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STABILITY IN FUNCTIONAL DIFFERENCE EQUATIONS WITH APPLICATIONS TO INFINITE DELAY VOLTERRA DIFFERENCE EQUATIONS

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Abstract. We consider a functional difference equation and use fixed point theory to obtain necessary and sufficient conditions for the asymptotic stability of its zero solution. At the end of the paper we apply our results to nonlinear Volterra infinite delay difference equations.

1. Introduction

It is well known that when studying stability of solutions, Lyapunov functions or functionals are the way to go. However, the stability results are as good as the Lyapunov functional that is being constructed, see [6]. Moreover, in most cases, Lyapunov functional will require severe conditions (see Theorem 1 below) on the terms in the equations in order for it to be decreasing along the solutions. For more on recent results regarding stability in difference equations we refer the reader to [1], [2], [3], [4], [5], [9] and [10]. For recent results on Volterra integro-differential equations, we refer the reader to [7–9] and the references therein.

Let $\mathbf{R} = (-\infty, \infty)$, $\mathbf{Z}^+ = [0, \infty)$ and $\mathbf{Z}^- = (-\infty, -1]$, respectively. To motivate the reader, we consider the delay difference equation

$$x(t+1) = a(t)x(t) + b(t)x(t - g(t)), \tag{1.1}$$

where $a, b, g : \mathbf{Z}^+ \to \mathbf{R}$, and $t - g(t) \in \mathbf{Z}$.

Theorem 1. *Suppose*

$$\triangle g(t) \le 0,\ g(t) > 0 \ \text{ for all } \ t \in \mathbf{Z}^+ \ \text{ and } \ t - g(t) \to \infty \ \text{ as } \ t \to \infty.$$

Also, suppose there is a $\delta > 0$ *such that*

$$|a(t)| + \delta < 1, \tag{1.2}$$

and

$$|b(t)| \le \delta. \tag{1.3}$$

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Then the zero solution of (1.1) *is asymptotically stable.*

Proof. Define the Lyapunov functional $V(t,x)$ by

$$V(t,x) = |x(t)| + \delta \sum_{s=t-g(t)}^{t-1} |x(s)|.$$

Then along solutions of (1.1) we have

$$\begin{aligned}
\triangle V &= |x(t+1)| - |x(t)| + \delta \sum_{s=t+1-g(t+1)}^{t} |x(s)| - \delta \sum_{s=t-g(t)}^{t-1} |x(s)| \\
&\leq |a(t)||x(t)| - |x(t)| + |b(t)|x(t-g(t))| \\
&\quad + \delta \sum_{s=t+1-g(t)}^{t} |x(s)| - \delta \sum_{s=t-g(t)}^{t-1} |x(s)| \\
&= \Big(|a(t)| + \delta - 1\Big)|x(t)| + \Big(|b(t)| - \delta\Big)|x(t-g(t))| \\
&\leq \Big(|a(t)| + \delta - 1\Big)|x(t)| \\
&\leq -\gamma |x(t)| \text{ for some positive constant } \gamma.
\end{aligned}$$

By referring to [2], it follows from the above relation that the zero solution of (1.1) is asymptotically stable. □

Remark 1. One of the difficulties that are associated with the above method is the construction of a suitable Lyapunov functional. Moreover, conditions (1.2) and (1.3) in Theorem 1 imply that

$$|a(t)| + |b(t)| < 1 \text{ for all } t \in \mathbb{Z}.$$

In this paper we concentrate on the delay functional difference equation

$$x(t+1) = a(t)x(t) + g(t, x_t), \tag{1.4}$$

where $a : \mathbf{Z}^+ \to \mathbf{R}$, and $g : \mathbf{Z}^+ \times \mathcal{C}$ is continuous with $\mathcal{C}$ being the Banach space of bounded functions $\phi : \mathbf{Z}^- \to \mathbf{R}$ with the supremum norm

$$||\phi|| = \sup_{t \in \mathbf{Z}^-} \{|\phi(t)|\} < \infty.$$

If $x_t \in \mathcal{C}$, then $x_t(s) = x(t+s)$ for $s \in \mathbf{Z}^-$.

We note that when the function $g(t,\phi)$ in not a linear function, then the search for a suitable Lyapunov function or functional becomes extremely difficult, without severe restrictions, see Theorem 1 or [6].

