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A GENERALIZATION OF MULTIPLICATION MODULES

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Abstract. For $M \in R$-Mod, $N \subseteq M$ and $L \in \sigma[M]$ we consider the product $N_M L = \sum_{f \in \mathrm{Hom}_R(M,L)} f(N)$. A module $N \in \sigma[M]$ is called an M-multiplication module if for every submodule L of N, there exists a submodule I of M such that $L = I_M N$. We extend some important results given for multiplication modules to M-multiplication modules. As applications we obtain some new results when M is a semiprime Goldie module. In particular we prove that M is a semiprime Goldie module with an essential socle and $N \in \sigma[M]$ is an M-multiplication module, then N is cyclic, distributive and semisimple module. To prove these results we have had to develop new methods.

Introduction

Multiplication modules were introduced by Barnard [3]. In that paper the author proves that, if R is commutative and a semilocal ring, then an R-module is a multiplication module if and only if, it is cyclic and that every finitely generated artinian multiplication module is cyclic. Multiplication modules have been studied by several authors [1, 2, 12, 13] and [15].

Later, a more general situation was considered in [17]. In that paper, the author works with noncommutative rings. A right R-module M is called a multiplication module if for every submodule N of M, there exists an ideal I of R such that $N = MI$. In [17] the author investigates and gives results of these multiplication modules. Some of these results extend the results given in [12].

In this paper we use the product of modules defined in [5] and we define the concept of M-multiplication module, where $M \in R\text{-}Mod$. A module $N \in \sigma[M]$ is an M-multiplication module if for every submodule L of N there exists a submodule I of M such that $I_M N = L$.

We investigate the properties of these modules and obtain results that extend the results (for rings) given by El-Bast, P. Smith in [12] and Tuganbaev in [17] to the context of modules. We emphasize that the techniques used by El-Bast

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and, P. Smith and Tuganbaev are not easily generalized in a manner useful for our purposes. We have thus had to develop new methods. Moreover with these methods we give new results when M is a semiprime Goldie module, see [9,10] and [11].

In order to do this, we organized the article into four sections. Section 1 is devoted to preliminary concepts to develop this investigation. Section 2 is devoted to defining and studying the concept of M-multiplication module and in Section 3 we give the main results of the paper. In Theorem 3.12 we prove that if $M = \bigoplus_{i=1}^{n} M_i$ is a progenerator in $\sigma[M]$ and N is an M-multiplication module, such that every M_i is FI-simple and generates $\sigma[M_i]$ with $\mathrm{Hom}_R(M_i, M_j) = 0$ for all $i \neq j$, then N is cyclic, distributive and semisimple module. In this section, we obtain some applications when M is a semiprime Goldie module. In Theorem 3.7, we prove that if M is projective in $\sigma[M]$ and M is a semiprime Goldie module with essential socle, then every M-multiplication module is cyclic, distributive and semisimple module. In particular, when R is a semiprime, left Goldie ring and N is an R-multiplication module, then N is cyclic, distributive and semisimple module. Examples are given to illustrate this theory.

In what follows, R will denote an associative ring with unity and R-Mod will denote the category of unitary left R-modules. Let M and X be R-modules. Then X is said to be M-generated if there exists an R-epimorphism from a direct sum of copies of M onto X. The category $\sigma[M]$ is defined as the full subcategory of R-Mod consisting of all R-modules X which are isomorphic to a submodule of an M-generated module. For details about the category $\sigma[M]$ we refer the reader [19]. The trace of M in X is defined to be $tr^M(X) = \sum_{f \in \mathrm{Hom}_R(M,X)} f(M)$, thus X is M-generated if and only if $tr^M(X) = X$.

If N is a fully invariant submodule of M, we write $N \subseteq_{FI} M$. If N is an essential submodule of M, we write $N \subseteq_{ess} M$. When a module has no non-zero fully invariant proper submodules, it is called an FI-simple module. An R-module M is a duo module if $N \subseteq_{FI} M$ for all N submodule of M. A module is said to be distributive if the lattice of its submodules is distributive, i.e., $(X + Y) \cap Z = X \cap Z + Y \cap Z$ for any of its submodules X, Y and Z.

1. Preliminaries

In the following, we present some results which will be used in this paper.

Definition 1.0 ([6, Definition 1.1]). Let $M \in R$-Mod. Let K be a submodule of M and $L \in R$-Mod. Put

$$K_M L = \sum \{f(K) \mid f \in \mathrm{Hom}(M, L)\}.$$

Remark 1.1. Note that given a submodule N of M, there exists a submodule $\overline{N} \subset M$ such that $\overline{N}$ is the least fully invariant submodule of M which contains N. In fact $\overline{N} = \sum \{f(N) \mid f \in \mathrm{Hom}(M, M)\}$. Notice that $\overline{N} = N_M M$. Also notice that if K and L are submodules of M, then

$$\sum\left\{f(\overline{K}) \mid f \in \mathrm{Hom}(M,L)\right\} = \sum\left\{f(K) \mid f \in \mathrm{Hom}(M,L)\right\}.$$

Therefore $\overline{K}_M L = K_M L$. It is not difficult to prove that if K is a submodule of M, then $K_M(\_)$ is a preradical over the ring R.

Proposition 1.2 ([6, Proposition 1.3])**.** *Let $M \in R$-Mod and K, K' be submodules of M. Then*:

1. *If $K \subset K'$, then $K_M X \subset K'_M X$ for every $X \in R$-Mod.*
2. *If $X \in R$-Mod and $Y \subseteq X$, then $K_M Y \subseteq K_M X$.*
3. *$M_M X = tr^M(X)$ for every $X \in R$-Mod.*
4. *$0_M X = 0$ for every $X \in R$-Mod.*
5. *$K_M X = 0$ if and only if $f(K) = 0$ for all $f \in \mathrm{Hom}(M,X)$.*
6. *If X, Y are submodules for any module $N \in R$-Mod, then $K_M X + K_M Y \subseteq K_M(X+Y)$.*
7. *If $\{K_i\}_{i\in I}$ is a family of submodules of M, then*
$$\left[\sum_{i\in I} K_i\right]_M N = \sum_{i\in I} K_{i\,M} N.$$
8. *If $\{X_i\}_{i\in I}$ is a family of R-modules, then*
$$K_M\left[\bigoplus_{i\in I} X_i\right] = \bigoplus_{i\in I} K_M X_i.$$

Notice that in (6) the equality is not true in general. See the example given in [6, Example 1.4].

Proposition 1.3. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $\{L_i\}_{i\in I}$ a family of submodules of a module $X \in \sigma[M]$. Then $N_M(\sum_{i\in I} L_i) = \sum_{i\in I} N_M L_i$.*

Proof. By Proposition 1.2(2), we have that $\sum_{i\in I} N_M L_i \subseteq N_M(\sum_{i\in I} L_i)$. We consider the canonical epimorphism $\varphi : \bigoplus_{i\in I} L_i \to \sum_{i\in I} L_i$ and let $f : M \to \sum_{i\in I} L_i$ be a morphism. As M is projective in $\sigma[M]$ there exists $\widehat{f} : M \to \bigoplus_{i\in I} L_i$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & M & & \\ & \overset{\widehat{f}}{\swarrow} & \downarrow f & & \\ \bigoplus_{i\in I} L_i & \overset{\varphi}{\to} & \sum_{i\in I} L_i & \to & 0 \end{array}$$

So $\varphi \circ \widehat{f} = f$. Thus $f(N) = \varphi(\widehat{f}(N))$. If $x \in N$, then $f(x) = \varphi(\widehat{f}(x))$. But $\widehat{f}(x) \in \bigoplus_{i\in I} L_i$. So there exists $r \in \mathbb{N}$ such that $\widehat{f}(x) = (l_1, l_2, \ldots, l_r)$. Hence we obtain $\pi_j(\widehat{f}(x)) = l_j$ where $\pi_j : \bigoplus_{i\in I} L_i \to L_j$ is the projection on L_j. Thus we have $f(x) = \varphi(\widehat{f}(x)) = \varphi(\pi_1 \circ \widehat{f}(x), \pi_2 \circ \widehat{f}(x), \ldots, \pi_r \circ \widehat{f}(x)) =$

$\pi_1 \circ \widehat{f}(x) + \pi_2 \circ \widehat{f}(x) + \cdots + \pi_r \circ \widehat{f}(x)$ with $\pi_j \circ \widehat{f} : M \to L_j$ and $1 \leq j \leq r$. Therefore $N_M(\sum_{i \in I} L_i) \subseteq \sum_{i \in I} N_M L_i$. $\square$

Definition 1.4. Let $M \in R$-Mod. If N is a submodule of M, then successive powers of N are defined as follows: First, $N^1 = N$. Then by induction, for any integer $k \geq 2$, we define $N^k = N_M(N^{k-1})$. We say that a submodule N of M is a idempotent submodule if $N_M N = N$.

