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ON CLEAN AND NIL CLEAN ELEMENTS IN SKEW T.U.P. MONOID RINGS

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Abstract. Let R be an associative ring with identity, M a t.u.p. monoid with only one unit and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. Let R be a reversible, M-compatible ring and $\alpha = a_1g_1 + \cdots + a_ng_n$ a non-zero element in skew monoid ring $R * M$. It is proved that if there exists a non-zero element $\beta = b_1h_1 + \cdots + b_mh_m$ in $R * M$ with $\alpha\beta = c$ is a constant, then there exist $1 \le i_0 \le n, 1 \le j_0 \le m$ such that $g_{i_0} = e = h_{j_0}$ and $a_{i_0}b_{j_0} = c$ and there exist elements $a, 0 \ne r$ in R with $\alpha r = ca$. As a consequence, it is proved that $\alpha \in R * M$ is unit if and only if there exists $1 \le i_0 \le n$ such that $g_{i_0} = e, a_{i_0}$ is unit and a_j is nilpotent for each $j \ne i_0$, where R is a reversible or right duo ring. Furthermore, we determine the relation between clean and nil clean elements of R and those elements in skew monoid ring $R * M$, where R is a reversible or right duo ring.

1. Introduction and preliminaries

Throughout the paper, unless mentioned otherwise, all rings are associated. We use the notation $\mathrm{U}(R)$, $\mathrm{Idem}(R)$, $\mathrm{nil}(R)$, $\mathrm{cln}(R)$, $\text{nil-cln}(R)$, $\mathrm{nil}_*(R)$, $\mathrm{nil}^*(R)$, $\text{L-rad}(R)$ and $\mathrm{J}(R)$ to denote the set of unit elements, idempotent elements, nilpotent elements, clean elements, nil clean elements, the prime radical, the upper nil radical, the Levitzki radical and the Jacobson radical of a ring R, respectively.

Recall that a ring R is *reduced* if it has no non-zero nilpotent element. According to Krempa [16] an endomorphism σ of a ring R is called *rigid* if $a\sigma(a) = 0$ implies that $a = 0$ for $a \in R$. A ring R is said to be σ-*rigid* if there exists a rigid endomorphism σ of R. Concept of σ-compatible rings introduced in [11, 13]. A ring R is called σ-*compatible* if for each $a, b \in R, ab = 0$ if and only if $a\sigma(b) = 0$. We now make a survey of several kinds of generalizations of reduced rings. A ring R is *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. A ring R is *semicommutative* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. On

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the other hand, a ring R is called *2-primal* if $\text{nil}_*(R) = \text{nil}(R)$ (see [3]). Shin in [25, Proposition 1.11] showed that a ring R is 2-primal if and only if every minimal prime ideal P of R is completely prime (i.e., R/P is a domain). A ring R is *weakly 2-primal* if $\text{nil}(R) = \text{L-rad}(R)$. A ring R is NI if $\text{nil}(R) = \text{nil}^*(R)$. It is known that reduced $\Rightarrow$ reversible $\Rightarrow$ semicommutative $\Rightarrow$ 2-primal $\Rightarrow$ weakly 2-primal $\Rightarrow$ NI, but the converse does not holed (see [7, 14]). Moreover, a ring is *right* (*resp., left*) *duo* if every right (resp., left) ideal is an ideal. Reversible as well as (one-sided) duo rings are semicommutative. A ring R is called *abelian* if each idempotent element of R is central. It is known that reversible rings and also semicommutative rings are abelian. But these implications are irreversible.

Let M be a monoid. In the following, we denote the identity element of M by e. A monoid M with a partial order $\leq$ is called *ordered monoid* if for any $a_1, a_2, b \in M$, $a_1 \leq a_2$ implies $a_1 b \leq a_2 b$ and $ba_1 \leq ba_2$. A *strictly ordered monoid* $(M, \leq)$ is an ordered monoid such that for any $a_1, a_2, b \in S$, $a_1 < a_2$ implies $a_1 b < a_2 b$ and $ba_1 < ba_2$. A strictly ordered monoid M is said to be *positively strictly ordered* if $m \geq e$ for all $m \in M$.

We use the following terminology. If A and B are non-empty subsets of a monoid M, then an element $s_0 \in AB = \{ab : a \in A, b \in B\}$ is said to be a *unique product element* (u.p. element for short) in the product of AB if it is uniquely presented in the form of $s = ab$ where $a \in A$ and $b \in B$.

Recall that a monoid M is called *unique product monoid* (*u.p. monoid* for short) if for any two non-empty finite subsets $A, B \subseteq M$ there exist $a \in A$ and $b \in B$ such that ab is u.p. element in the product of AB. Also, a monoid M is said to be *two unique product monoid* (or simply t.u.p. monoid) if for any two non-empty finite subsets $A, B \subseteq M$ with $|A| + |B| > 2$ there exist two u.p. elements in the product of AB. Strojnowski in [26, Theorem 1] proved that a group is t.u.p. if and only if it is u.p. Clearly, each strictly totally ordered monoid is t.u.p. monoid.

Assume that R is a ring, M a monoid and $\omega : M \to \text{End}(R)$ a monoid homomorphism. For each $g \in M$ we denote the image of g by ω_g (i.e., $\omega(g) = \omega_g$). Then all finite formal combinations $\sum_{i=1}^{n} a_i g_i$, with point-wise addition and multiplication induced by $(ag)(bh) = (a\omega_g(b))gh$ form a ring that is called *skew monoid ring* and it is denoted by $R * M$. Let R be a ring, M a monoid and $\omega : M \to \text{End}(R)$ a monoid homomorphism. We say that R is *M-compatible* (resp. *M-rigid*) if ω_g is compatible (resp. rigid) for any $g \in M$. The construction of skew monoid ring generalizes some classical ring construction such as polynomial rings, skew polynomial rings, Laurent polynomial rings, skew Laurent polynomial rings and monoid rings. Hence any result on skew monoid ring has its counterpart in each of the subclasses.

Unit, idempotent and nilpotent elements play an important role in noncommutative ring theory. An element $a \in R$ is said to be *clean* if it present as the sum of a unit and an idempotent. Rings in which every element is sum of unit and idempotent are called *clean*, have been extensively studied. The concept of clean ring originally defined by Nicholson [21]. In recent decades many authors studied this class of ring, many generalization and variation of it. Anderson and Camilo in [2] defined uniquely clean ring in commutative rings. An element $a \in R$ is said to be *uniquely clean* if it uniquely present as the sum of a unit and an idempotent. A ring R is called *uniquely clean* if every element is uniquely clean. Later, Nicholson and Zhou [22] extend the notation of uniquely clean in noncommutative ring.

