K$ for some index set I.*

ii) *$Spec(M) = \emptyset$ or $Spec(M) = \{S_{\mathfrak{p}}(0) \oplus D \mid D \subsetneqq N$ and $D \simeq \bigoplus_{j\in J} K$ for some index set $J\}$.*

iii) *If $P \in Spec(M)$, then $(P : M) = \mathfrak{p}$.*

Proof. Since $\mathfrak{p} \subseteq \mathrm{ann}_R(M)$, M is an $\frac{R}{\mathfrak{p}}$- module and we have $M_t = S_{\mathfrak{p}}(0)$ as $\frac{R}{\mathfrak{p}}$-module. Now the proof is similar to the proof of Proposition 1.3. □

Remark 2.2. Let R be a Noetherian ring, $\mathfrak{p} \in \mathrm{Spec}(R)$ and M be an injective R-module. We put $M(\mathfrak{p}) = \bigoplus_{i\in I} E(\frac{R}{\mathfrak{p}})$ such that the number of indecomposable summands in the decomposition of $M(\mathfrak{p})$ equals $\dim_{k(\mathfrak{p})}\mathrm{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}})$, where $k(\mathfrak{p}) = \frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}}$. Let $\{\mathfrak{p}_i \mid i \in \Omega\} \subseteq \mathrm{Spec}(R)$ be the set of all prime ideals $\mathfrak{p}$ of R such that $\dim_{k(\mathfrak{p})}\mathrm{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}}) \neq 0$. By [1, Theorem 3.2.8], we have $M \simeq \bigoplus_{i\in\Omega} M(\mathfrak{p}_i)$. It is easy to show that there exist submodules N_i of $M (i \in \Omega)$ such that $M = \bigoplus_{i\in\Omega} N_i$ and for every $i \in \Omega$, $N_i \simeq M(\mathfrak{p}_i)$.

Lemma 2.3. *With the notation as Remark 2.2, we have the following:*

i) *If $P \in Spec(M(\mathfrak{p}))$, then $\mathfrak{p} \subseteq (P : M(\mathfrak{p}))$.*

ii) *If $\mathfrak{p} \in Max(R)$, then $Spec(M(\mathfrak{p})) = \{P \subsetneqq M(\mathfrak{p}) \mid \mathfrak{p}M(\mathfrak{p}) \subseteq P\}$.*

Proof. By Remark 2.2, $M(\mathfrak{p}) = \bigoplus_{i\in I} E(\frac{R}{\mathfrak{p}})$. Let $j \in I$ and $B_j = \bigoplus_{i\in I} A_i$ such that $A_j = E(\frac{R}{\mathfrak{p}})$ and for every $i \in I \setminus \{j\}$, $A_i = 0$. We have $M(\mathfrak{p}) = \bigoplus_{i\in I} B_i$. Let $P \in \mathrm{Spec}(M(\mathfrak{p}))$ and $Q_i = P \bigcap B_i (i \in I)$. Then $Q_i = B_i$ or $Q_i \in \mathrm{Spec}(B_i)$. Since $B_i \simeq E(\frac{R}{\mathfrak{p}})(i \in \Omega)$, by Lemma 1.1(ii), for every $i \in I$, we have $\mathfrak{p} \subseteq (Q_i : B_i)$. So $\bigoplus_{i\in I} Q_i \subseteq P$ implies that $\mathfrak{p} \subseteq (\bigoplus_{i\in I} Q_i : M(\mathfrak{p})) \subseteq (P : M(\mathfrak{p}))$.

ii) The proof is similar to the proof of Lemma 1.1(v). □

In the following result, we give a charactrization of prime submodules of injective modules over Artinian rings.

Proposition 2.4. *Let R be an Artinian ring. Let M be an injective R-module and $M = \bigoplus_{i\in\Omega} N_i$ be as in Remark 2.2. Then $Spec(M) = \{P \subsetneqq M \mid P = \bigoplus_{i\in\Omega} P_i$ such that for every $i \in \Omega$, $P_i \leq N_i$ and there exists a unique $j \in \Omega$ such that $\mathfrak{p}_j N_j \subseteq P_j \neq N_j$ and for every $i \in \Omega \setminus \{j\}$, $P_i = N_i\}$.*