Remark 1.5. Let $M \in R\text{-}Mod$. In [4, Definition 1.1] is defined the annihilator in M of a class $\mathcal{C}$ of R-modules as $Ann_M(\mathcal{C}) = \bigcap_{K \in \Omega} K$, where $\Omega = \{K \subseteq M \mid \text{there exists } W \in \mathcal{C} \text{ and } f \in \operatorname{Hom}(M, W) \text{ with } K = \ker f\}$.

If the class $\mathcal{C}$ consists of a single module X we use the notation $Ann_M(X)$ and $Ann_M(X) = \cap \{\ker f \mid f \in \operatorname{Hom}(M, X)\}$. Also notice that if $N \subseteq M$, then $Ann_M(M/N) \subseteq N$. We proved in [6, Proposition 1.9] that if $M \in R$-Mod and $\mathcal{C}$ is a class of left R-modules, then

$$Ann_M(\mathcal{C}) = \sum \{N \subset M \mid N_M X = 0 \text{ for all } X \in \mathcal{C}\}.$$

In addition we also proved that

$$Ann_M(\mathcal{C}) = \sum \{N \subseteq_{FI} M \mid N_M X = 0 \text{ for all } X \in \mathcal{C}\}.$$

2. M-multiplication modules

Definition 2.0. Let $M \in R\text{-}Mod$ and $N \in \sigma[M]$. N is called an M-multiplication module if for every submodule L of N, there exists a submodule I of M such that $I_M N = L$.

Remark 2.1. If N is an M-multiplication module and L is a submodule of N, then there exists $I \subseteq M$ such that $I_M N = L$. By Remark 1.1, we have that $\overline{I}_M N = I_M N = L$, where $\overline{I} = I_M M$ is a fully invariant submodule of M. So I in Definition 2.0 can be chosen as a fully invariant submodule of M. We also note that if $M = R$ and $N \in R$-Mod, then N is an R-multiplication module in the sense of Definition 2.0 if and only if N is a multiplication module in the sense of [17].

Example 2.2. (a) Let $M \in R\text{-}Mod$ such that M is a duo module. Then M is an M-multiplication module. In fact let N be a submodule of M. As $N_M M = \sum_{f: M \to M} f(N)$ and N is a fully invariant submodule of M, then $N_M M \subseteq N$. On the other hand, it is clear that $N \subseteq N_M M$. Hence we have that $N_M M = N$.

(b) Let R be a ring such that $\{S_j\}_{j \in J}$ is a family of non isomorphic simple R-modules. Consider $M = \bigoplus_{j \in J} S_j$. Then N is an M-multiplication module for all submodules N of M. In fact as $N \subseteq M$, then there exists $J' \subseteq J$ such that $N = \bigoplus_{j' \in J'} S_{j'}$. Now if $L \subseteq N$ there exists $J'' \subseteq J'$ such that $L = \bigoplus_{j'' \in J''} S_{j''}$. Since $L \subseteq_{FI} M$, then $L_M N = L$.

(c) Let $M \in R\text{-}Mod$ and N a maximal submodule of M. Then $S = M/N$ is an M-multiplication module. In fact, as $S \in \sigma[M]$ and $\pi : M \to S$ is the natural projection, then we have that $M_M S = S$.

Remark 2.3. If N, $H \in \sigma[M]$ such that N is an M-multiplication module and $N \cong H$, then H is an M-multiplication module.

Proposition 2.4. *Let M be an R-module and $N \in \sigma[M]$. Then the following conditions hold.*

(i) *If N is an M-multiplication module, then N is a duo module.*
(ii) *If M is a duo module and projective in $\sigma[M]$, then M/K is M-multiplication module for all submodules K of M.*

Proof. (i) Let L be a submodule of N. As N is an M-multiplication module, then there exists a fully invariant submodule I of M such that $I_M N = L$. As $I_M N = \sum_{f:M\to N} f(I)$, then $L = \sum_{f:M\to N} f(I)$. Therefore, if $h : N \to N$, then $h(L) = h(\sum_{f:M\to N} f(I)) = \sum_{f:M\to N}(h \circ f)(I) \subseteq L$, and this proves that L is a fully invariant submodule of N.

(ii) Let L/K be a submodule of M/K. We claim that $L_M(M/K) = L/K$. In fact we know that $L_M(M/K) = \sum_{f\in \mathrm{Hom}_R(M,M/K)} f(L)$. Now let $f : M \to M/K$ be a morphism. As M is projective in $\sigma[M]$ there exists $\widehat{f} : M \to M$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & M & & \\ & \overset{\widehat{f}}{\swarrow} & \downarrow f & & \\ M & \overset{\pi}{\to} & M/K & \to & 0 \end{array}$$

where π is the natural projection. Thus $\pi \circ \widehat{f} = f$. Since M is a duo module, then $\widehat{f}(L) \subseteq L$. So we have that $f(L) = \pi \circ \widehat{f}(L) \subseteq \pi(L) = L/K$. Hence, we obtain that $L_M(M/K) \subseteq L/K$. On the other hand we know that $\pi \in \mathrm{Hom}_R(M, M/K)$. Thus $\pi(L) = L/K$. Therefore, $L/K \subseteq L_M(M/K)$. So we obtain that $L_M(M/K) = L/K$. Thus M/K is an M-multiplication module. □

The converse of Proposition 2.4(i) is not true in general. Consider the following example.

Example 2.5. Let $M = \mathbb{Z}_{p^\infty}$. Now let N be a proper submodule of M, it is clear that N is a duo module and $N \in \sigma[M]$. If $N = \mathbb{Z}_{p^n}$, then $f = 0$ for all morphisms $f : \mathbb{Z}_{p^\infty} \to \mathbb{Z}_{p^n}$. Thus $I_M N = I_{\mathbb{Z}_{p^\infty}} \mathbb{Z}_{p^n} = \sum_{f:\mathbb{Z}_{p^\infty}\to\mathbb{Z}_{p^n}} f(I) = 0$ for all submodules I of M. Therefore, N is not an M-multiplication module for all $N \subsetneq M$. We also note that by Example 2.2(a), M is the only one M-multiplication submodule of M.

Notice that each simple R-module S is a multiplication module in the sense given in [17]. But in general, is not true that each simple R-module $S \in \sigma[M]$

is an M-multiplication module. In fact, if $M = \mathbb{Z}_{p^\infty}$ as in Example 2.5 and $S = \mathbb{Z}_p$, then $I_M S = 0$ for all submodules $I \subseteq M$. So S is not an M-multiplication module.

On the other hand, note that if S is a simple module such that $S \in \sigma[M]$ and there exists a nonzero morphism $f : M \to S$, then S is an M-multiplication module. In fact, since S is a simple module, then $M_M S = S$. Hence S is an M-multiplication module.

