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FINITE p-GROUPS IN WHICH THE NORMALIZER OF EVERY NON-NORMAL SUBGROUP IS CONTAINED IN ITS NORMAL CLOSURE

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Abstract. In this paper, finite p-groups G satisfying $N_G(H) \leq H^G$ for every non-normal subgroup H of G are completely classified. This solves a problem proposed by Y. Berkovich.

1. Introduction

All groups considered in this paper are finite. It is well-known that the normality of subgroups plays an important role in the research of group theory. But not every subgroup is normal. If H is a non-normal subgroup of a p-group G, then we have

$$H < N_G(H) < G \text{ and } H < H^G < G.$$

It is a way to measure the degree of the normality of H by using $N_G(H)$ or H^G. Many authors have developed their work in this line. For example, Lv, Zhou and Yu in [4] studied the p-group G with $|\langle a\rangle^G : \langle a\rangle| \leq p^m$ for every cyclic subgroup $\langle a\rangle$ of G, and Zhang and Guo in [7] investigated the p-groups whose non-normal cyclic subgroups have small index in their normalizers, and Zhao and Guo in [8] determined the p-groups in which the normal closures of the non-normal cyclic subgroups have small index. Y. Berkovich has proposed the following problem:

Problem 1.1 ([1, Problem 439])**.** Study the p-groups G such that, whenever H is a non-normal subgroup of G, then $N_G(H) \leq H^G$.

This problem connects normalizers with normal closures, and the condition $N_G(H) \leq H^G$ indicates that H has low degree of normality in some sense.

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In the following, we will classify the p-groups in Problem 1.1 completely. For convenience, such groups are called $\mathcal{P}$-groups. The main results are:

Theorem 1.2. *A 2-group G is a $\mathcal{P}$-group if and only if G is of one of the following types*:

(1) *a Dedekind 2-group*;
(2) *a maximal class 2-group*;
(3) $\langle a, b \mid a^{2^2} = b^{2^n} = 1, [b, a] = b^{-2}\rangle$, *where* $n \geq 2$;
(4) $\langle a, b \mid a^{2^2} = b^{2^n} = 1, [b, a] = b^{2^{n-1}-2}\rangle$, *where* $n \geq 3$.

Moreover, except for Q_8 (Q_8 is of type (1) and type (2)), groups of different types, or of same type but with different values of parameters, are not isomorphic.

Theorem 1.3. *Let p be an odd prime. Then a p-group G is a $\mathcal{P}$-group if and only if G is of one of the following types*:

(1) *an abelian p-group*; (2) $M_p(2, 1)$; (3) $M_p(1, 1, 1)$; (4) $M_p(2, 2)$.

The meanings of $M_p(2, 1)$, $M_p(1, 1, 1)$ and $M_p(2, 2)$ see Lemma 2.2.

2. Preliminaries

In this section, we first recall some basic concepts and notations, and then give some basic results which are useful in the sequel.

We use D_{2^n}, Q_{2^n}, SD_{2^n}, C_{p^n} and C_p^n to denote the dihedral group of order 2^n, the generalized quaternion group of order 2^n, the semi-dihedral group of order 2^n, the cyclic group of order p^n and the elementary abelian group of order p^n, respectively. We use $A * B$, $A \times B$ and $A - B$ to denote the central product, the direct product and the set $\{x \mid x \in A, \text{but}\, x \notin B\}$ of a group A and a group B. We also use $d(G)$ and $c(G)$ to denote the minimal number of generators of a group G and the nilpotent class of G. If G is a p-group, then $\Omega_{\{i\}}(G) = \{g \in G \mid g^{p^i} = 1\}$, $\mho_{\{i\}}(G) = \{g^{p^i} \mid g \in G\}$, $\Omega_i(G) = \langle \Omega_{\{i\}}(G)\rangle$ and $\mho_i(G) = \langle \mho_{\{i\}}(G)\rangle$, respectively. All other terminology and notation not mentioned here are standard.

Definition 2.1 ([1, §1, Definition 2])**.** A group G of order p^m is said to be of maximal class if $m > 2$ and $c(G) = m - 1$.

Lemma 2.2 ([5])**.** *Let G be a minimal non-abelian p-group. Then G is isomorphic to one of the following groups*:

(1) $Q_8 = \langle a, b \mid a^4 = 1, b^2 = a^2, a^b = a^{-1}\rangle$;
(2) $M_p(n, m) = \langle a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}}\rangle$, *where* $n \geq 2$, $m \geq 1$;
(3) $M_p(n, m, 1) = \langle a, b \mid a^{p^n} = b^{p^m} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1\rangle$, *where* $n \geq m \geq 1$, *and if* $p = 2$, *then* $m + n \geq 3$.