Recently, Diesl [9] modified the definition of clean ring and obtained an interesting new concept he called nil clean. Following [9], an element a in R is said to be (*strongly*) *nil-clean* if there exists an idempotent e and a nilpotent b in R such that $a = e + b$ (and $eb = be$). A ring R is called (*strongly*) *nil clean* if every element of R is (strongly) nil clean. Nil clean and strongly nil clean rings are naturally connected to clean and strongly clean ring. Kosan et al. [15] proved that an element a is strongly nil clean if and only if a is clean and $a - a^2$ is nilpotent, and that a ring R is strongly nil clean if and only if $R/\mathrm{J}(R)$ is boolean and $\mathrm{J}(R)$ is nil.

The aim of this paper is to determine some type of elements such as idempotents, units, cleans, and nil cleans in a skew monoid ring $R * M$.

In Section 2, we show that if $0 \neq \alpha = a_1g_1 + \cdots + a_ng_n$ is an idempotent in $R * M$ then there exists $1 \leq i \leq n$ such that $g_i = e$ and $a_i - f \in \mathrm{nil}(R)$ for some idempotent $f \in R$ and $a_j \in \mathrm{nil}(R)$ for all $j \neq i$, where R is a 2-primal and M-compatible ring and M is a u.p. monoid. Also, we prove that the set of idempotent elements in $R * M$ are coincide to the set of idempotents in R when the base ring R is a semicommutative ring and M is a u.p. monoid.

In Section 3, we recall the definition of two unique product monoid (t.u.p. monoid) and bring some examples about t.u.p. monoids.

In Section 4, first we prove some results which concern the constant products of elements in skew monoid ring $R * M$, then we study unit elements of skew monoid ring $R * M$. It is shown that a non-zero element $\alpha = a_1g_1 + \cdots + a_ng_n$ of $R * M$ is unit, if there exists $1 \leq i \leq n$ such that $g_i = e$ and a_i is unit in R and a_j is nilpotent for all $j \neq i$, where M is a t.u.p. monoid with only one unit and R is a M-compatible reversible or right duo ring. Therefore we can determine (strongly) clean and (strongly) nil clean elements in skew monoid ring $R * M$.

2. Idempotent elements in skew monoid ring over semicommutative ring

Idempotent elements play an important role in the study of noncommutative rings. In this section, we determine construction of idempotents in skew monoid ring $R * M$ and relation between idempotent elements in R and skew monoid ring $R * M$ over semicommutative ring R when M is a u.p. monoid.

Lemma 2.1 ([10, Lemma 2.8]). *Let R be a ring, M a monoid and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism such that R is M-compatible. Then for each* *elements* $a_1, a_2, \dots, a_n \in R$ *we have* $a_1 a_2 \cdots a_n \in \mathrm{nil}(R)$ *if and only if*

$$\omega_{m_1}(a_1)\omega_{m_2}(a_2)\cdots\omega_{m_n}(a_n) \in \mathrm{nil}(R)$$

for all elements $m_1, m_2, \dots, m_n \in M$.

Let R be a ring, M a monoid and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. According to [10, Definition 3.1], R is said to be *skew Armendarize ring relative to* M (or simply *skew* M*-Armendariz*) if whenever non-zero elements $\alpha = \sum_{i=1}^{n} a_i g_i, \beta = \sum_{j=1}^{m} b_j h_j$ in $R * M$ satisfy $\alpha\beta = 0$, then $a_i \omega_{g_i}(b_j) = 0$ for each $1 \le i \le n$ and $1 \le j \le m$.

Proposition 2.2 ([10, Proposition 3.3])**.** *Let* R *be a ring,* M *a monoid and* $\omega : M \to \mathrm{End}(R)$ *a monoid homomorphism. If* R *is* M*-rigid, then* R *is skew* M*-Armendariz.*

Theorem 2.3 ([10, Theorem 4.4.])**.** *Let* R *be a* 2*-primal ring,* M *a u.p. monoid and* $\omega : M \to \mathrm{End}(R)$ *a monoid homomorphism. If* R *is* M*-compatible, then* $\mathrm{nil}(R * M) = \mathrm{nil}(R) * M$.

Let I be a nil ideal in R and $a \in R$ be such that $\bar{a} \in \bar{R} := R/I$ is an idempotent. Then by [17, Theorem 21.28] there exists an idempotent $f \in aR$ such that $\bar{f} = \bar{a} \in \bar{R}$. Also, let σ be an endomorphism of a ring R. According to [13, Lemma 2.2], σ is rigid if and only if R is reduced and σ is compatible. Thus for a monoid M and monoid homomorphism $\omega : M \to \mathrm{End}(R)$, R is M-rigid if and only if R is reduced and M-compatible.

Proposition 2.4. *Let* R *be a* 2*-primal ring,* M *a u.p. monoid and* $\omega : M \to \mathrm{End}(R)$ *a monoid homomorphism. Let* R *be a* M*-compatible ring and* $0 \ne \alpha = a_1 g_1 + \cdots + a_n g_n$ *an idempotent element of* $R * M$*. Then there exists* $1 \le i \le n$ *such that* $g_i = e, \overline{a_i} = \overline{f} \in R/\mathrm{nil}(R)$ *for some idempotent* $f \in R$ *and* $a_j \in \mathrm{nil}(R)$ *for all* $j \ne i$.

Proof. Since R is NI, $\overline{R} = R/\mathrm{nil}(R)$ is reduced. We define $\overline{\omega} : M \to \mathrm{End}(\overline{R})$ such that $\overline{\omega}(m) = \overline{\omega}_m; \overline{\omega}_m(\overline{a}) = \overline{\omega_m(a)}$. Assume that $\bar{a}\overline{\omega}_m(\bar{b}) = \bar{0}$. Then $a\omega_m(b) \in \mathrm{nil}(R)$ and so by Lemma 2.1, $ab \in \mathrm{nil}(R)$. Thus $\overline{R} = R/\mathrm{nil}(R)$ is M-compatible (i.e., $\overline{\omega}_m$ is compatible for each $m \in M$), which implies that $\overline{R}$ is M-rigid and so it is skew M-Armendariz, by Proposition 2.2. The element $\overline{\alpha} = \overline{a}_1 g_1 + \cdots + \overline{a}_n g_n$ is non-zero. Otherwise, if $\overline{\alpha} = \bar{0}$, then $a_i \in \mathrm{nil}(R)$ for each $1 \le i \le n$, which implies that $\alpha \in \mathrm{nil}(R) * M$. Therefore $\alpha \in \mathrm{nil}(R * M)$, by Theorem 2.3, which is a contradiction. From $\overline{\alpha}^2 = \overline{\alpha}$ we have $(\overline{\alpha} - \bar{1}e)\overline{\alpha} = (\overline{a}_1 g_1 + \cdots + \overline{a}_n g_n - \bar{1}e)\overline{\alpha} = \bar{0}$. Therefore $\overline{a}_i = \overline{a}_i \bar{1} = \bar{0}$, if $g_i \ne e$ for each $1 \le i \le n$, since R is skew M-Armendariz. This means $\overline{\alpha} = 0$, which is a contradiction. Thus we can assume that there exists $1 \le i \le n$ such that $g_i = e$. Without loss of generality assume that $i = 1$. Thus $\bar{0} = \overline{\alpha}(\overline{\alpha} - \bar{1}) = \overline{\alpha}((\overline{a_1 - 1})e + \overline{a}_2 g_2 + \cdots + \overline{a}_n g_n)$ which implies that $\overline{a}_1^2 = \overline{a}_1$ and $a_i \in \mathrm{nil}(R)$ for all $i \ne 1$, since $\overline{R}$ is skew M-Armendariz. Then by [17, Theorem 21.28], there exists an idempotent $f \in R$ such that $\overline{a}_1 = \overline{f} \in \overline{R}$, as desired. □

Now we can determine the relation between idempotent elements of R and $R*M$ when M is u.p. monoid and R is a semicommutative and M-compatible ring.