Proposition 2.6. *Let $M \in R$-Mod be a generator projective in $\sigma[M]$ and $N \in \sigma[M]$. The following conditions are equivalent*:

(i) *N is an M-multiplication module.*

(ii) *If $B \subseteq_{FI} M$ and $B \subseteq Ann_M(N)$, then N is an $\frac{M}{B}$-multiplication module.*

Proof. (i)⇒(ii) It is clear that $B \subseteq Ann_M(N)$ implies $B_M N = 0$. By [7, Proposition 1.5], it follows $N \in \sigma\left[\frac{M}{B}\right]$. Let $L \subseteq N$ be a submodule, as N is an M-multiplication module, then there exists $I \subseteq_{FI} M$ such that $L = I_M N$. We will prove that $(\frac{I+B}{B})_{\frac{M}{B}} N = L$. Let $f : M \to N$ be a morphism. As $B_M N = 0$, then $h(B) = 0$ for all $h : M \to N$. Thus, $B \subseteq \ker f$. Hence we can define $\overline{f} : M/B \to N$ such that $\overline{f}(m + B) = f(m)$. Therefore, $\overline{f}(\frac{I+B}{B}) = f(I + B) = f(I)$. On the other hand we know that $(\frac{I+B}{B})_{\frac{M}{B}} N = \sum_{g:\frac{M}{B}\to N} g(\frac{I+B}{B})$, then $L = I_M N = \sum_{f:M\to N} f(I) \subseteq \sum_{g:\frac{M}{B}\to N} g(\frac{I+B}{B})$. Thus $L \subseteq (\frac{I+B}{B})_{\frac{M}{B}} N$. Now if $g : \frac{M}{B} \to N$ and $\pi : M \to \frac{M}{B}$ is the natural projection, then $g \circ \pi : M \to N$ and $(g \circ \pi)(I) = g(\pi(I)) = g(\frac{I+B}{B}) \subseteq \sum_{f:M\to N} f(I) = I_M N = L$. Therefore $(\frac{I+B}{B})_{\frac{M}{B}} N = \sum_{g:\frac{M}{B}\to N} g(\frac{I+B}{B}) \subseteq L$, and we have proved $(\frac{I+B}{B})_{\frac{M}{B}} N = L$.

(ii)⇒(i) Suppose that N is an $\frac{M}{B}$-multiplication module and let L be a submodule of N, then there exists $I/B \subseteq M/B$ such that $(\frac{I}{B})_{\frac{M}{B}} N = L$. We will prove that $L = I_M N$. In fact let $g : \frac{M}{B} \to N$ be a morphism. Now we consider the morphism $g \circ \pi : M \to N$ with π the natural projection. So we have that $(g \circ \pi)(I) = g(I/B)$. $L = (\frac{I}{B})_{\frac{M}{B}} N = \sum_{g:\frac{M}{B}\to N} g(\frac{I}{B}) \subseteq \sum_{f:M\to N} f(I) = I_M N$, then $L \subseteq I_M N$.

On the other hand let $f : M \to N$ be a morphism. As $B \subseteq Ann_M(N)$, then we can define $\overline{f} : M/B \to N$ as above, and $\overline{f}(\frac{I}{B}) = f(I)$. Hence $I_M N = \sum_{f:M\to N} f(I) \subseteq \sum_{g:\frac{M}{B}\to N} g(\frac{I}{B}) = (\frac{I}{B})_{\frac{M}{B}} N = L$. Thus $L = I_M N$. □

Definition 2.7. Let $M \in R$-Mod be a projective module in $\sigma[M]$ and $N \in \sigma[M]$. If X and Y are submodules of N, we define,

$$(Y : X) = \sum \{K \subseteq M \mid K_M X \subseteq Y\}.$$

Remark 2.8. By Proposition 1.2, we have that $(Y : X)$ is the maximal submodule of M such that $(Y : X)_M X \subseteq Y$. Moreover by Remark 1.1, we have that $K_M X = \overline{K}_M X$, where $\overline{K}$ is the smallest fully invariant submodule of M that contains K. So $(Y : X) = \sum\{K \subseteq_{F.I} M \mid K_M X \subseteq Y\}$. Therefore $(Y : X) \subseteq_{F.I} M$. Now if $L \subseteq N$ is a submodule, then $(L : N) = Ann_M(N/L)$.

In fact as $(L:N)_M N \subseteq L$, then by [4, Proposition 5.5] we have that

$$(L:N)_M(N/L) = 0.$$

Thus $(L:N) \subseteq Ann_M(N/L)$. Since $[Ann_M(N/L)]_M (N/L) = 0$, then by [4, Proposition 5.5] $Ann_M(N/L)_M N \subseteq L$. Hence $Ann_M(N/L) \subseteq (L:N)$. So we obtain that $(L:N) = Ann_M(N/L)$.

Proposition 2.9. *Let $M \in R$-Mod be a projective module in $\sigma[M]$ and $N \in \sigma[M]$. Then the following conditions are equivalent*:

(1) *N is an M-multiplication module.*

(2) *$L \subseteq (L:N)_M N$ for all submodules L of N.*

(3) *$L = (L:N)_M N = [Ann_M(N/L)]_M N$ for all submodules L of N.*

Proof. (1) $\Rightarrow$ (2) Let L be a submodule of N. As N is an M-multiplication module, then there exists $I \subseteq_{FI} M$ such that $L = I_M N$. So $I \subseteq (L:N)$. Now by Proposition 1.2, we have that $L = I_M N \subseteq (L:N)_M N$.

(2) $\Rightarrow$ (3) If L is a submodule of N, it is clear $(L:N)_M N \subseteq L$. Thus by 2) we have that $L = (L:N)_M N$. Moreover, by Remark 2.8, we have that $(L:N) = Ann_M(N/L)$. Therefore $(L:N)_M N = [Ann_M(N/L)]_M N$ and so $L = (L:N)_M N = [Ann_M(N/L)]_M N$.

(3) $\Rightarrow$ (1) Since $(L:N)$ is a submodule of N and $L = (L:N)_M N$ for all submodules L of N, then N is an M-multiplication module. □

Proposition 2.10. *Let $M \in R$-Mod be a projective module in $\sigma[M]$ and $N \in \sigma[M]$. If N is an M-multiplication module, then N/N' is an M-multiplication module for all submodules N' of N.*

Proof. Let $\widetilde{L} = L/N'$ be a submodule of $N/N' = \widetilde{N}$. As N is an M-multiplication module, then by Proposition 2.9 $[Ann_M(N/L)]_M N = L$. We claim that $\left[Ann_M(\widetilde{N}/\widetilde{L})\right]_M \widetilde{N} = \widetilde{L}$. In fact we know that $\left[Ann_M(\widetilde{N}/\widetilde{L})\right]_M \widetilde{N} = \sum_{f:M\to\widetilde{N}} f(Ann_M(\widetilde{N}/\widetilde{L}))$. Now let $f: M \to \widetilde{N}$ be a morphism. Since M is projective in $\sigma[M]$, then there exists a morphism $\overline{f}: M \to N$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & M & & \\ & \overset{\overline{f}}{\swarrow} & \downarrow f & & \\ N & \overset{\pi}{\to} & \widetilde{N} & \to & 0 \end{array}$$

where π is the natural projection.

Thus $\pi \circ \overline{f} = f$ and $(\pi \circ \overline{f})(Ann_M(\widetilde{N}/\widetilde{L})) = f(Ann_M(\widetilde{N}/\widetilde{L}))$. Moreover it is clear that $Ann_M(\widetilde{N}/\widetilde{L}) = Ann_M(N/L)$. Since N is an M-multiplication module, then by Proposition 2.9 we have that

$$L = [Ann_M(N/L)]_M N = \sum_{h:M\to N} h(Ann_M(N/L)).$$

Hence

$$\begin{aligned}\widetilde{L} = \pi(L) &= \pi(\sum_{h:M\to N} h(Ann_M(N/L))) \\ &= \sum_{\pi\circ h:M\to N/N'} (\pi\circ h)(Ann_M(N/L)) \\ &\subseteq \sum_{f:M\to \overline{N}} f(Ann_M(\widetilde{N}/\widetilde{L})) = \left[Ann_M(\widetilde{N}/\widetilde{L})\right]_M \widetilde{N}.\end{aligned}$$

Therefore $\overline{L} \subseteq \left[Ann_M(\widetilde{N}/\widetilde{L})\right]_M \widetilde{N}$. On the other hand as $\overline{f} : M \to N$, then $\overline{f}(Ann_M(N/L)) \subseteq \sum_{h:M\to N} h(Ann_M(N/L)) = [Ann_M(N/L)]_M N = L$. Hence we obtain that $\pi(\overline{f}(Ann_M(N/L))) \subseteq \pi(L) = L/N' = \widetilde{L}$. Thus $f(Ann_M(N/L)) \subseteq \widetilde{L}$ for all morphism $f : M \to \widetilde{N}$. So $\sum_{f:M\to\overline{N}} f(Ann_M(N/L)) \subseteq \widetilde{N}$. Therefore $[Ann_M(N/L)]_M \widetilde{N} \subseteq \widetilde{L}$.

Hence $[Ann_M(N/L)]_M \widetilde{N} = \widetilde{L}$. So by Proposition 2.9, we have the result. □

Notice that if N is an M-multiplication module, then $f(N)$ is also an M-multiplication module, for all $f \in End_R(N)$.