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2. Stability

In this section, we use fixed point theory to obtain necessary and sufficient conditions for the asymptotic stability of the zero solution of (1.4). Throughout this paper we assume $g(t, 0) = 0$ so that $x = 0$ is a solution of (1.4). For every positive $\beta > 0$, we define the set

$$\mathcal{C}(\beta) = \{\phi \in \mathcal{C} : ||\phi|| \leq \beta\}.$$

Given a function $\psi : \mathbf{Z} \to \mathbf{Z}$, we define $||\psi||^{[s,t]} = \sup\{|\psi(u)| : s \leq u \leq t\}$. Moreover, for $D > 0$ a sequence $x : (-\infty, D] \to \mathbf{R}$ is called a solution of (1.4) through $(t_0, \phi) \in \mathbf{Z}^+ \times \mathcal{C}$ if $x_{t_0} = \phi$ and x satisfies (1.4) on $[t_0, D]$. Due to the importance of the next result, we summarize it in the following lemma.

Lemma 1. *Suppose that $a(t) \neq 0$ for all $t \in \mathbb{Z}^+$. Then $x(t)$ is a solution of equation (1.4) if and only if*

$$x(t) = \phi(t_0) \prod_{s=t_0}^{t-1} a(s) + \sum_{s=t_0}^{t-1} \prod_{u=s+1}^{t-1} a(u)\, g(s, x_s) \text{ for } t \geq t_0. \tag{2.1}$$

The proof of Lemma 1 follows easily from the variation of parameters formula and hence we omit it.

In preparation for our next theorem we let $L > 0$ be a constant, $\delta_0 \geq 0$ and $t_0 \geq 0$. Let $\phi \in \mathcal{C}(\delta_0)$ be fixed and set

$$S = \Big\{x : \mathbf{Z} \to \mathbf{R} : x_{t_0} = \phi, x_t \in \mathcal{C}(L) \text{ for } t \geq t_0,\ x(t) \to 0 \text{ as } t \to \infty\Big\}.$$

Then, S is a complete metric space with metric

$$\rho(x, y) = \sup_{t \geq t_0} |x(t) - y(t)|.$$

Define the mapping $P : S \to S$ by

$$\big(Px\big)(t) = \phi(t) \text{ if } t \leq t_0$$

and

$$\big(Px\big)(t) = \phi(t_0) \prod_{s=t_0}^{t-1} a(s) + \sum_{s=t_0}^{t-1} \prod_{u=s+1}^{t-1} a(u)\, g(s, x_s) \text{ for } t \geq t_0.$$

It is clear that for $\varphi \in S$, $P\varphi$ is continuous.

Theorem 2. *Assume the existence of positive constants α, L, and a sequence $b : \mathbf{Z}^+ \to [0, \infty)$ such that the following conditions hold:*

(i) $a(t) \neq 0$ *for all* $t \in \mathbb{Z}^+$.

(ii) $\displaystyle\sum_{s=0}^{t-1} \Big| \prod_{u=s+1}^{t-1} a(u) \Big| b(s) \leq \alpha < 1$ *for all* $t \in \mathbb{Z}^+$.

(iii) $|g(t, \phi) - g(t, \psi)| \leq b(t)||\phi - \psi||$ *for all* $\phi, \psi \in \mathcal{C}(L)$.

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(iv) *For each* $\epsilon > 0$ *and* $t_1 \geq 0$, *there exists a* $t_2 > t_1$ *such that for* $t > t_2, x_t \in \mathcal{C}(L)$ *imply*

$$|g(t, x_t)| \leq b(t)\Big(\epsilon + ||x||^{[t_1, t-1]}\Big).$$

Then the zero solution of (1.4) *is asymptotically stable if and only if*

(v) $\Big|\prod_{s=0}^{t-1} a(s)\Big| \to 0$ *as* $t \to \infty$.