Theorem 2.5. *Let R be a semicommutative and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism, where M is a u.p. monoid. If R is M-compatible, then* $\mathrm{Idem}(R*M) = \mathrm{Idem}(R)$.

Proof. Let $\alpha = a_1g_1+\cdots+a_ng_n \in \mathrm{Idem}(R*M)$. By Proposition 2.4 there exists $1 \le i \le n$ such that $g_i = e, \overline{a_i} = \overline{f} \in R/\mathrm{nil}(R)$ for some idempotent $f \in R$ and $a_j \in \mathrm{nil}(R)$ for all $j \neq i$. Hence $a_i = f + t$ where t is a nilpotent element of R. Let $i = 1$ and $\alpha_1 = te+a_2g_2+\cdots+a_ng_n$. Then $\alpha = f+\alpha_1$. Since R is 2-primal, $\alpha_1 \in \mathrm{nil}(R)*M = \mathrm{nil}(R*M)$, by Theorem 2.3. If $\alpha_1 \neq 0$, then there exists non-negative integer k such that ${\alpha_1}^k = 0 \neq \alpha_1^{k-1}$. By the proof of [1, Theorem 2.41] any compatible homomorphism is idempotent-stabilizing, thus $\omega_{g_i}(f) = f$ and so $\alpha_1 f = f\alpha_1$. Since $\alpha^2 = \alpha$, we have $0 = (f+\alpha_1)(1-f-\alpha_1) = \alpha_1-2f\alpha_1-\alpha_1^2$. Then ${\alpha_1}^2 = (1-2f)\alpha_1$. By multiplying ${\alpha_1}^{k-2}$ from right hand side to ${\alpha_1}^2 = (1-2f)\alpha_1$ we have $0 = {\alpha_1}^k = (1-2f)\alpha_1^{k-1}$. Since $1-2f$ is invertible, $\alpha_1^{k-1} = 0$, which is a contradiction. Hence $\alpha_1 = 0$ and so $\alpha = f$. □

If R is a semicommutative ring, M a u.p. monoid and $\omega : M \to \mathrm{End}(R)$ identity homomorphism (i.e., ω_m is identity endomorphism of R for each $m \in M$), then the skew monoid ring $R*M$ is isomorphic to monoid ring $R[M]$. As a result of Theorem 2.5 we have $\mathrm{Idem}(R[M]) = \mathrm{Idem}(R)$.

Corollary 2.6. *Let R be a semicommutative and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism, where M is a u.p. monoid. If R is M-compatible, then $R*M$ is abelian.*

Let M be a monoid generated by $\{x\}$ and σ be an endomorphism of R. Let $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism such that $\omega_{x^i} = \sigma^i$ for each $x^i \in M$. Then the skew monoid ring $R*M$ is isomorphic to skew polynomial ring $R[x;\sigma]$.

Corollary 2.7. *Let R be a semicommutative ring and σ an endomorphism of R. If R is a σ-compatible ring, then* $\mathrm{Idem}(R[x;\sigma]) = \mathrm{Idem}(R)$ *and $R[x;\sigma]$ is abelian.*

Note that if M, N are two u.p. monoids, then $M \times N$ is also u.p. monoid, by [18, Lemma 1.13]. On the other hand, for a ring R and indeterminates x and y we have $R[x,y] \cong R[x][y]$. Thus, we have the following result.

Corollary 2.8. *Let R be a semicommutative ring. Then* $\mathrm{Idem}(R[x_1,x_2,\ldots,x_n]) = \mathrm{Idem}(R)$ *and $R[x_1,x_2,\ldots,x_n]$ is abelian.*

3. Two unique product monoids

As mentioned in the introduction, a monoid M is said to be t.u.p. if for any two non-empty finite subsets $A, B \subseteq M$ with $|A| + |B| > 2$ there exist two u.p. elements in the product of AB. We recall the following definitions from [19].

A partially ordered set $(M, \leq)$ is called *artinian* if every strictly decreasing sequence of elements of M is finite, and $(M, \leq)$ is called *narrow* if every subset of a pairwise order-incomparable elements of M is finite. Thus $(M, \leq)$ is artinian and narrow if and only if every non-empty subset of M has at least one but only a finite number of minimal elements. Let $(M, \leq)$ be an ordered monoid. We say that $(M, \leq)$ is an *artinian narrow unique product monoid* (or a.n.u.p. monoid) if for every two artinian and narrow subsets A and B of M there exists a *u.p.* element in the product of AB. An ordered monoid $(M, \leq)$ is called *minimal artinian narrow unique product monoid* (or m.a.n.u.p. monoid) if for every two artinian and narrow subsets A and B of M there exist $a \in \min(A)$ and $b \in \min(B)$ such that ab is a u.p. element of AB. A monoid M is said to be *totally orderable* if $(M, \leq)$ is an ordered monoid for some total order $\leq$ (any two different elements of M are comparable). An ordered monoid $(M, \leq)$ is said to be *quasi-totally ordered* (and $\leq$ is called a quasi-total order on M) if $\leq$ can be refined to an order $\preceq$ with respect to which M is a strictly totally ordered monoid.

One could define an a.n.t.u.p. monoid in the obvious way:

Definition 3.1. An ordered monoid M is an *a.n.t.u.p. monoid* if for every two artinian narrow subsets A, B of M with $|A| + |B| > 2$ there exist at least two u.p. elements in the product AB. Also, an ordered monoid M is said to be *m.a.n.t.u.p. monoid* if for every two artinian narrow subsets A, B of M with $|A| + |B| > 2$ there exist at least two u.p. elements a_1b_1, a_2b_2 in the product AB such that $a_1, a_2 \in \min(A)$ and $b_1, b_2 \in \min(B)$.