Corollary 2.11. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $N \in \sigma[M]$. Suppose that N is an M-multiplication module and L is a direct summand of N. Then L is an M-multiplication module.*

Proof. It follows from Remark 2.3 and Proposition 2.10. □

Lemma 2.12. *Let $M \in R$-Mod and $N \in \sigma[M]$ such that N is an M-multiplication module. Then N is an M-generated module.*

Proof. Since N is an M-multiplication module, then there exists a submodule I of M, such that $I_M N = N$. As $I_M N = \sum_{f:M\to N} f(I)$, then $N = \sum_{f:M\to N} f(I)$. On the other hand we know that the trace of M in N is $tr^M(N) = \sum_{g:M\to N} g(M)$. So $f(I) \subseteq f(M)$ for all $f \in \mathrm{Hom}_R(M, N)$. Hence $N = \sum_{f:M\to N} f(I) \subseteq \sum_{g:M\to N} g(M) \subseteq N$. Therefore $N = tr^M(N)$. So N is M-generated. □

Proposition 2.13. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $N \in \sigma[M]$ an M-multiplication module. Let L be a proper submodule of N and $S = End_R(N/L)$. If $f \in S$ is idempotent, then f is central.*

Proof. We have that N is an M-multiplication module, then N/L is an M-multiplication module. Thus N/L is a duo module. Hence it is easy to show that f is central. □

Proposition 2.14. *Let M be a FI-simple module. If $N \in \sigma[M]$ is an M-multiplication module, then N is a simple module.*

Proof. Let L be a submodule of N. As N is an M-multiplication module, then there exists $I \subseteq M$ such that $I_M N = L$. By Remark 1.1, we know that I is a fully invariant submodule of M. Since M is FI-simple, then $I = 0$ or $I = M$. If $I = 0$, then $I_M N = 0$. If $I = M$, then $L = I_M N = M_M N = \sum_{f:M\to N} f(M) = tr^M(N)$. By Lemma 2.12, we have that $tr^M(N) = N$. Therefore $L = N$. Hence N is a simple module. □

Note that in Proposition 2.14, N is isomorphic to M/K where K is a maximal submodule of M. In fact as N is a simple module, then by Lemma 2.12, there exists a non zero morphism $f : M \to N$. Since N is a simple module, then $N \cong M/\ker f$. Hence $\ker f$ is the maximal submodule of M.

Lemma 2.15. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $N \in \sigma[M]$ an M-multiplication module. If I is a maximal fully invariant submodule of M and $I_M N \neq N$, then $I_M N$ is a maximal submodule of N. Moreover there exists a cyclic submodule X of N such that $M = I + Ann_M(N/X)$.*

Proof. Let L be a submodule of N such that $I_M N \subseteq L$. As N is an M-multiplication module, then by Remark 1.1, there exists a fully invariant submodule J of M such that $J_M N = L$. If $J \subseteq I$, then by Proposition 1.2, we have that $J_M N \subseteq I_M N$. Hence we have that $L \subseteq I_M N \subseteq L$. Thus $I_M N = L$. If $J \nsubseteq I$, then $I + J = M$ since $I + J$ is a fully invariant submodule of M. Hence we obtain that $(I + J)_M N = M_M N$. As M generates N, then $M_M N = N$. On the other hand by Proposition 1.2, $N = (I + J)_M N = I_M N + J_M N$. Since $J_M N = L$, then $N = I_M N + L$. But $I_M N \subseteq L$. Therefore $N = L$. Now by Proposition 1.14, there exists X a cyclic submodule of N such that $Ann_M(N/X) \nsubseteq I$. Since $Ann_M(N/X)$ is a fully invariant submodule of M and I is a maximal fully invariant submodule of M, then $M = I + Ann_M(N/X)$. □

Definition 2.16. Let $M \in R$-Mod. We say that M has commutative multiplication of fully invariant submodules if $N_M L = L_M N$ for all fully invariant submodules N and L of M.

Example 2.17. (a) If S is an FI-simple R-module, then S has commutative multiplication of fully invariant submodules.

(b) If M is a duo and co-semisimple R-module , then M has commutative multiplication of submodules. In fact by [11, Proposition 5.4], we have that every submodule N of M is idempotent. Therefore $(N \cap L)^2 = N \cap L$ for all N and L submodules of M. Since N is a fully invariant submodule of M, then $N_M L \subseteq N \cap L$. On the other hand by Proposition 1.2, we have that $(N \cap L)^2 = [N \cap L]_M [N \cap L] \subseteq N_M L$. Hence we obtain that $N_M L = N \cap L$. Therefore $N_M L = L_M N$ for all submodules N and L of M. So M has commutative multiplication of fully invariant submodules. Note that in Examples 2.2(b) the module $M = \bigoplus_{j \in J} S_j$ is a duo and co-semisimple R-module.

(c) If R is a commutative ring and M is a multiplication R-module, then $N_M L = L_M N$ for all submodules N and L of M.

Proposition 2.18. *Let $M \in R$-Mod be projective in $\sigma[M]$. Suppose that M has commutative multiplication of fully invariant submodules. If N is an M-multiplication module and I is a fully invariant submodule of M such that $I_M N = N$, then $I_M L = L$ for all $L \subseteq N$.*

Proof. Let $L \subseteq N$. As N is an M-multiplication module, then by Remark 1.1, there exists a fully invariant submodule J of M such that $L = J_M N$. As $I_M N = N$, then $L = J_M(I_M N) = (J_M I)_M N$ since M is projective in $\sigma[M]$. As M has commutative multiplication of fully invariant submodules, then $L = (J_M I)_M N = (I_M J)_M N = I_M(J_M N) = I_M L$. □

Proposition 2.19. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $N \in \sigma[M]$ an M-multiplication module. Suppose that M has commutative multiplication of fully invariant submodules. If P is a fully invariant submodule of M, then the following conditions are equivalent*:

(i) $N = P_M N$.
(ii) $L = P_M L$ *for all submodules* L *of* N.
(iii) $X = P_M X$ *for all cyclic submodule* X *of* N.

Proof. (i)⇒(ii) By Proposition 2.18.
(ii)⇒(iii) It is clear.
(iii)⇒(i) As $N = \sum_{n \in N} Rn$, then $P_M N = P_M(\sum_{n \in N} Rn)$. By Proposition 1.3, we have that $P_M(\sum_{n \in N} Rn) = \sum_{n \in N} P_M(Rn)$. Now by (iii) $\sum_{n \in N} P_M(Rn) = \sum_{n \in N} Rn = N$. □

3. Main results

Proposition 3.0. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $N \in \sigma[M]$. The following conditions are equivalent*:

(i) *N is an M-multiplication module.*
(ii) *For every cyclic submodule X of N, there exists a submodule I of M such that $X = I_M N$.*
(iii) *For every submodule X of N, there exists a set $\{X_j\}_{j \in J}$ of submodules of X and a set $\{I_j\}_{j \in J}$ of fully invariant submodules of M such that $X = \sum_{j \in J} X_j$ and $X_j = (I_j)_M N$ for each $j \in J$.*

Proof. (i)⇒(ii) It is clear.
(ii)⇒(iii) Let X be a submodule of M, and let $\{X_j\}_{j \in J}$ be the set of all cyclic submodules of X. By hypothesis, we know that for every X_j there exists a submodule I_j of M such that $X_j = (I_j)_M N$ for each $j \in J$. Since $X = \sum_{j \in J} X_j$, then $\{X_j\}_{j \in J}$ and $\{I_j\}_{j \in J}$ are the required sets.
(iii)⇒(i) Let X be a submodule of N. By (iii) there exists a set $\{X_j\}_{j \in J}$ of submodules of X and a set $\{I_j\}_{j \in J}$ of fully invariant submodules of M such that $X = \sum_{j \in J} X_j$ and $X_j = (I_j)_M N$ for each $j \in J$. Denote $I = \sum_{j \in J} I_j$, then I is a fully invariant submodule of M. Thus we obtain that $X = \sum_{j \in J} X_j = \sum_{j \in J} (I_j)_M N$. By Proposition 1.2, we have that $X =$

$\sum_{j\in J}(I_j)_M N = (\sum_{j\in J} I_j)_M N = I_M N$. So N is an M-multiplication module. □