Proof. Suppose (v) hold and let $K = \sup_{t \geq t_0}\Big|\prod_{s=t_0}^{t-1} a(s)\Big|$ for $t_0 \in \mathbf{Z}^+$. Then $K > 0$ due to (i). Choose $\delta_0 > 0$ such that $\delta_0 K + \alpha L \leq L$. Then for $x \in S$ and for fixed $\phi \in \mathcal{C}(\delta_0)$ we have

$$\begin{aligned} |\big(Px\big)(t)| &\leq |\phi(t_0)|\Big|\prod_{s=t_0}^{t-1} a(s)\Big| + \sum_{s=t_0}^{t-1}\Big|\prod_{u=s+1}^{t-1} a(u)\Big| b(s)||x_s|| \\ &\leq \delta_0 K + \alpha L \leq L \quad \text{for} \quad t \geq t_0. \end{aligned}$$

Hence, $\big(Px\big) \in \mathcal{C}(L)$. Next we show that $\big(Px\big)(t) \to 0$ as $t \to \infty$. Let $x \in S$. As a consequence of $x(t) \to 0$ as $t \to \infty$, there exists $t_1 > t_0$ such that $|x(t)| < \epsilon$ for all $t \geq t_1$. Moreover, since $|x(t)| \leq L$, for all $t \in \mathbf{Z}$, by (iv) there is a $t_2 > t_1$ such that for $t > t_2$ we have

$$|g(t, x_t)| \leq b(t)\Big(\epsilon + ||x||^{[t_1, t-1]}\Big).$$

Thus, for $t \geq t_2$, we have

$$\begin{aligned} \Big|\sum_{s=t_0}^{t-1}\prod_{u=s+1}^{t-1} a(u)\, g(s, x_s)\Big| &\leq \sum_{s=t_0}^{t_2-1}\Big|\prod_{u=s+1}^{t-1} a(u)\Big|\, |g(s, x_s)| \\ &\quad + \sum_{s=t_2}^{t-1}\Big|\prod_{u=s+1}^{t-1} a(u)\Big|\, |g(s, x_s)| \\ &\leq \sum_{s=t_0}^{t_2-1}\Big|\prod_{u=s+1}^{t-1} a(u)\Big|\, ||x_s|| \\ &\quad + \sum_{s=t_2}^{t-1}\Big|\prod_{u=s+1}^{t-1} a(u)\Big|\, b(s)\Big(\epsilon + ||x||^{[t_1, s-1]}\Big) \\ &\leq \sum_{s=t_0}^{t_2-1}\Big|\prod_{u=s+1}^{t_2-1} a(u)\Big|\Big|\prod_{u=t_2}^{t-1} a(u)\Big|\, ||x_s|| + 2\alpha\epsilon \\ &\leq \alpha L\Big|\prod_{u=t_2}^{t-1} a(u)\Big| + 2\alpha\epsilon. \end{aligned}$$

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By (v), there exists $t_3 > t_2$ such that

$$\delta_0\Big|\prod_{u=s+1}^{t-1} a(u)\Big| + L\Big|\prod_{u=t_2}^{t_3-1} a(u)\Big| < \epsilon.$$

Thus, for $t \geq t_3$, we have

$$|\big(Px\big)(t)| \leq \delta_0\Big|\prod_{u=s+1}^{t-1} a(u)\Big| + \alpha L\Big|\prod_{u=t_2}^{t-1} a(u)\Big| + 2\alpha\epsilon < 3\epsilon.$$

Hence, $\big(Px\big)(t) \to 0$ as $t \to \infty$. Left to show that $\big(P\varphi\big)(t)$ is a contraction under the maximum norm. Let $\zeta, \eta \in S$. Then

$$\begin{aligned}\Big|(P\zeta)(t) - (P\eta)(t)\Big| &\leq \sum_{s=t_0}^{t-1} \Big|\prod_{u=s+1}^{t-1} a(u)\Big| \, |g(s,\zeta_s) - g(s,\eta_s)| \\ &\leq \sum_{s=t_0}^{t-1} \Big|\prod_{u=s+1}^{t-1} a(u)\Big| b(s) \, |\zeta_s - \eta_s| \\ &\leq \alpha\rho(\zeta,\eta).\end{aligned}$$

Or,

$$\rho(P\zeta, P\eta) \leq \alpha\rho(\zeta,\eta).$$

Thus, by the contraction mapping principle P has a unique fixed point in S which solves (1.4) with $\phi \in \mathcal{C}(\delta_0)$ and $x(t) = x(t,t_0,\phi) \to 0$ as $t \to \infty$. We are left with showing that the zero solution of (1.4) is stable. Let $\epsilon > 0, \epsilon < L$ be given and choose $0 < \delta < \epsilon$ so that $\delta K + \alpha\epsilon < \epsilon$. We claim that $|x(t)| < \epsilon$ for all $t \geq t_0$. Notice that by the choice of δ we have $|x(t_0)| < \epsilon$. Let $t^* \geq t_0 + 1$ be such that $|x(t^*)| \geq \epsilon$ and $|x(s)| < \epsilon$ for $t_0 \leq s \leq t^* - 1$. If $x(t) = x(t,t_0,\phi)$ is a solution for (1.4) with $||\phi|| < \delta$, then