Proof. Let $\Sigma = \{P \subsetneqq M \mid P = \bigoplus_{i\in\Omega} P_i$ such that for every $i \in \Omega$, $P_i \leq N_i$ and there exists a unique $j \in \Omega$ such that $\mathfrak{p}_j N_j \subseteq P_j \neq N_j$ and for every $i \in \Omega\setminus\{j\}$, $P_i = N_i\}$. We show that $\text{Spec}(M) = \Sigma$. Let $P \in \Sigma$. So $P = \bigoplus_{i\in\Omega} P_i$ such that for every $i \in \Omega$, $P_i \leq N_i$ and there exists a unique $j \in \Omega$ such that $\mathfrak{p}_j N_j \subseteq P_j \neq N_j$ and for every $i \in \Omega \setminus \{j\}$, $P_i = N_i$. Since $N_j \simeq M(\mathfrak{p}_j)$, by Lemma 2.3(ii), $P_j \in \text{Spec}(N_j)$. It is easy to see that $(P : M) = \mathfrak{p}_j \in \text{Max}(R)$ and hence $P \in \text{Spec}(M)$. Conversely, let $P \in \text{Spec}(M)$ and for every $i \in \Omega$, $P_i = P \bigcap N_i$. We prove that $P = \bigoplus_{i\in\Omega} P_i$. Assume that Ω is a finite set and $|\Omega| = n$. By induction on n, we prove that $P = \bigoplus_{i=1}^{n} P_i$. Let $n = 2$. Then $M = N_1 \oplus N_2$. Clearly $P_1 \oplus P_2 \subseteq P$. If $P_1 = N_1$ and $P_2 = N_2$, then $P = M$, which is a contradiction. Assume that $N_2 \neq P_2$. So $(P_2 : N_2) = \mathfrak{p}_2$ and $(P_1 : N_1) = \mathfrak{p}_1$ or R. Since $\mathfrak{p}_1 \neq \mathfrak{p}_2$, there exists $r \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$. Put $y = x_1 + x_2 \in P$, where $x_1 \in N_1$ and $x_2 \in N_2$. We have $ry = rx_1 + rx_2 \in P$ and $rx_1 \in P_1$. So $rx_2 = ry - rx_1 \in P \bigcap N_2 = P_2$ and hence $x_2 \in P_2$. Therefore $x_1 = y - x_2 \in P \bigcap N_1 = P_1$. So $y \in P_1 \oplus P_2$ and we have $P = P_1 \oplus P_2$. Let $k \in \mathbb{N}$ and suppose the claim is true for $n = k - 1$. Let $M = \bigoplus_{i=1}^{k} N_i$. Clearly $\bigoplus_{i=1}^{k} P_i \subseteq P$. For every $i \in \{1, \ldots, k\}$, we have $(P_i : N_i) = \mathfrak{p}_i$ or R. Since $P \neq M$, there exists $i \in \{1, \ldots, k\}$ such that $(P_i : N_i) = \mathfrak{p}_i$ and there exists $j \in \{1, \ldots, k\}$ such that $(P_j : N_j) \nsubseteq \bigcap_{i=1, i\neq j}^{k} (P_i : N_i)$. Let $j = 1$ and $r \in (P_1 : N_1) \setminus \bigcap_{i=2}^{k} (P_i : N_i)$. Put $y = x_1 + \cdots + x_k \in P$, where $x_i \in N_i (1 \leq i \leq k)$. We prove that $x_i \in P_i (1 \leq i \leq k)$. Assume that $N = \bigoplus_{i=2}^{k} N_i$ and $D = P \bigcap N$. If $D = N$, then for every $i \in \{2, \ldots, k\}$ $P_i = N_i$ and hence $P_1 \neq N_1$. So $(P_1 : N_1) \subseteq \bigcap_{i=2}^{k} (P_i : N_i) = R$, which is a contradiction. Therefore $D \neq N$ and hence $D \in \text{Spec}(N)$. By assumption of induction, we have $D = \bigoplus_{i=2}^{k} P_i$. Now put $y' = x_2 + \cdots + x_k$. We have $ry = rx_1 + \cdots + rx_k \in P$ and $rx_1 \in P_1 \subseteq P$. So $ry' = ry - rx_1 \in P \bigcap N = D$. Since $r \notin \bigcap_{i=2}^{k} (P_i : N_i)$, $r \notin (D : N)$ and thus $y' \in D$. Thus $x_i \in P_i (2 \leq i \leq k)$ and hence $x_i \in P_i (1 \leq i \leq k)$. Therefore $P = \bigoplus_{i=1}^{k} P_i$. Then for every $n \in \mathbb{N}$ with $|\Omega| = n$, we have $P = \bigoplus_{i=1}^{n} P_i$. Now we show that $P = \bigoplus_{i\in\Omega} P_i$. Clearly $\bigoplus_{i\in\Omega} P_i \subseteq P$. Let $z \in P$. There exist $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \Omega$ such that $z = \Sigma_{j=1}^{n} x_{i_j}$, where $x_{i_j} \in N_{i_j}$. Assume that, $N = \bigoplus_{j=1}^{n} N_{i_j}$ and $D = P \bigcap N$. We have $D \in \text{Spec}(N)$ or $D = N$. By the above argument, we have $x_{i_j} \in P_{i_j} (1 \leq j \leq n)$ and hence $z \in \bigoplus_{i\in\Omega} P_i$. So $P = \bigoplus_{i\in\Omega} P_i$. Now let $i, j \in \Omega$, $i \neq j$, $P_i \neq N_i$ and $P_j \neq N_j$. Since $\mathfrak{p}_i \neq \mathfrak{p}_j$, there exist $r \in \mathfrak{p}_i \setminus \mathfrak{p}_j$, $x_i \in N_i \setminus P_i$. Let $x_j \in P_j$ and $t = x_i + x_j$. So $rt = rx_i + rx_j \in P$. Since $r \notin \mathfrak{p}_j$, $r \notin (P : M)$. On the other hand, $x_i \notin P_i$ and hence $t \notin P$, which is a contradiction. Therefore $P \in \Sigma$ and we have $\text{Spec}(M) = \Sigma$. □