Proposition 3.1. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $N \in \sigma[M]$. Suppose that $N = \bigoplus_{j\in J} N_j$. Then the following conditions are equivalent:*

(i) *N is an M-multiplication module.*

(ii) (a) *N is a duo module.*

(b) *For every $j \in J$, the module N_j is an M-multiplication module and there exists a fully invariant submodule B_j of M such that $N_j = (B_j)_M N$.*

(iii) (a) *$L = \bigoplus_{j\in J}(L \cap N_j)$ for every submodule L of N.*

(b) *For every $j \in J$ the module N_j is an M-multiplication module and there exists a fully invariant submodule B_j of M such that $N_j = (B_j)_M N$.*

(iv) *For every subset J' of J, the module $\bigoplus_{j'\in J'} N_{j'}$ is an M-multiplication module such that $\bigoplus_{j'\in J'} N_{j'} = (B_{J'})_M N$ for some a fully invariant submodule $B_{J'}$ of M.*

Proof. (i)⇒(ii) By Proposition 2.4, N is a duo module and by Corollary 2.11, every N_j is an M-multiplication module. Moreover, since N is an M-multiplication module it is clear that there exists a fully invariant submodule B_j of M with $N_j = (B_j)_M N$ for all $j \in J$.

(ii)⇒(iii) Let L be a submodule of N and let $\pi_j : N \to N_j$ be the natural projection. Since N is duo module, then L is a fully invariant submodule of N, then $\pi_j(L) \subseteq L$. Now let $l \in L$. As $L \subseteq N$, then $l = n_1 + n_2 + \cdots + n_r$ with $n_j \in N_j$. Hence we obtain that $\pi_j(l) = n_j \in N_j \cap L$. Thus $l = \pi_1(l) + \pi_2(l) + \cdots + \pi_r(l) \subseteq (N_1 \cap L) \oplus \cdots \oplus (N_r \cap L) = \pi_1(L) \oplus \pi_2(L) \oplus \cdots \oplus \pi_r(L)$. Therefore we obtain that $L = \bigoplus_{j\in J}(L \cap N_j)$.

(iii)⇒(i) Let L be a submodule of N. By (iii) we know that $L = \bigoplus_{j\in J}(L \cap N_j)$. Let $L_j = L \cap N_j$. As every N_j is an M-multiplication module, then there exists a fully invariant submodule C_j of M such that $L_j = (C_j)_M N_j$. Moreover by (iii) there exists a fully invariant submodule B_j of M with $N_j = (B_j)_M N$ for all $j \in J$. Thus we obtain that $L_j = (C_j)_M N_j = (C_j)_M [(B_j)_M N]$. Since M is projective in $\sigma[M]$, then

$$L_j = (C_j)_M [(B_j)_M N] = [(C_j)_M B_j]_M N.$$

On the other hand we have that $(C_j)_M B_j$ is a fully invariant submodule of M. Now let $D_j = (C_j)_M B_j$. Since $L = \bigoplus_{j\in J} L_j$, then

$$L = \bigoplus_{j\in J} [(C_j)_M B_j]_M N = \bigoplus_{j\in J} (D_j)_M N.$$

Now by Proposition 1.2, we have that $L = \bigoplus_{j\in J}(D_j)_M N = (\bigoplus_{j\in J} D_j)_M N$. As $\bigoplus_{j\in J} D_j$ is a fully invariant submodule of M, then N is an M-multiplication module.

(i)⇒(iv) As $N = \bigoplus_{j\in J} N_j$ is an M-multiplication module, then for all subset J' of J there exists a fully invariant submodule $B_{J'}$ of M such that $\bigoplus_{j'\in J'} N_{j'} = (B_{J'})_M N$. Furthermore we have that $\bigoplus_{j'\in J'} N_{j'}$ is a direct summand of N, so by Corollary 2.11, $\bigoplus_{j'\in J'} N_{j'}$ is an M-multiplication module.

(iv)⇒(i) Let X be a cyclic submodule of N. As $N = \bigoplus_{j\in J} N_j$, then there exists a subset J' of J such that $X \subseteq \bigoplus_{j'\in J'} N_{j'}$. By hypothesis we know that $\bigoplus_{j'\in J'} N_{j'}$ is an M-multiplication module such that $\bigoplus_{j'\in J'} N_{j'} = (B_{J'})_M N$ for some fully invariant submodule $B_{J'}$ of M. Since $\bigoplus_{j'\in J'} N_{j'}$ is an M-multiplication module, then there exists a fully invariant submodule $C_{J'}$ of M such that $X = (C_{J'})_M(\bigoplus_{j'\in J'} N_{j'})$. Thus we obtain that $X = (C_{J'})_M\left[(B_{J'})_M N\right] = \left[(C_{J'})_M(B_{J'})\right]_M N$. By Proposition 3.0, N is an M-multiplication module. □

Remark 3.2. If N is an M-multiplication module and $L \subseteq N$ is a submodule, then there exists $I \subseteq_{FI} M$ such that $L = I_M N$. Now let $I' = I + Ann_M(N)$. Hence $I'_M N = \left[I + Ann_M(N)\right]_M N$. By Proposition 1.2, we have that $\left[I + Ann_M(N)\right]_M N = (I_M N) + (Ann_M(N)_M N) = I_M N = L$. So we can suppose that $Ann_M(N) \subseteq I$. Moreover, if $\mathfrak{F}_L = \{I \subseteq_{FI} M \mid I_M N = L\}$, then $J = \sum_{I\in\mathfrak{F}_L} I$ is a fully invariant submodule of M and by Proposition 1.2, $J_M N = (\sum_{I\in\mathfrak{F}_L} I)_M N = \sum_{I\in\mathfrak{F}_L} I_M N = L$. Therefore, there exists a maximal fully invariant submodule J of M such that $J_M N = L$.

In the next proposition we characterize this module.

Proposition 3.3. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $N \in \sigma[M]$ an M-multiplication module. If L is a submodule of N, then $Ann_M(N/L)$ is the maximal submodule of M such that $Ann_M(N/L)_M N = L$.*

Proof. It is clear that $Ann_M(N/L)_M(N/L) = 0$. By [6, Proposition 5.5], we have that $Ann_M(N/L)_M N \subseteq L$. Now, as N is an M-multiplication module, then there exists $I \subseteq_{FI} M$ such that $I_M N = L$. Now by [4, Proposition 5.5], we have that $I_M(N/L) = 0$. Thus $I \subseteq Ann_M(N/L)$. Hence by Proposition 1.2, we obtain that $L = I_M N \subseteq Ann_M(N/L)_M N \subseteq L$. Therefore $Ann_M(N/L)_M N = L$. □

Corollary 3.4. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $N \in \sigma[M]$. Suppose that $N = \bigoplus_{i\in I} N_i$ with $card(I) \geq 2$ and that N is an M-multiplication module, then $N_i = \left[Ann_M(\bigoplus_{j\neq i} N_j)\right]_M N_i$ for every $i \in I$.*

Proof. Let $i \in I$, we know that $\bigoplus_{j\neq i} N_j \cong N/N_i$. Thus $Ann_M(\bigoplus_{j\neq i} N_j) = Ann_M(N/N_i)$. Now by Proposition 3.3, we have that $Ann_M(N/N_i)_M N = N_i$. So $\left[Ann_M(\bigoplus_{j\neq i} N_j)\right]_M N = N_i$. On the other hand by Proposition 1.2, we obtain that

$$\begin{aligned}\left[Ann_M(\bigoplus_{j\neq i} N_j)\right]_M N &= \left[Ann_M(\bigoplus_{j\neq i} N_j)\right]_M (\bigoplus_{i\in I} N_i)\\ &= \bigoplus_{i\in I}\left[Ann_M(\bigoplus_{j\neq i} N_j)_M N_i\right]\\ &= Ann_M(\bigoplus_{j\neq i} N_j)_M N_i.\end{aligned}$$

Therefore $\left[Ann_M(\bigoplus_{j\neq i} N_j)\right]_M N_i = N_i$ for every $i \in I$. □

Proposition 3.5. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $N \in \sigma[M]$ an M-multiplication module. Suppose that $N = X \oplus Y$. The following assertions hold*:

(i) *Let P in $\sigma[M]$ and $\alpha : P \to X$ and $\beta : P \to Y$ epimorphisms, then the homomorphism $\alpha + \beta : P \to N$ is an epimorphism.*

(ii) *If the modules X and Y are cyclic modules, then N is a cyclic module.*

Proof. (i) Let $Z = (\alpha+\beta)(P)$. Now we consider $\pi_X : N \to X$ and $\pi_Y : N \to Y$ to be the natural projections. Thus $\pi_X(Z) = \pi_X((\alpha+\beta)(P)) = \alpha(P) = X$. Analogously $\pi_Y(Z) = \beta(P) = Y$. By Proposition 3.1(iii), we have that $Z = (X\cap Z)\oplus(Y\cap Z)$. Thus $X = \pi_X(Z) = X\cap Z \subseteq Z$ and $Y = \pi_Y(Z) = Y\cap Z \subseteq Z$. Hence we obtain that $N = X \oplus Y = (X \cap Z) \oplus (Y \cap Z) \subseteq Z \subseteq N$. Therefore $N = Z = (\alpha + \beta)(P)$. So $\alpha + \beta : P \to N$ is an epimorphism.