If $(M, \leq)$ is an ordered monoid the following implications hold:

$$
\begin{array}{ccc}
 & M \text{ is a commutative, torsion free, cancellative monoid} & \\
 & \downarrow & \\
 & (M,\leq) \text{ is quasi-totally ordered monoid} & \\
\swarrow & & \searrow \\
(M,\leq) \text{ is a m.a.n.u.p. monoid} & \longleftarrow & (M,\leq) \text{ is a m.a.n.t.u.p. monoid} \\
\downarrow & & \downarrow \\
(M,\leq) \text{ is an a.n.u.p. monoid} & \longleftarrow & (M,\leq) \text{ is an a.n.t.u.p. monoid} \\
\downarrow & & \downarrow \\
M \text{ is a u.p. monoid} & \longleftarrow & M \text{ is a t.u.p. monoid}
\end{array}
$$

Marks, Mazurek and Ziembowski [19] showed that left implications in the above diagram are irreversible. To prove quasi-totally ordered monoids are m.a.n.t.u.p. note that by assumption $\leq$ can be refined to a total order $\preceq$ such that $(M, \preceq)$ is a strictly ordered monoid. If A and B are artinian and narrow subsets of $(M, \leq)$, then the sets $\min(A), \min(B)$ are finite, and thus there exist $a_1 \in \min(A) (b_1 \in \min(B))$ which is smallest under the total order $\preceq$ and $a_2 \in \min(A) (b_2 \in \min(B))$ which is greatest under $\preceq$. Now, clearly a_1b_1 and a_2b_2 are two u.p. elements of AB. The remaining implications are obvious.

Clearly, any t.u.p. monoid is u.p. The following example shows that the converse is not true in general.

Example 3.2 ([24, Example 13 of Chap. 10])**.** Let M be the monoid generated by $x_1, x_2, x_3, X_1, X_2, X_3$ subject to the following relations:

$$x_1X_1 = x_2X_3, \qquad x_1X_2 = x_3X_1, \qquad x_1X_3 = x_2X_2, \qquad x_3X_2 = x_2X_1.$$

As shown in [24], M is a u.p. monoid. Let A=$\{x_1, x_2, x_3\}$ and B=$\{X_1, X_2, X_3\}$. Clearly, x_3X_3 is the only u.p. element in the product AB. Hence M is not t.u.p.

Marks et al. in [19, Example 2.7] showed that the monoid mentioned in Example 3.2 is m.a.n.u.p. Therefore m.a.n.u.p. monoids are not t.u.p. in general.

Our next example shows that *t.u.p.* dose not imply *a.n.u.p.*

Example 3.3. Let M be the monoid generated by $\{x_i \mid i \in \mathbb{N}\} \cup \{X_j \mid j \in \mathbb{N}\}$ with the following relations:

$$x_iX_j = \begin{cases} x_{i-2}X_{i-2} & \text{if} \quad i \geq 3 \quad \text{and} \quad j = i + (-1)^{i+1} \\ x_jX_i & \text{otherwise.} \end{cases}$$

Hence $x_iX_j = x_jX_i$ for any $i \neq j$ except for the following products:

$$x_3X_4 = x_1X_1, \qquad x_4X_3 = x_2X_2, \qquad x_5X_6 = x_3X_3, \qquad x_6X_5 = x_4X_4,$$

and so on. Marks et al. [19, Example 2.6], showed that M admits a strict total ordering (and hence is t.u.p. monoid) but $(M, \leq)$ is not a.n.u.p.

4. Unit elements in skew monoid ring over reversible or right duo ring

It is well known that a polynomial over commutative ring R is unit if and only if its constant term is a unit in R and other coefficients are nilpotent. Chen [5, Example 2.8] showed that the conclusion is not true for noncommutative ring in general. Also, in [6] he generalized the constant-product theorem for a commutative polynomial ring (which is proved in [14]) to skew polynomial ring $R[x;\sigma]$, where R is a reversible ring which is σ-compatible for an endomorphism σ of R. Also he proved a skew polynomial $f(x)$ in $R[x;\sigma]$ is a unit if and only if its constant term is a unit in R and other coefficients are all nilpotent, when R is a weakly 2-primal ring which is σ-compatible. In this section, we generalize these results to skew monoid ring $R * M$.

For an element $\alpha = a_1g_1 + \cdots + a_ng_n \in R * M$ with $a_i \neq 0$ for each i, we say that length $\alpha = n$ and denote it by $\ell(\alpha)$.

Lemma 4.1. *Let R be a ring and M a t.u.p. monoid with only one unit and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. Let R be M-rigid and $\alpha = a_1g_1 + \cdots + a_ng_n$, $\beta = b_1h_1 + \cdots + b_mh_m$ non-zero elements of $R * M$ with $\alpha\beta = c \in R$. Then there exist i_0, j_0 such that $g_{i_0} = e = h_{j_0}, a_{i_0}b_{j_0} = c$ and $a_ib_j = 0$ for all $i + j \neq i_0 + j_0$.*

Proof. Let $\alpha = a_1g_1 + \cdots + a_ng_n$, $\beta = b_1h_1 + \cdots + b_mh_m$ be non-zero elements of $R * M$ with $\ell(\alpha) = n$ and $\ell(\beta) = m$. If $c = 0$, then $a_i\omega_{g_i}(b_j) = 0$ for each i, j, since R is skew M-Armendariz by [10, Proposition 3.3]. Thus $a_ib_j = 0$ for each i, j, as desired. Now, suppose that $c \neq 0$. We proceed by induction on $m + n$. For $m = 1$ or $n = 1$ the result is clear. Now, let $m, n > 1$ and suppose that the result is true for all the smaller values than $m + n$. Since M is a t.u.p. monoid, there exist $1 \leq i_0, i_1 \leq n, 1 \leq j_0, j_1 \leq m$ such that $g_{i_0}h_{j_0}$ and $g_{i_1}h_{j_1}$ are t.u.p. elements in the product of two subsets $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_m\}$ of M. Since M has only one unit, from $\alpha\beta = c$ we have $g_{i_0} = e = h_{j_0}, a_{i_0}\omega_{g_{i_0}}(b_{i_0}) = c$ and $a_i\omega_{g_{i_1}}(b_{j_1}) = 0$, which implies that $a_{i_0}b_{j_0} = c$ and $a_{i_1b_{j_1}} = 0$, since $\omega_{g_{i_0}} = id_R$ and R is M-compatible. Without loss of generality we can assume that $i_0 = j_0 = 1, i_1 = n$ and $j_1 = m$. Thus $a_nb_m = 0 = b_ma_n$ since R is reduced. Then $b_mc = b_m\alpha\beta = (b_ma_1e + b_ma_2g_2 + \cdots + b_ma_{n-1}g_{n-1})\beta$. Let $\alpha_1 = b_m\alpha$. Then $b_mc = \alpha_1\beta$. Therefore by induction hypothesis we have $b_ma_1b_m = b_ma_2b_m = \cdots = b_ma_{n-1}b_m = 0$. Thus $b_ma_i = 0$ and so $a_1b_m = a_2b_m = \cdots = a_nb_m = 0$, since R is reduced. Then $c = \alpha\beta = \alpha(\beta - b_mg_m) = \alpha\beta_1$. Since $\ell(\alpha) + \ell(\beta_1) \leq m + n - 1$, then by induction hypothesis we get $a_1b_1 = c$ and $a_ib_j = 0$ for each $i + j > 2$. □

The following example shows that the assumption "M has only one unit" in Theorem 4.1 is not superfluous. Note that if M is a monoid generated by $\{x, x^{-1}\}$ and ω be the identity homomorphism, then $R * M$ is isomorphic to Laurent polynomial ring $R[x, x^{-1}]$.