$$\begin{aligned}|x(t^*)| &\leq \delta\,\Big|\prod_{s=t_0}^{t^*-1} a(s)\Big| + \sum_{s=t_0}^{t^*-1}\Big|\prod_{u=s+1}^{t^*-1} a(u)\Big| b(s)||x_s|| \\ &\leq \delta K + \alpha\epsilon < \epsilon,\end{aligned}$$

which contradicts the definition of t^*. Thus $|x(t)| < \epsilon$ for all $t \geq t_0$ and hence the zero solution of (1.4) is asymptotically stable.

Conversely, suppose (v) does not hold. Then by (i) there exists a sequence $\{t_n\}$ such that for positive constant q,

$$\Big(\Big|\prod_{u=0}^{t_n-1} a(u)\Big|\Big)^{-1} = q \;\text{ for }\; n = 1,2,3,\dots.$$

Now by (ii) we have that

$$\sum_{s=0}^{t_n-1}\Big|\prod_{u=s+1}^{t_n-1} a(u)\Big| b(s) \leq \alpha,$$

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from which we get that

$$\Big(\Big|\prod_{u=0}^{t_n-1}a(u)\Big|\Big)^{-1}\sum_{s=0}^{t_n-1}\Big|\prod_{u=s+1}^{t_n-1}a(u)\Big|b(s)\leq\alpha\Big(\Big|\prod_{u=0}^{t_n-1}a(u)\Big|\Big)^{-1}.$$

This simplifies to

$$\sum_{s=0}^{t_n-1}\Big(\Big|\prod_{u=0}^{s}a(u)\Big|\Big)^{-1}b(s)\leq\alpha q.$$

Thus the sequence $\left\{\sum_{s=0}^{t_n-1}\Big(\Big|\prod_{u=0}^{s}a(u)\Big|\Big)^{-1}b(s)\right\}$ is bounded and hence there is a convergent subsequence. Thus, for the sake of keeping a simple notation we may assume that

$$\lim_{n\to\infty}\sum_{s=0}^{t_n-1}\Big(\Big|\prod_{u=0}^{s}a(u)\Big|\Big)^{-1}b(s)=\omega$$

for some positive constant ω. Next we may choose a positive integer $\widetilde{n}$ large enough so that

$$\sum_{s=t_{\widetilde{n}}}^{t_n-1}\Big(\Big|\prod_{u=0}^{s}a(u)\Big|\Big)^{-1}b(s)<\frac{1-\alpha}{2K^2}$$

for all $n\geq\widetilde{n}$.

Consider the solution $x(t,t_{\widetilde{n}},\phi)$ with $\phi(s)=\delta_0$ for $s\leq\widetilde{n}$. Then, $|x(t)|\leq L$ for all $n\geq\widetilde{n}$ and

$$\begin{aligned}|x(t)|&\leq\delta_0\,\Big|\prod_{s=t_{\widetilde{n}}}^{t-1}a(s)\Big|+\sum_{s=t_{\widetilde{n}}}^{t-1}\Big|\prod_{u=s+1}^{t-1}a(u)\Big|b(s)||x_s||\\&\leq\delta_0K+\alpha||x_t||.\end{aligned}$$

This implies

$$|x(t)|\leq\frac{\delta_0K}{1-\alpha}\ \text{ for all }\ t\geq t_{\widetilde{n}}.$$