(ii) As X and Y are cyclic modules, then there exist epimorphisms $\alpha : R \to X$ and $\beta : R \to Y$. So by (i) we have that $\alpha + \beta : R \to N$ is an epimorphism. Thus N is a cyclic module. □

Notice that if N is an M-multiplication module and $N = \bigoplus_{i=1}^n X_i$ with $X_i \cong M/K_i$ for every $1 \leq i \leq n$, then there are epimophisms $\pi_i : M \to X_i$. Hence by Proposition 3.5(i), we have an epimorphism $\Pi : M \to N$. So N is a quotient of M.

Corollary 3.6. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $N \in \sigma[M]$ an M-multiplication module. If N is a direct sum of finitely many cyclic modules, then N is a cyclic module.*

Proof. The proof follows from Proposition 3.5(ii). □

Let $\mathcal{S}$ denote the class of simple modules in $\sigma[M]$, and let $\mathcal{J}$ denote the radical cogenerated by $\mathcal{S}$, so that for an R-module $N \in \sigma[M]$, the submodule $\mathcal{J}(N)$ is the intersection of all maximal submodules of N. If N has no maximal submodules, $\mathcal{J}(N) = N$.

Note that if $N \in \sigma[M]$ is finitely generated, then $\mathcal{J}(N)$ is a superfluous submodule in N. Also notice that $\mathcal{J}(M)_M X \subseteq \mathcal{J}(X)$ for all $X \in R$-Mod, in fact as $\mathcal{J}(M)_M X = \sum_{f:M\to X} f(\mathcal{J}(N))$ and $\mathcal{J}$ is a radical, then $f(\mathcal{J}(M)) \subseteq \mathcal{J}(X)$ for all morphism $f : M \to X$. Moreover if X is a finitely generated R-module, then $\mathcal{J}(M)_M X$ is a superfluous submodule of X since $\mathcal{J}(X)$ is a superfluous submodule in X.

Theorem 3.7. *Let $M \in R$-Mod be projective in $\sigma[M]$ and $N \in \sigma[M]$ an M-multiplication module. Suppose that N is an artinian module, then the following assertions hold*:

(i) *$N/\mathcal{J}(N)$ is a cyclic module.*
(ii) *If $\mathcal{J}(N)$ is a superfluous submodule of N, then N is a cyclic module.*
(iii) *If N is a finitely generated module, then N is a cyclic module.*

Proof. (i) We know by Proposition 2.10 that $N/\mathcal{J}(N)$ is an M-multiplication module. Since $N/\mathcal{J}(N)$ is an artinian module and $\mathcal{J}(N/\mathcal{J}(N)) = 0$, then by [16, Corollary 1.8, p. 180], $N/\mathcal{J}(N)$ is a semisimple module of finite length. Hence we obtain that $N/\mathcal{J}(N)$ is a finitely generated module. Thus $N/\mathcal{J}(N)$ is a direct sum of finitely many simple modules. Therefore $N/\mathcal{J}(N)$ is a direct sum of finitely many cyclic modules. By Corollary 3.6, we have that N is a cyclic module.

(ii) By (i) we obtain that $N/\mathcal{J}(N)$ is a cyclic module. Thus there exists an element $x + \mathcal{J}(N) \in N/\mathcal{J}(N)$ such that $R(x + \mathcal{J}(N)) = N/\mathcal{J}(N)$. Therefore $Rx + \mathcal{J}(N) = N$, but $\mathcal{J}(N)$ is a superfluous submodule of N. So we obtain that $Rx = N$ is a cyclic module.

(iii) If N is a finitely generated module, then $\mathcal{J}(N)$ is a superfluous submodule of N. Now by (ii) N is a cyclic submodule. □

Lemma 3.8. *Let $\{M_1, M_2, \ldots, M_n\}$ be a family of R-modules such that $\bigoplus_{i=1}^n M_i$ is projective in $\sigma[\bigoplus_{i=1}^n M_i]$. Suppose that M_i generates $\sigma[M_i]$ for all $i = 1, 2, \ldots, n$. If $\mathrm{Hom}_R(M_i, M_j) = 0$ for all $i \neq j$, then $\mathrm{Hom}_R(N_i, L_j) = 0$ for all $N_i \in \sigma[M_i]$ and $L_j \in \sigma[M_j]$.*

Proof. Since N_i is M_i-generated and L_j is M_j-generated, then there exists X_i and X_j sets such that $N_i \cong \frac{M_i^{(X_i)}}{K_i}$ and $L_j \cong \frac{M_j^{(X_j)}}{K_j}$. Now let $f : N_i \to L_j$ be a non-zero morphism. Now let $\left\{u_x : M_i \to M_i^{(X_i)} \mid x \in X_i\right\}$ be the inclusions and $\pi_i : M_i^{(X_i)} \to \frac{M_i^{(X_i)}}{K_i}$ the canonical projection. As $f \neq 0$ there exists $x \in X_i$ such that $F_x = f \circ \pi_i \circ u_x \neq 0$. As $\bigoplus_{i=1}^n M_i$ is projective in $\sigma[\bigoplus_{i=1}^n M_i]$, then M_i is projective in $\sigma[\bigoplus_{i=1}^n M_i]$ for all $i = 1, 2, \ldots, n$. Therefore there exists a morphism $\overline{F} : M_i \to M_j^{(X_j)}$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & M_i & & \\ & \overset{\overline{F}}{\swarrow} & \downarrow F_x & & \\ M_j^{(X_j)} & \overset{\pi_j}{\to} & \frac{M_j^{(X_j)}}{K_j} & \to & 0 \end{array}$$

Hence we obtain that $\overline{F} \neq 0$. So $\mathrm{Hom}_R(M_i, M_j) \neq 0$, this is a contradiction. □

Notice that if we have the conditions of Lemma 3.8, then:

Proposition 3.9. *Let $\{M_1, M_2, \ldots, M_n\}$ be a family of R-modules, such that $M = \bigoplus_{i=1}^n M_i$ is a progenerator in $\sigma[\bigoplus_{i=1}^n M_i]$. Suppose that M_i generates $\sigma[M_i]$ for all $1 \leq i \leq n$ and $\mathrm{Hom}_R(M_i, M_j) = 0$ for all $i \neq j$. Then $N = \bigoplus_{i=1}^n [(M_i)_{M_i} N]$ for all $N \in \sigma[M]$.*

Proof. Let $N \in \sigma[M]$. Since M is a generator of $\sigma[M]$, then $N = M_M N$. By Proposition 1.2, we have that $N = M_M N = (\bigoplus_{i=1}^n M_i)_{(\bigoplus_{i=1}^n M_i)} N = \sum_{i=1}^n \left[(M_i)_{(\bigoplus_{i=1}^n M_i)} N\right]$. Now we claim that $(M_j)_{(\bigoplus_{i=1}^n M_i)} N = (M_j)_{M_j} N$. In fact we know that $(M_j)_{(\bigoplus_{i=1}^n M_i)} N = \sum_{f: \bigoplus_{i=1}^n M_i \to N} f(M_j) = \sum_{f: M_j \to N} f(M_j)$ $= (M_j)_{M_j} N$ for all $1 \leq i \leq n$. Thus we obtain that $N = \sum_{i=1}^n (M_i)_{M_i} N$.