Example 4.2. Let R be a ring with $|\mathrm{Idem}(R)| \geq 3$. Let a be a nontrivial idempotent element of R and $\alpha = ax^{-1} + (1 - a)x, \beta = (1 - a)x^{-1} + ax \in R[x, x^{-1}]$. Then $\alpha\beta = 1$ but any product of coefficients α and β is not identity.

Lemma 4.3 ([6, Lemma 2.3])**.** *Let R be a 2-primal ring which is σ-compatible for an endomorphism σ of R. If P is any minimal prime ideal of R, then both $\sigma(P)$ and $\sigma^{-1}(P)$ are contained in P.*

Chen [6, Theorem 2.4] proved that for a reversible ring R which is σ-compatible, if $f = a_0 + a_1x + \cdots + a_nx^n$, $g = b_0 + b_1x + \cdots + b_mx^m$ are non-zero elements of $R[x; \sigma]$ such that $gf = c$ is a constant element of R, then $b_0a_0 = c$ and there exist non-zero elements a and r in R such that $rf(x) = ac$ with $r = b_pa$ for some $p, 0 \leq p \leq m$, and a is either one or a product of at most m coefficients from $f(x)$. In the next theorem we extend this result to skew monoid ring $R * M$, where R is a reversible ring and M is a t.u.p. monoid.

Theorem 4.4. *Let R be a reversible ring, M a t.u.p. monoid with only one unit and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism and $\alpha = a_1g_1 + \cdots + a_ng_n$ a non-zero element of $R * M$. Let R be a M-compatible ring. If there is non-zero element $\beta = b_1h_1 + \cdots + b_mh_m \in R * M$ and $c \in R$ with $\alpha\beta = c$, then there exist $1 \leq i_0 \leq n$, $1 \leq j_0 \leq m$ such that $g_{i_0} = e = h_{j_0}$, $a_{i_0}b_{j_0} = c$ and there exist elements a and $0 \neq r$ in R with $r\beta = ac$. Furthermore, if a_{i_0} is a unit in R, then b_j is nilpotent for all $j \neq j_0$.*

Proof. First we prove that the conclusion is true for any element $\beta \in R*M$ with $\ell(\beta)=1$. Let $0 \neq \beta = b_1h_1$. Then $c = \alpha\beta = \alpha b_1h_1$. Therefore, there exist $1 \leq i \leq n$ such that $g_i = e = h_1$ and $a_i\omega_{g_i}(b_1) = c$, since any u.p. monoid is cancelative by [4, Lemma 1.1]. Therefore $a_ib_1 = c$, since $g_i = e$ and $\omega_{g_i} = id_R$. Hence $a_i\beta = c$. In this case $r = a_i$ and $a = 1$.

Now, assume that $\ell(\beta) = m \geq 1$. We proceed by induction on $\ell(\alpha) = n$. If $\ell(\alpha) = 1$, then $\alpha = a_1g_1$. Since $\alpha\beta = a_1g_1(b_1h_1 + \cdots + b_mh_m) = c$, there exist $1 \leq j_1, j_2 \leq m$ such that $g_1 = e = h_{j_1}$, $a_1\omega_{g_1}(b_{j_1}) = c$ and $a_1\omega_{g_1}(b_{j_2}) = 0$. Without loss of generality we can assume that $j_1 = 1$ and $j_2 = m$. Thus $a_1b_1 = c$ and $a_1b_m = 0$. Hence $c = \alpha\beta = a_1e(b_1e + b_2h_2 + \cdots + b_{m-1}h_{m-1})$ and so there exist $a, r \neq 0$ in R such that $r\beta = ac$, as desired. Thus, we assume that it is true for any element has length less than $\ell(\alpha) = n$ with $n \geq 2$. From $\alpha\beta = c$, there exist non-zero integers $1 \leq i_0, i_1 \leq n$ and $1 \leq j_0, j_1 \leq m$ such that $g_{i_0}h_{j_0}$ and $g_{i_1}h_{j_1}$ are two u.p. elements in the product of two subsets $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_m\}$ of M. Since M has only one unit $g_{i_0} = e = h_{j_0}, a_{i_0}\omega_{g_{i_0}}(b_{j_0}) = c$ and $a_{i_1}\omega_{g_{i_1}}(b_{j_1}) = 0$. Therefore $a_{i_0}b_{j_0} = c$ and $a_{i_1}b_{j_1} = 0$, since $\omega_{g_{i_0}} = id_R$ and R is M-compatible. Without loss of generality, we can assume that $i_0 = j_0 = 1$ and $i_1 = n, j_1 = m$. Thus $g_1 = e = h_1, a_1b_1 = c$ and $a_nb_m = 0$. We consider the following two cases.

Case 1: Let $c = 0$. We can conclude it with a similar argument as used in the proof of [12, Proposition 1.2].

Case 2: Let $c \neq 0$. If $\alpha b_k = 0$ for each $k \neq 1$, then $a_1b_k = 0$ for each $k \neq 1$, since R is M-compatible. Hence $a_1\beta = c$. In this case $r = a_1$ and $a = 1$. Now, assume that there exists positive integer $k \neq 1$ such that $\alpha b_k \neq 0$. We consider the following two sub-cases.

Sub-case 1: Suppose that $k = m$. Thus $\alpha b_m \neq 0 \neq b_m\alpha$, since R is reversible and M-compatible. Therefore, $b_m\alpha$ is a non-zero element in $R*M$ which it has length less than $\ell(\alpha)$ satisfying $b_m\alpha\beta = b_mc$. By induction hypothesis, there is $a, 0 \neq r$ in R such that $r\beta = ab_mc$, as desired.