On the other hand, for $n\geq\widetilde{n}$, we also have

$$\begin{aligned}|x(t)|&\geq\delta_0\,\Big|\prod_{s=t_{\widetilde{n}}}^{t_n-1}a(s)\Big|-\sum_{s=t_{\widetilde{n}}}^{t-1}\Big|\prod_{u=s+1}^{t_n-1}a(u)\Big|b(s)||x_s||\\&\geq\delta_0\,\Big|\prod_{s=t_{\widetilde{n}}}^{t_n-1}a(s)\Big|-\frac{\delta_0K}{1-\alpha}\Big|\prod_{u=0}^{t_n-1}a(u)\Big|\sum_{s=t_{\widetilde{n}}}^{t-1}\Big|\Big(\prod_{u=0}^{s}a(u)\Big)^{-1}\Big|b(s)\\&=\delta_0\,\Big|\prod_{s=t_{\widetilde{n}}}^{t_n-1}a(s)\Big|-\frac{\delta_0K}{1-\alpha}\Big|\prod_{u=0}^{t_{\widetilde{n}}-1}a(s)\Big|\prod_{u=t_{\widetilde{n}}}^{t_n-1}a(s)\Big|\sum_{s=t_{\widetilde{n}}}^{t-1}\Big|\Big(\prod_{u=0}^{s}a(u)\Big)^{-1}\Big|b(s)\\&\geq\Big|\prod_{s=t_{\widetilde{n}}}^{t_n-1}a(s)\Big|\Big(\delta_0-\frac{\delta_0K}{1-\alpha}K\sum_{s=t_{\widetilde{n}}}^{t-1}\Big|\Big(\prod_{u=0}^{s}a(u)\Big)^{-1}\Big|b(s)\Big)\end{aligned}$$

$$\geq \Big|\prod_{s=t_{\tilde{n}}}^{t_n-1} a(s)\Big|(\delta_0 - \frac{\delta_0 K}{1-\alpha} K \frac{1-\alpha}{2K^2}) = \frac{\delta_0}{2}\Big|\prod_{s=t_{\tilde{n}}}^{t_n-1} a(s)\Big|$$
$$= \frac{\delta_0}{2}\Big|\prod_{u=0}^{t_n-1} a(s)\Big|\Big(\Big|\prod_{u=0}^{t_{\tilde{n}}-1} a(s)\Big|\Big)^{-1} \to \frac{\delta_0}{2} q/q \neq 0 \ \text{ as } \ n \to \infty.$$

Hence, condition (v) is necessary. This completes the proof. □

3. Infinite delay Volterra equations

In this section we apply the results of the previous section to nonlinear Volterra infinite delay equations of the form

$$x(t+1) = a(t)x(t) + \sum_{s=-\infty}^{t-1} G(t,s,x(s)), \tag{3.1}$$

where $a : \mathbf{Z}^+ \to \mathbf{R}$ and $G : \Omega \times \mathbf{R} \to \mathbf{R}$, $\Omega = \{(t,s) \in \mathbf{Z}^2 : t \geq s\}$ and G is continuous in x. We prove the following theorem which gives necessary and sufficient conditions for the stability of the zero solution of (3.1).

Theorem 3. *Assume the existence of positive constants α, L, and a sequence $p : \Omega \to \mathbf{R}^+$ such that the following conditions hold*:

(I) $a(t) \neq 0$ *for all* $t \in \mathbb{Z}^+$,

(II) $\sup_{t\in\mathbf{Z}^+} \sum_{s=0}^{t-1}\Big|\prod_{u=s+1}^{t-1} a(u)\Big|\sum_{\tau=0}^{s-1} p(s,\tau) \leq \alpha < 1$ *for all* $t \in \mathbb{Z}^+$,

(III) *If* $|x|, |y| \leq L$, *then*
$$|G(t,s,x) - G(t,s,y)| \leq p(t,s)|x-y|$$
and $G(t,s,0) = 0$ *for all* $(t,s) \in \Omega$,

(IV) *For each* $\epsilon > 0$ *and* $t_1 \geq 0$, *there exists a* $t_2 > t_1$ *such that for* $t \geq t_2$, *implies*
$$\sum_{s=-\infty}^{t_1-1} p(t,s) \leq \epsilon \sum_{s=-\infty}^{t-1} p(t,s).$$

Then the zero solution of (3.1) *is asymptotically stable if and only if*

(V) $\Big|\prod_{s=0}^{t-1} a(s)\Big| \to 0$ *as* $t \to \infty$.