Theorem 2.5. *Let R be a Noetherian ring and M be an injective R-module. Then $Spec(M) = \{P \subsetneqq M \mid (P : M) = \mathfrak{q} \in Spec(R)$ and $\frac{M}{P}$ is a K-module, where K is the quotient field of $\frac{R}{\mathfrak{q}}\}$.*

Proof. Let $\Sigma = \{P \subsetneqq M \mid (P : M) = \mathfrak{q} \in Spec(R)$ and $\frac{M}{P}$ is a K-module, where K is the quotient field of $\frac{R}{\mathfrak{q}}\}$. We show that $\mathrm{Spec}(M) = \Sigma$. Let $P \in \Sigma$. We have $(P : M) = \mathfrak{q} \in \mathrm{Spec}(R)$ and $\frac{M}{P}$ is a K-module. Then $\{P\}$ is a $\{0\}$-prime submodule of K-vector space $\frac{M}{P}$. So $\{P\}$ is a $\{0\}$-prime submodule of $\frac{R}{\mathfrak{q}}$-module $\frac{M}{P}$ and hence P is a $\mathfrak{q}$-prime submodule of R-module M. Thus $P \in \mathrm{Spec}(M)$. Conversely, let $P \in \mathrm{Spec}(M)$. There exists $\mathfrak{q} \in Spec(R)$ such that $(P : M) = \mathfrak{q}$. By Remark 2.2, there exist an index set Ω and a subset $\{\mathfrak{p}_i \mid i \in \Omega\}$ of $\mathrm{Spec}(R)$ and submodules N_i of $M(i \in \Omega)$ such that $M = \bigoplus_{i\in\Omega} N_i$, where $N_i \simeq M(\mathfrak{p}_i)(i \in \Omega)$. Let $\Omega' = \{i \in \Omega \mid N_i \nsubseteq P\}$. If $\Omega' = \emptyset$, then $P = M$, which is a contradiction. So $\Omega' \neq \emptyset$. Put $A = \bigoplus_{i\in\Omega'} N_i$ and $B = \bigoplus_{i\in\Omega\setminus\Omega'} N_i$, then $M = A \oplus B$. Clearly $B \leq P$. Let $P_i = P \bigcap N_i$ $(i \in \Omega')$. We have $\bigoplus_{i\in\Omega'} P_i \subseteq P$. Since $P_i \bigcap B = \{0\}$ $(i \in \Omega')$, $(\bigoplus_{i\in\Omega'} P_i) \cap B = \{0\}$ and hence $(\bigoplus_{i\in\Omega'} P_i) \oplus B \subseteq P$. So by Lemma 2.3(i), $\bigcap_{i\in\Omega'} \mathfrak{p}_i \subseteq \mathfrak{q}$. Now we prove that $\frac{E}{P}$ is a K-module. At first, we define R-homomorphism $f_s : A \longrightarrow A$ by $f_s(\{x_i\}_{i\in\Omega'}) = \{sx_i\}_{i\in\Omega'}$, where $s \in R \setminus \mathfrak{q}$. By Lemma 1.4, it is easy to see that f_s is an automorphism of A. For every $r \in R$, $s \in R \setminus \mathfrak{q}$, $x \in M$, we put $\overline{r} = r + \mathfrak{q}$, $\overline{s} = s + \mathfrak{q}$ and $\overline{x} = x + P$. Let $x = a + b$, where $a \in A$ and $b \in B$. Since for every $a \in A$, f_s is an automorphism of A, there exists a unique $y \in A$ such that $sy = a$. Now we define the map $K \times \frac{M}{P} \longrightarrow \frac{M}{P}$ by $\frac{\overline{r}}{\overline{s}} \cdot (\overline{a} + \overline{b}) = \overline{ry}$, where $sy = a$. By reasoning similar to the proof of Theorem 1.5, this map is well-defined and hence $\frac{M}{P}$ is a K-module. Therefore $P \in \Sigma$ and $\mathrm{Spec}(M) = \Sigma$. □

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