Sub-case 2: Suppose that $k \neq m$ and $\alpha b_k \neq 0$. Then there exists $1 \leq t \leq m$ such that $\alpha b_r \neq 0$ for each $1 \leq r \leq t$ and $\alpha b_s = 0$ for each $t < s \leq m$ (we can rewrite β if it is necessary). This implies that $c = \alpha\beta = \alpha(b_1h_1 + \cdots + b_th_t)$. Thus, there exist $1 \leq i \leq n$ and $1 \leq r \leq t$ such that a_ib_r is u.p. element in the product of $\{g_1 = e, g_2, \ldots, g_n\}$ and $\{h_1 = e, h_2, \ldots, h_t\}$ and $a_i\omega_{g_i}(b_r) = 0$, since R is t.u.p with only one unit and $g_1 = e = h_1$. Therefore $a_ib_r = 0$, since R is M-compatible. Without loss of generality we can assume that $i = n$ and $r = t$. Therefore $a_nb_t = b_ta_n = 0$. This implies that $b_t\alpha$ has length less than $\ell(\alpha)$ satisfying $b_t\alpha\beta = b_tc$. Hence, there exist elements $a', 0 \neq r \in R$ such that $r\beta = a'b_tc$. In this case, $a = a'b_t$.

Now assume that a_1 is unit in R. We prove b_j is nilpotent for all $j > 1$. Let P be a minimal prime ideal of R and $\overline{R} = R/P$. We define $\overline{\omega} : M \to \mathrm{End}(R/P)$ with $\overline{\omega}(m) = \overline{\omega}_m$; $\overline{\omega}_m(\overline{a}) = \overline{\omega_m(a)}$. Thus $\overline{R} * M$ is a skew monoid ring with monoid homomorphism $\overline{\omega}$. Since reversible rings are 2-primal, P is a completely prime ideal of R and so $\overline{R} = R/P$ is a domain. We prove that $\overline{R}$ is M-compatible. If $\overline{a}\overline{b} = 0$ for $a, b \in R$, then $ab \in P$. Since P is completely prime, $a \in P$ or $b \in P$. This implies that $a\omega_m(b) \in P$, by Lemma 4.3, and so $\overline{a}\overline{\omega}_m(\overline{b}) = 0$. Conversely, assume that $\overline{a}\overline{\omega}_m(\overline{b}) = \overline{0}$. This means that $a\omega_m(b) \in P$ and so $a \in P$ or $\omega_m(b) \in P$. Hence, by Lemma 4.3 we have $ab \in P$ and so $\overline{a}\overline{b} = \overline{0}$.

If $\overline{\beta} = 0$, then $b_j \in P$ for all $1 \leq j \leq m$, as desired. Now assume that $\overline{\beta} \neq 0$. Since a_1 is unit in R, $\overline{\alpha} \neq 0$. Since $\overline{\alpha}\overline{\beta} = \overline{c}$ in $\overline{R} * M$, then there exist elements $\overline{a}, 0 \neq \overline{r} \in R$ such that $\overline{r}\overline{\beta} = \overline{a}\overline{c}$. This means $\overline{r}\overline{b_j} = 0$ for all $j \neq 1$. Therefore $rb_j \in P$ and since $r \notin P$, we have $b_j \in P$ for all $j \neq 1$. Since P is an arbitrary minimal prime ideal of R, so $b_j \in \text{nil}_*(R)$ for all $j \neq 1$. This implies that $b_j \in \text{nil}(R)$ for all $j \neq 1$, since $\text{nil}(R) = \text{nil}_*(R)$. □

Theorem 4.5. *Let R be a reversible ring, M a t.u.p. monoid with only one unit, $\omega : M \to \text{End}(R)$ a monoid homomorphism and $\alpha = a_1g_1 + \cdots + a_ng_n$ a non-zero element of $R*M$. If there is non-zero element $\beta = b_1h_1 + \cdots + b_mh_m \in R * M$ and $c \in R$ with $\alpha\beta = c$, then there exist $1 \leq i_0 \leq n, 1 \leq j_0 \leq m$ such that $g_{i_0} = e = h_{j_0}$, $a_{i_0}b_{j_0} = c$ and there exist elements a and $0 \neq r$ in R with $\alpha r = ca$. Furthermore, if b_{j_0} is a unit in R, then a_i is nilpotent for each $i \neq i_0$.*

Proof. By a similar argument as used in the proof of Theorem 4.4 one can prove it. □

Let M be the monoid generated by $\{x\}$ and σ be an endomorphism of R. Let $\omega : M \to \text{End}(R)$ a monoid homomorphism such that $\omega_{x^i} = \sigma^i$ for each $i \geq 0$. Then it is clear that the skew monoid ring $R * M$ is isomorphic to skew polynomial ring $R[x; \sigma]$.

Corollary 4.6 ([6, Theorem 2.4])**.** *Let R be a reversible ring which is σ-compatible for an endomorphism σ of R and $f = a_0 + a_1x + \cdots + a_nx^n$ a non-zero element in $R[x; \sigma]$. If there is a non-zero skew polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ in $R[x; \sigma]$ with $gf = c$ is a constant, then $b_0a_0 = c$ and there exist elements a and $0 \neq r$ in R such that $rf(x) = ac$. Furthermore, if b_0 is unit in R, then $a_1, a_2, \ldots, a_n$ are all nilpotent.*

Nielsen in [23] called a ring R is right (resp. left) *McCoy* if for each pair of non-zero polynomials $f, g \in R[x]$ providing $fg = 0$, then there exists a non-zero element r in R such that $fr = 0$ (resp. $rg = 0$). The author in [12] extend the concept of McCoy ring to monoid ring $R[M]$. Let R be a ring, M a monoid and α, β be non-zero elements in $R[M]$. Then R is said to be *right M-McCoy* if $\alpha\beta = 0$, then there exists a non-zero element $r \in R$ such that $\alpha r = 0$. Left M-McCoy is defined similarly. Finally, in [20] the McCoy condition was extended to skew monoid ring $R * M$. According to [20, definition 2.16] R is called *right skew M-McCoy*, if for each pair of non-zero elements $\alpha = a_1g_1 + \cdots + a_ng_n$ and $\beta = b_1h_1 + \cdots + b_mh_m$ in $R * M$, $\alpha\beta = 0$ implies that $\alpha r = 0$ for some

$r \in R$. Left skew M-McCoy rings are defined similarly. If R is both right and left skew M-McCoy, then we say that R is *skew M-McCoy*.

As a consequence of Theorems 4.4 and 4.5 we have the following results.