Now as $(M_i)_{M_i} N \subseteq M_i$, then by [18, Proposition 3.10], the sum

$$\sum_{i=1}^n (M_i)_{M_i} N$$

is direct. Therefore we have that $N = \bigoplus_{i=1}^n [(M_i)_{M_i} N]$. □

Notice that in general for a family $\{X_i\}_{i=1}^n \subseteq R$-Mod and $N \in \sigma[\bigoplus_{i=1}^n X_i]$, the sum $\sum_{i=1}^n (X_i)_{M_i} N$ is not a direct sum. Consider the following example.

Example 3.10. Let $R = \mathbb{Z}$, $M_1 = \mathbb{Z}_2$, $M_2 = \mathbb{Q}$, $M = \mathbb{Z}_2 \oplus \mathbb{Q}$ and $N = \mathbb{Z}_{2^\infty}$. We have that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Q}) = 0$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_2) = 0$. We know that there exists an epimorphism $f : \mathbb{Q} \to \mathbb{Z}_{2^\infty}$. Thus we have that $\mathbb{Q}_{\mathbb{Q}}(\mathbb{Z}_{2^\infty}) = \mathbb{Z}_{2^\infty}$. Hence we also obtain that $\mathbb{Z}_{2^\infty} = (\mathbb{Z}_2 \oplus \mathbb{Q})_{\mathbb{Z}_2 \oplus \mathbb{Q}} \mathbb{Z}_{2^\infty}$. On the other hand by Proposition 1.2, we have that $(\mathbb{Z}_2 \oplus \mathbb{Q})_{\mathbb{Z}_2 \oplus \mathbb{Q}} \mathbb{Z}_{2^\infty} = (\mathbb{Z}_2)_{\mathbb{Z}_2 \oplus \mathbb{Q}} \mathbb{Z}_{2^\infty} + (\mathbb{Q})_{\mathbb{Z}_2 \oplus \mathbb{Q}} \mathbb{Z}_{2^\infty} = (\mathbb{Z}_2)_{\mathbb{Z}_2} \mathbb{Z}_{2^\infty} + \mathbb{Q}_{\mathbb{Q}} \mathbb{Z}_{2^\infty} = \mathbb{Z}_2 + \mathbb{Z}_{2^\infty}$. But this sum is not a direct sum.

Remark 3.11. In [14] the authors showed that, if M is an R-module and N is a fully invariant submodule of M, then there exists a preradical r such that $r(M) = N$. Now if $M = \bigoplus_{i=1}^n M_i$, we know that $N = r(\bigoplus_{i=1}^n M_i) = \bigoplus_{i=1}^n r(M_i)$ and each $r(M_i)$ is a fully invariant submodule of M_i for all $i = 1, 2, \ldots, n$.

Theorem 3.12. *Let $\{M_1, M_2, \ldots, M_n\}$ be a family of R-modules, such that $M = \bigoplus_{i=1}^n M_i$ is a projective generator in $\sigma[M]$. Suppose that M_i generates $\sigma[M_i]$ for all $1 \leq i \leq n$ and $\mathrm{Hom}_R(M_i, M_j) = 0$ for all $i \neq j$. If M_i is FI-simple for all $1 \leq i \leq n$ and $N \in \sigma[M]$ is an M-multiplication module, then N is cyclic, distributive and semisimple module.*

Proof. By Proposition 3.9, we have that $N = \bigoplus_{i=1}^n ((M_i)_{M_i} N)$. Note that $(M_i)_{M_i} N \in \sigma[M_i]$. Now we claim that $(M_j)_{M_j} N$ is an M_j-multiplication module for each $1 \leq j \leq n$. In fact let $L \subseteq (M_j)_{M_j} N$. Now by Corollary 2.11, we have that $(M_j)_{M_j} N$ is an M-multiplication module. Therefore there exists a fully invariant submodule I of M, such that $I_M((M_j)_{M_j}N) = L$. We denote $N_j = (M_j)_{M_j}N$. Thus $I_M N_j = L$. By Remark 3.11, we have that $I = \bigoplus_{i=1}^{n} I_i$, where each I_i is a fully invariant submodule of M_i. Hence $L = (\bigoplus_{i=1}^{n} I_i)_M N_j$. On the other hand by Proposition 1.2, we have that $L = (\bigoplus_{i=1}^{n} I_i)_M N_j = \sum_{i=1}^{n}(I_i)_M N_j$. As $(I_k)_M N_j = (I_k)_{(\bigoplus_{i=1}^{n} M_i)}N_j = \sum_{f:(\bigoplus_{i=1}^{n} M_i)\to N_j} f(I_k) = \sum_{f:M_k\to N_j} f(I_k) = (I_k)_{M_k}N_j$. Thus $L = \sum_{i=1}^{n}(I_i)_{M_i}N_j$. Now analogously to the proof of Proposition 3.9 we obtain that the sum $\sum_{i=1}^{n}(I_i)_{M_i}N_j$ is direct. Thus

$$L = \bigoplus_{i=1}^{n}(I_i)_{M_i}N_j.$$

On the other hand we know that $L \subseteq (M_j)_{M_j}N = N_j$. But as $N_j = (M_j)_{M_j}N = tr^{M_j}(N) \in \sigma[M_j]$. Hence we obtain that $L \in \sigma[M_j]$. Moreover $(I_i)_{M_i}N_j \subseteq (M_i)_{M_i}N_j = tr^{M_i}(N_j) \in \sigma[M_i]$. Now by Lemma 3.8, we have that $(I_i)_{M_i}N_j = 0$ for all $j \neq i$. Therefore we obtain that $L = (I_j)_{M_j}N_j = (I_j)_{M_j}((M_j)_{M_j}N)$. So $(M_j)_{M_j}N$ is an M_j-multiplication module for each $1 \leq j \leq n$.

Now as M_j is FI-simple, then by Proposition 2.14, we have that $(M_j)_{M_j}N$ is a simple module. Thus $N = \bigoplus_{i=1}^{n}(M_i)_{M_i}N$ is a semisimple module. Now by Corollary 3.6, we obtain that N is cyclic. Moreover as $(M_i)_{M_i}N \in \sigma[(M_i)_{M_i}N]$, then by Lemma 3.8, we have that $\mathrm{Hom}_R((M_i)_{M_i}N, (M_j)_{M_j}N) = 0$ for all $i \neq j$. Thus $(M_i)_{M_i}N \ncong (M_j)_{M_j}N$. So by [17, Proposition 2.18(1)], we have that N is a distributive module. □

Remark 3.13. If M is an R-module such that $M/\mathcal{J}(M)$ is an artinian module, then by [16, p. 180, Proposition 1.7], we have that $M/\mathcal{J}(M)$ is a semisimple of finite length. Therefore $M/\mathcal{J}(M) = \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_n$, where each $\mathcal{S}_i$ is a homogeneous component. Thus $\sigma[M/\mathcal{J}(M)]$ consists of semisimple modules and $M/\mathcal{J}(M)$ is a progenerator of $\sigma[M/\mathcal{J}(M)]$. Moreover every $\mathcal{S}_i$ is a generator of the category $\sigma[\mathcal{S}_i]$ and $\mathrm{Hom}_R(\mathcal{S}_i, \mathcal{S}_j) = 0$ for all $i \neq j$. On the other hand we know that every $\mathcal{S}_i$ is an FI-simple module. Note that the family $\{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\}$ satisfies the condition of Theorem 3.3.

Proposition 3.14. *Let $M \in R$-Mod projective in $\sigma[M]$ such that $M/\mathcal{J}(M)$ is an artinian module. If $N \in \sigma[M]$ is an M-multiplication module, then the following assertions hold.*

(i) $\frac{N}{\mathcal{J}(M)_M N}$ *is a cyclic, distributive and semisimple module.*

(ii) *If $\mathcal{J}(M)_M N$ is a superfluous submodule of N, then N is a cyclic module.*

Proof. (i) Since N is an M-multiplication module, then by Proposition 2.10, we have that $\frac{N}{\mathcal{J}(M)_M N}$ is an M-multiplication module. Now by [4, Proposition 5.5], we obtain that $\mathcal{J}(M)_M(\frac{N}{\mathcal{J}(M)_M N}) = 0$. Thus $\mathcal{J}(M) \subseteq Ann_M(\frac{N}{\mathcal{J}(M)_M N})$. As $\mathcal{J}(M)$ is a fully invariant submodule of M, then by Proposition 2.6, we have that $\frac{N}{\mathcal{J}(M)_M N}$ is an $M/\mathcal{J}(M)$-multiplication module. Now by Remark

3.13 and Theorem 3.12, we obtain that $\frac{N}{\mathcal{J}(M)_M N}$ is a cyclic, distributive and semisimple module.