Proof. We only need to verify that (iii) and (iv) of Theorem 2 hold. First we remark that due to condition (iii) we have that $|G(t,s,x)| \leq p(t,s)L$. Equation (3.1) can be put in the form of Equation (1.4) by letting

$$g(t,\phi) = \sum_{s=-\infty}^{-1} G(t, t+s, \phi(s)).$$

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To verify (iii) we let $b(t) = \sum_{s=-\infty}^{t-1} p(t,s)$ and then for any functions $\phi, \varphi \in \mathcal{C}(L)$, we have

$$\begin{aligned}
|g(t,\phi) - g(t,\varphi)| &\leq \Big| \sum_{s=-\infty}^{-1} G(t, t+s, \phi(s)) - \sum_{s=-\infty}^{-1} G(t, t+s, \varphi(s)) \Big| \\
&\leq \sum_{s=-\infty}^{-1} p(t, t+s)\ ||\phi - \varphi|| \\
&= b(t)||\phi - \varphi||.
\end{aligned}$$

Next we verify (iv). Let $\epsilon > 0$ and $t_1 \geq 0$ be given. By (IV) there exists a $t_2 > t_1$ such that

$$L \sum_{s=-\infty}^{t_1-1} p(t,s) < \epsilon \sum_{s=-\infty}^{t-1} p(t,s) \ \text{ for all } \ t > t_2.$$

Let $x_t \in \mathcal{C}(L)$ and for $t > t_2$ we have

$$\begin{aligned}
|g(t,x_t)| &\leq \sum_{s=-\infty}^{t_1-1} \big|G(t,s,x(s))\big| + \sum_{s=t_1}^{t-1} \big|G(t,s,x(s))\big| \\
&\leq \sum_{s=-\infty}^{t_1-1} Lp(t,s) + \sum_{s=t_1}^{t-1} p(t,s)|x(s)| \\
&\leq \epsilon \sum_{s=-\infty}^{t-1} p(t,s) + \sum_{s=t_1}^{t-1} p(t,s)||x||^{[t_1,t-1]} \\
&\leq b(t)\Big(\epsilon + ||x||^{[t_1,t-1]}\Big).
\end{aligned}$$

This implies that (iv) is satisfied, and hence by Theorem 2, the zero solution of (3.1) is asymptotically stable if and only if (v) holds. □

We end the paper with the following example.

Example 1. Consider the difference equation

$$x(t+1) = \frac{1}{2^t}x(t) + \sum_{s=-\infty}^{t-1} 2^{s-t}x(s), \ n \geq 0. \tag{3.2}$$

In this example we take $t_0 = 0$. We observe that $a(t) = \frac{1}{2^t}$, and $G(t,s,x) = 2^{s-t}x(s)$. We make sure all the conditions of Theorem 3 are satisfied. Thus,

$$\prod_{s=0}^{t-1} \frac{1}{2^s} \to 0 \ \text{ as } t \to \infty,$$

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and hence condition (V) is satisfied. It is clear that $p(t,s) = 2^{s-t}$. Next we make sure condition (II) is satisfied.

$$\begin{aligned}\sup_{t\in\mathbf{Z}^+}\sum_{s=0}^{t-1}\Big|\prod_{u=s+1}^{t-1}a(u)\Big|\sum_{\tau=0}^{s-1}p(s,\tau) &= \sup_{t\in\mathbf{Z}^+}\sum_{s=0}^{t-1}\Big|\prod_{u=s+1}^{t-1}2^{-u}\Big|\sum_{\tau=0}^{s-1}2^{s-\tau}\\ &\le \sup_{t\in\mathbf{Z}^+}\sum_{s=0}^{t-1}\Big|\prod_{u=s+1}^{t-1}2^{-u}(1-2^{-s})\\ &\le \sup_{t\in\mathbf{Z}^+}\sum_{s=0}^{t-1}2^{1-t}(1-2^{-s})\Big|\\ &\le 2^{1-t}[-2^{1-t}+2+\frac{4^{1-t}}{3}-4/3]\\ &\le 2/3 \ \text{ for all } \ t\in\mathbb{Z}^+.\end{aligned}$$

Hence (II) is satisfied. Left to show (IV) is satisfied. Let $t_1 \ge 0$ be given. Then

$$\begin{aligned}\sum_{s=-\infty}^{t_1-1}p(t,s) &= \sum_{s=-\infty}^{t_1-1}2^{-t+s}\\ &= 2^{-t}[2^{t_1}-2^{-\infty}]\\ &\le 2^{t-t_2}\\ &= 2^{-t_2}\sum_{s=-\infty}^{t-1}2^{-t+s}\\ &\le \epsilon\sum_{s=-\infty}^{t-1}p(t,s), t\ge t_2\ge t_1.\end{aligned}$$

Thus all the conditions of Theorem 2 are satisfied and the zero solution of (3.2) asymptotically stable.

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