Corollary 4.7. *Let R be a reversible, M a t.u.p. monoid with only one unit and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. If R is a M-compatible ring, then R is skew M-McCoy.*

Corollary 4.8. *Let R be a reversible ring, M a quasi-totally ordered monoid with only one unit and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. If R is a M-compatible ring, then R is skew M-McCoy.*

Corollary 4.9 ([23, Theorem 2])**.** *Let R be a reversible ring. Then R is McCoy.*

Lemma 4.10. *Let R be a semicommutative ring, M a t.u.p. monoid with only one unit and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. Let R be a M-compatible ring and $0 \neq \alpha = a_1g_1+\cdots+a_ng_n, 0 \neq \beta = b_1h_1+\cdots+b_mh_m \in R*M$ with $\alpha\beta = c \in R$. Then $a_{i_1}\cdots a_{i_t}\beta = 0$ or $a_{i_1}\cdots a_{i_t}a_{i_{t+1}}\beta = a_{i_1}a_{i_2}\cdots a_{i_t}c$, for some $\{i_1,\dots,i_t,i_{t+1}\} \subseteq \{1,\dots,n\}$.*

Proof. First assume that $c = 0$. Then $a_{i_1}\cdots a_{i_t}\beta = 0$ for some $\{i_1,\dots,i_t\} \subseteq \{1,\dots,n\}$ and $1 \leq t \leq m$, by [12, Lemma 1.19].

Now, assume that $a_{i_1}a_{i_2}\cdots a_{i_t}c \neq 0$ for any $\{i_1,\dots,i_t\} \subseteq \{1,\dots,n\}$. There exist $1 \leq i_0, i_1 \leq n$ and $1 \leq j_0, j_1 \leq m$ such that $g_{i_0}h_{j_0}$ and $g_{i_1}h_{j_1}$ are two u.p. elements in the product of two subsets $\{g_1,\dots,g_n\}$ and $\{h_1,\dots,h_m\}$ of M. Since $\alpha\beta = c$ and M has only one unit, $g_{i_0} = e = h_{j_0}$, $a_{i_0}\omega_{g_{i_0}}(b_{j_0}) = c$ and $a_{i_1}\omega_{g_{i_1}}(b_{j_1}) = 0$, which implies that $a_{i_0}b_{j_0} = c$ and $a_{i_1}b_{j_1} = 0$, since R is M-compatible and $\omega_{g_{i_0}} = id_R$. Without loss of generality, we can assume that $i_0 = j_0 = 1$ and $j_1 = m$. Since R is semicommutative and M-compatible, we have $a_{i_1}c = a_{i_1}\alpha\beta = (a_{i_1}a_1e + a_{i_1}a_2g_2 + \cdots + a_{i_1}a_ng_n)(b_1e + b_2h_2 + \cdots + b_{m-1}h_{m-1})$. Thus there exist $1 \leq i_2 \leq n$ and $1 \leq j_2 \leq m$ such that $g_{i_2}h_{j_2}$ is u.p. in the product of $\{g_1 = e, g_2,\dots,g_n\}$ and $\{h_1 = e, h_2,\dots,h_{m-1}\}$ and $a_{i_1}a_{i_2}\omega_{g_{i_2}}(b_{j_2}) = 0 = a_{i_1}a_{i_2}b_{j_2}$. Without loss of generality, we can assume that $j_2 = m-1$. Again, since R is semicommutative we have $a_{i_1}a_{i_2}c = a_{i_1}a_{i_2}\alpha(b_1e + b_2h_2 + \cdots + b_{m-2}h_{m-2})$. Continuing this process we can prove $a_{i_1}a_{i_2}\cdots a_{i_t}c = a_{i_1}a_{i_2}\cdots a_{i_t}\alpha b_1e$ for some $\{i_1,\dots,i_t\} \subseteq \{1,\dots n\}$. Thus $a_{i_1}a_{i_2}\cdots a_{i_t}a_1\beta = a_{i_1}a_{i_2}\cdots a_{i_t}c$, as desired. □

As a consequence of Lemma 4.10 we have the following result.

Corollary 4.11. *Let R be a semicommutative ring which is σ-compatible for an endomorphism σ of R and $f = a_0 + a_1x + \cdots + a_nx^n$, $g = b_0 + b_1x + \cdots + b_mx^m$ be non-zero elements of $R[x;\sigma]$ with $fg = c \in R$. Then $a_0b_0 = c$ and $a_{i_1}a_{i_2}\cdots a_{i_t}a_{i_{t+1}}g = 0$ or $a_{i_1}a_{i_2}\cdots a_{i_t}a_{i_{t+1}}g = a_{i_1}a_{i_2}\cdots a_{i_t}c$ for some $\{i_1,\dots,i_t\} \subseteq \{1,\dots,n\}$.*

For a ring R and element $a \in R$ we denote the set of right (resp. left) annihilators of a over R by $ann_r^R(a)$ (resp. $ann_r^R(a)$). Let $c = 0$ in the Corollary

4.1. Then by a similar argument as used in the proof of [23, Theorem 4] we can conclude the following result.

Corollary 4.12. *Let R be a semicommutative ring and σ an endomorphism of R. Given $f(x)g(x) = c \in R$ with non-zero skew polynomials $f(x), g(x)$ in $R[x;\sigma]$, then (at least) on of $ann_r^{R[x;\sigma]}(f(x)) \cap R \neq 0$ or $ann_r^{R[x;\sigma]}(g(x)) \cap R \neq 0$.*

Let R be a ring and M a u.p. monoid. Cheon and Kim in [8, Theorem 3] proved that $\mathrm{nil}_*(R[M]) = \mathrm{nil}_*(R)[M]$. Also, they proved that for a ring R and a u.p. monoid M, R is 2-primal if and only if monoid ring $R[M]$ is 2-primal [8, Theorem 5]. By a similar argument as used in the [8, Theorem 5] with small changes one can prove the following lemma.

Lemma 4.13. *Let R be a ring, M a u.p. monoid and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. Let R be a M-compatible ring. Then R is 2-primal if and only if the skew monoid ring $R * M$ is 2-primal.*

Now we determine all of the unit elements of $R * M$, when R is a reversible or right duo ring and M a t.u.p. monoid with only one unit.

Proposition 4.14. *Let R be a reversible or right duo ring, M a t.u.p. monoid with only one unit and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. Let R be a M-compatible ring. Then $0 \neq \alpha = a_1g_1 + \cdots + a_ng_n \in R * M$ is unit if and only if there exists $1 \leq i_0 \leq n$ such that $g_{i_0} = e, a_{i_0} \in U(R)$ and a_i is nilpotent for each $i \neq i_0$.*

Proof. Assume that α is unit. Then there exist $0 \neq \beta = b_1h_1 + \cdots + b_mh_m$ such that $\beta\alpha = 1 = \alpha\beta$. Since M is u.p. monoid with only one unit, there exist i_0, j_0 such that $g_{i_0}h_{j_0}$ is a u.p. element in the product of two subsets $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_m\}$ of M, $g_{i_0} = e = h_{j_0}$ and $a_{i_0}\omega_{g_{i_0}}(b_{j_0}) = 1$. Therefore $a_{i_0}b_{j_0} = 1$ since $\omega_{g_{i_0}} = id_R$. Without loss of generality, we can assume that $i_0 = j_0 = 1$. Thus $a_1b_1 = 1$ and so b_1 is unit in R. Hence, if R is a reversible ring, then a_i is nilpotent for all $i \neq 1$, by Theorem 4.5.