(ii) By (i) we have that $\frac{N}{\mathcal{J}(M)_M N}$ is a cyclic module. So there exists $\overline{x} = x + \mathcal{J}(M)_M N$ such that $\frac{N}{\mathcal{J}(M)_M N} = R\overline{x} = \frac{Rx+\mathcal{J}(M)_M N}{\mathcal{J}(M)_M N}$. Therefore we have that $Rx + \mathcal{J}(M)_M N = N$. As $\mathcal{J}(M)_M N$ is superfluous, then $Rx = N$. Hence we obtain that N is a cyclic module. □

Theorem 3.15. *Let $M \in R$-Mod projective in $\sigma[M]$ such that $M/\mathcal{J}(M)$ is an artinian module. If $N \in \sigma[M]$ is finitely generated and an M-multiplication module, then N is a cyclic module.*

Proof. Since N is finitely generated, then $\mathcal{J}(N)$ is a superfluous submodule of N. As $\mathcal{J}(M)_M N \subseteq \mathcal{J}(N)$, then $\mathcal{J}(M)_M N$ is a superfluous submodule of N. So by Proposition 3.14, we obtain that N is a cyclic module. □

We apply the results obtained in the previous sections to prime modules given in [6, Definition 1.1] and we obtain new results

Definition 3.16. Let $M \in R$-Mod and $N \neq M$ a fully invariant submodule of M. We say that N is prime in M if for any K, L fully invariant submodules of M we have that $K_M L \subseteq N$ implies that $K \subseteq N$ or $L \subseteq N$. We say that M is a prime module if 0 is prime in M.

Notice that if $I \subseteq R$ is an ideal, then I is prime in R in the sense of Definition 3.17 if and only if I is a prime ideal.

Definition 3.17. Let $M \in R$-Mod and $0 \neq N \in \sigma[M]$ such that $\mathrm{Hom}_R(M, N) \neq 0$. We say N is prime in $\sigma[M]$ if $Ann_M(K) = Ann_M(N)$ for all submodules K of N such that $\mathrm{Hom}_R(M, K) \neq 0$.

We note that, if $M = R$, then N is prime in $\sigma[M] = R$-Mod in the sense of Definition 3.16 if and only if N is prime in the sense given by Patrick F. Smith in [15].

Note that if S is a simple module in $\sigma[M]$ and $\mathrm{Hom}_R(M, S) \neq 0$, then S is prime in $\sigma[M]$.

Proposition 3.18. *Let $M \in R$-Mod be a progenerator in $\sigma[M]$ and $N \in \sigma[M]$ an M-multiplication module. The following conditions are equivalent.*

(i) *N is prime in $\sigma[M]$.*

(ii) *$Ann_M(N)$ is prime in M.*

Proof. (i)⇒(ii) As M is a generator in $\sigma[M]$, then $\mathrm{Hom}_R(M, X) \neq 0$ for all $0 \neq X \in \sigma[M]$. Now let K, L be submodules of M such that $K_M L \subseteq Ann_M(N)$. Suppose that $L \nsubseteq Ann_M(N)$. Thus $0 \neq L_M N \subseteq N$. Since N is prime in $\sigma[M]$, then $Ann_M(L_M N) = Ann_M(N)$. But we have that $(K_M L)_M N = 0$. As M is projective in $\sigma[M]$, then $K_M(L_M N) = 0$. Thus $K \subseteq Ann_M(L_M N)$. Hence we obtain that $K \subseteq Ann_M(N)$. Therefore $Ann_M(N)$ is prime in M.

(ii)⇒(i) Let $0 \neq K \subseteq N$. Clearly we have that $Ann_M(N) \subseteq Ann_M(K)$. As N is an M-multiplication module, then there exists a fully invariant submodule B of M such that $K = B_M N$. Since $Ann_M(K)_M K = 0$, then $(Ann_M(K))_M(B_M N) = 0$. As M is projective in $\sigma[M]$, then

$$(Ann_M(K)_M B)_M N = 0.$$

Thus $Ann_M(K)_M B \subseteq Ann_M(N)$. Since $Ann_M(N)$ is prime in M, then $Ann_M(K) \subseteq Ann_M(N)$ or $B \subseteq Ann_M(N)$. If $B \subseteq Ann_M(N)$, then $0 = B_M N = K$, it is a contradiction. □

Notice that (i) implies (ii) is not true in general. We consider the example 1.12 given in [6]. In that example $M = E(S)$ where S is the only one simple module. We have that S is prime in $\sigma[M]$. But M does not have submodules prime in M, then $Ann_M(S) = S$ is not prime in M. In that example M is not projective in $\sigma[M]$. We also note that (ii) implies (i) is not true in general. We consider the following example:

Example 3.19. Let $M = \mathbb{Z}$. We have that $Ann_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}) = 0$, this is prime in $\mathbb{Z}$. But $Ann_{\mathbb{Z}}(\mathbb{Z}_p) = p\mathbb{Z} \neq 0$. Hence we obtain that $\mathbb{Z}_{p^\infty}$ is not prime in $\sigma[\mathbb{Z}]$. Note that $\mathbb{Z}_{p^\infty}$ is not a $\mathbb{Z}$-multiplication module.

The following definitions were given in [8, Definition 3.1] and [10, Definition 2.3]. We include them here for the convenience of the reader.

Let $M \in R$-Mod. For a subset $X \subseteq End_R(M)$, let $\mathcal{A}_X = \cap\{\ker f \mid f \in X\}$. Now we consider the set $\mathcal{A}_M = \{\mathcal{A}_X \mid X \subseteq End_R(M)\}$.

Definition 3.20. Let $M \in R$-Mod. We say M satisfies the ascending chain condition (ACC) on left annihilators if $\mathcal{A}_M$ satisfies ACC.

For more details about these modules see [9].

Notice that if $M = R$, then R satisfies ACC on annihilators in the sense of Definition 3.20 if and only if R satisfies ACC on left annihilators in the usual sense for a ring R.

Also note that if K is a submodule of M, we have that

$$Ann_M(K) = \bigcap_{f \in X} \{\ker f \mid X = \mathrm{Hom}_R(M, K)\}.$$

Since $\mathrm{Hom}_R(M, K) \subseteq End_R(M)$, then $Ann_M(K) = \mathcal{A}_X$, where $X = \mathrm{Hom}_R(M, K)$. Hence $Ann_M(K) \in \mathcal{A}_M$.

Definition 3.21. Let $M \in R$-Mod. We say that M is a Goldie module if it satisfies ACC on left annihilators and has finite uniform dimension.

Notice that if $M = R$, then R is a Goldie module in the sense of Definition 3.21 if and only if R is left Goldie ring in the usual sense.

Proposition 3.22. *Let M be projective in $\sigma[M]$ such that M has a non-zero socle. Suppose that M is prime and satisfies ACC on left annihilators. If N is an M-multiplication module, then N is a simple module.*

Proof. By [11, Proposition 3.11], we have that M is FI-simple. Hence by Proposition 2.14, we obtain that N is a simple module. □

Corollary 3.23. *Let R be a ring such that R has a non-zero socle. Suppose that R is a prime ring and satisfies ACC on left annihilators. If N is an R-multiplication module, then N is a simple module.*

Theorem 3.24. *Let M be projective in $\sigma[M]$ with an essential socle. Suppose that M is a semiprime Goldie module. If N is an M-multiplication module, then N is cyclic, distributive and semisimple module.*

Proof. By [11, Corollary 2.12], we have that M is a semisimple artinian module. Thus $M = \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_n$, where each $\mathcal{S}_i$ is a homogeneous component. Moreover every $\mathcal{S}_i$ is FI-simple for all $1 \leq i \leq n$. Thus M satisfies the conditions of Theorem 3.12. Hence we obtain that N is cyclic, distributive and semisimple. □

Notice that in the particular case that R is a ring with an essential socle and R is a semiprime left Goldie ring: If N is an R-multiplication module, then N is a cyclic, distributive and semisimple module.

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