Now, assume that R is a right duo ring. Since R is NI, $R/\mathrm{nil}(R)$ is reduced. For each $m \in M$, $\omega_m(\mathrm{nil}(R)) \subseteq \mathrm{nil}(R)$. Thus we can assume that $\omega_m \in \mathrm{End}(R/\mathrm{nil}(R))$. Hence we define $\overline{\omega} : M \to \mathrm{End}(R/\mathrm{nil}(R))$ such that $\overline{\omega}(m) = \omega_m$. Thus $\overline{R} = R/\mathrm{nil}(R)$ is M-compatible and so is M-rigid. Since $\overline{\alpha}\overline{\beta} = \overline{1}$ in $\overline{R} * M$ and M is t.u.p. monoid with only one unit, there exist $1 \leq i_1 \leq n, 1 \leq j_1 \leq m$ such that $g_{i_1} = e = h_{j_1}, \overline{a}_{i_1}\overline{b}_{j_1} = \overline{1}$ and $\overline{a}_i\overline{b}_j = 0$ for each $i + j \neq i_1 + j_1$, by Lemma 4.1. In the other hand, M has only one unit and $g_1 = e = h_1$, so $i_1 = j_1 = 1$. Thus $\overline{a}_1\overline{b}_1 = \overline{1}$ and $\overline{a}_i\overline{b}_j = \overline{0}$ for each $i + j > 2$. Hence, $\overline{a}_i\overline{b}_1 = \overline{0}$ for all $i \geq 2$, which implies that a_i is nilpotent in R for all $i \geq 2$, since b_1 is unit in R.

Conversely, assume that a_1 is unit and $a_2, \ldots, a_n$ are nilpotent. Thus $a_2g_2 + \cdots + a_ng_n \in \mathrm{nil}(R) * M = \mathrm{nil}(R * M)$, by Theorem 2.3. Since reversible and right duo rings are 2-primal, by Lemma 4.13 the skew monoid ring $R * M$ is 2-primal. On the other hand, every 2-primal ring is weakly 2-primal, thus $\mathrm{nil}(R * M) =$

L-rad$(R*M)$. Also, we have L-rad$(R*M) \subseteq J(R*M)$ by [17, Lemma 10.32], so $a_2g_2+\cdots+a_ng_n \in J(R*M)$. Thus $\alpha \in U(R*M)$. □

Corollary 4.15. *Let R be a reversible or right duo ring, M a t.u.p. monoid with only one unit and $\omega : R \to \mathrm{End}(R)$ a monoid homomorphism. Let R be a M-compatible ring. Then* $\mathrm{U}(R*M) = \{a_1g_1+a_2g_2+\cdots+a_ng_n \mid n \geq 1, g_1 = e, a_1 \in \mathrm{U}(R)$ *and* a_i *is nilpotent for all* $i \neq 1\}$.

As a consequence we obtain a generalization of [6, Corollary 2.9], which is just the first part of the following corollary. Not that, if M, N are two t.u.p. monoids, then by a similar argument as used in the proof of [18, Lemma 1.13], one can show that $M \times N$ is also t.u.p. Thus we have the second part of the following result.

Corollary 4.16. *Let R be a reversible or right duo ring and σ an endomorphism of R. If R is σ-compatible, then*

(1) $\mathrm{U}(R[x;\sigma]) = \mathrm{U}(R) + \mathrm{nil}(R[x])x$.
(2) *A non-zero element $f \in R[x_1,\ldots,x_n]$ is unit if and only if the constant term of f is unit and any other coefficients are nilpotent.*

Now, we are in the position to determine clean elements of skew monoid ring $R*M$.

Proposition 4.17. *Let R be a reversible or right duo ring, M a t.u.p. monoid with only one unit and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. Let R be a M-compatible ring. Then* $\mathrm{cln}(R*M) = \{a_1g_1+a_2g_2+\cdots+a_ng_n \mid n \geq 1,\ g_1 = e, a_1 \in \mathrm{cln}(R)$ *and* a_i *is nilpotent for each* $i \neq 1\}$.

Proof. It follows from Theorem 2.5 and Corollary 4.15. □

Let R be a reversible (or right duo) M-compatible ring, M a t.u.p. monoid with only one unit and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. Then by Corollary 2.6, the skew monoid ring $R*M$ is abelian. Hence every clean element of $R*M$ is also strongly clean.

Corollary 4.18. *Let R be a reversible or right duo ring and σ an endomorphism of R. If R is σ-compatible, then*

(1) $\mathrm{cln}(R[x;\sigma]) = \mathrm{cln}(R) + \mathrm{nil}(R[x;\sigma])x = \mathrm{cln}(R) + (\mathrm{nil}(R)[x;\sigma])x$.
(2) *A non-zero element $f \in R[x_1,\ldots,x_n]$ is clean if and only if the constant term of f is clean and any other coefficients are nilpotent.*

Let R be a reversible or right duo ring, M a positively strictly totally ordered monoid or a t.u.p. monoid with only one unit, and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. Let $\alpha = 1g \in R*M$. Since $1 \notin \mathrm{nil}(R)$ then $\alpha = 1g \notin \mathrm{cln}(R*M)$. Hence, the skew monoid ring $R*M$ is never clean. As a consequence, for a σ-compatible ring R, the skew polynomial ring $R[x;\sigma]$ is never clean and so the polynomial ring $R[x]$ is never clean [22, Proposition 13].

In the following, we determine nil clean elements of skew monid ring $R*M$.

Corollary 4.19. *Let R be a semicommutative, M a u.p. monoid and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. Let R be a M-compatible ring. Then $\alpha \in R * M$ is nil clean if and only if $\alpha = a_1 g_1 + \cdots + a_n g_n$ such that $g_1 = e, a_1 \in \mathrm{nil} - \mathrm{cln}(R)$ and $a_i \in \mathrm{nil}(R)$ for all $i \geq 2$.*

Proof. It follows from Theorem 2.5 and Corollary 2.3. □

Let R be a semicommutative and M-compatible ring, M a u.p. monoid and $\omega : M \to \mathrm{End}(R)$ a monoid homomorphism. By Corollary 2.6, the skew monoid ring $R * M$ is abelian. Hence every nil clean element of $R * M$ is also strongly nil clean.

Corollary 4.20. *Let R be a semicommutative ring and σ an endomorphism of R. If R is σ-compatible, then*

(1) $\text{nil-cln}(R[x;\sigma]) = \{a_0 + a_1 x + \cdots + a_n x^n \mid n \geq 0,\ a_0 \in \text{nil-cln}(R)$ *and* $a_i \in \mathrm{nil}(R)$ *for* $1 \leq i \leq n\}$.
(2) *A non-zero element $f \in R[x_1, \ldots, x_n]$ is nil clean if and only if the constant term of f is nil clean and any other coefficients are nilpotent.*

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