Lemma 2.3 ([1, §1, Lemma 1.4])**.** *Let G be a p-group and $N \trianglelefteq G$. If N has no abelian G-invariant subgroups of type (p, p), then N is either cyclic or* *isomorphic to one of the groups D_{2^n}, Q_{2^n} and SD_{2^n}. If, in addition, $N \leq \Phi(G)$, then N is cyclic. In particular, if G has no abelian normal subgroups of type (p,p), then G is either cyclic or isomorphic to one of the groups D_{2^n}, Q_{2^n} and SD_{2^n}.*

Lemma 2.4 ([2, Satz.III, Theorem 11.9])**.**

(1) *If G is a non-abelian 2-group such that $G/G' \cong C_2^2$, then G is one of the groups D_{2^n}, Q_{2^n} and SD_{2^n}.*
(2) *If G is a 2-group of maximal class, then G is one of the groups D_{2^n}, Q_{2^n} and SD_{2^n}.*
(3) *A 2-group G is of maximal class if and only if G is a non-abelian 2-group with $G/G' \cong C_2^2$.*

Lemma 2.5. *Let G be a nontrivial 2-group. If G is not of maximal class, then there exists a nontrivial subgroup $N \leq Z(G)$ such that G/N is not of maximal class.*

Proof. If G is abelian, then the lemma is clear. Now assume that G is non-abelian. Then $G/G' \ncong C_2^2$ by Lemma 2.4, and there exists a nontrivial subgroup N of G such that $N \leq G' \cap Z(G)$. Since $(G/N)/(G'/N) \cong G/G'$, it follows from Lemma 2.4 once more that G/N is not of maximal class. □

Lemma 2.6. *Let N be a normal subgroup of a $\mathcal{P}$-group G. Then G/N is also a $\mathcal{P}$-group.*

Proof. For any subgroup H/N of G/N, if $H/N \ntrianglelefteq G/N$, then $H \ntrianglelefteq G$ and so $N_G(H) \leq H^G$. Noticing that

$$N_{G/N}(H/N) = N_G(H)/N \leq H^G/N = (H/N)^{G/N},$$

we see G/N is also a $\mathcal{P}$-group. □

Lemma 2.7. *Let G be a non-Dedekind p-group. If $d(G) \geq 3$ and $|G'| = p$, then G is a non-$\mathcal{P}$-group.*

Proof. Since G is not a Dedekind group, there exist elements $a, b \in G$ such that $\langle b \rangle \ntrianglelefteq G$ and $[a,b] \neq 1$. Now write $A = \langle a, b \rangle$. Then it follows from $|G'| = p$ that $A' = G'$ and $A \trianglelefteq G$. By [6, Lemma 2.2], A is a minimal non-abelian group and so $G = A * C_G(A)$ by [1, §4, Lemma 4.2]. Clearly $C_G(A) \leq N_G(\langle b \rangle)$ and $\langle b \rangle^G \leq A$. If $C_G(A) \leq \langle b \rangle^G$, then $G = A$, in contradiction to the condition $d(G) \geq 3$. Therefore G is a non-$\mathcal{P}$-group. □

Lemma 2.8. *Suppose that a, b and x are elements of a 2-group G, where $x \in Z(G)$ and $o(x) = 2$.*

(1) *If $[b,a] = b^{-2}x^i$ with $i = 0$ or 1, then $[b,a^2] = 1$;*
(2) *If $b^{2^{n+1}} = 1$, and $[b,a] = b^{2^{n-1}-2}x^j$, where $n \geq 3$ and $j = 0$ or 1, then $[b,a^2] = b^{2^n}$.*

Proof. (1) From $[b,a]=b^{-2}x^i$, we get $b^a=b^{-1}x^i$. So $b^{a^2}=(b^{-1}x^i)^a=(b^{-1}x^i)^{-1}x^i=b$.

(2) Clearly, we have $b^a=b^{2^{n-1}-1}x^j$, and it follows that

$$b^{a^2}=(b^{2^{n-1}-1}x^j)^a=(b^{2^{n-1}-1}x^j)^{2^{n-1}-1}x^j=b^{(2^{n-1}-1)^2}=b^{-2^n+1}=b^{2^n}b.$$

Hence $[b,a^2]=b^{2^n}$. □

3. The classification of $\mathcal{P}$-groups

In this section, we first give some properties of $\mathcal{P}$-groups, and then classify $\mathcal{P}$-groups.

Lemma 3.1. *Let G be a minimal non-abelian p-group. Then G is a $\mathcal{P}$-group if and only if $|G|=p^3$ or $G\cong M_p(2,2)$.*

Proof. " $\Leftarrow$ " If $|G|=p^3$, then $|H|=p$ for any non-normal subgroup H of G and so $N_G(H)\trianglelefteq G$, which indicates that $H^G=N_G(H)$. If $G\cong M_p(2,2)$, then $\Omega_1(G)=Z(G)$ and so $|H|=p^2$ for any non-normal subgroup H of G. Similarly, we have $N_G(H)=H^G$. Hence the sufficiency holds.

" $\Rightarrow$ " Let G be a $\mathcal{P}$-group and suppose that $|G|>p^3$. By Lemma 2.2, G is one of the following groups:

(a) $G=\langle a,b \mid a^{p^n}=b^{p^m}=1,[a,b]=a^{p^{n-1}}\rangle$, where $n+m\geq 4$ and $n\geq 2, m\geq 1$;
(b) $G=\langle a,b \mid a^{p^n}=b^{p^m}=c^p=1,[a,b]=c,[c,a]=[c,b]=1\rangle$, where $n+m\geq 3$ and $n\geq m\geq 1$.

Assume G is type (a). If $n=2$ and $m>2$, then $\langle ab^p\rangle\ntrianglelefteq G$ and $\langle ab^p\rangle^G=\langle ab^p,a^p\rangle$. Clearly $a\in N_G(\langle ab^p\rangle)$ but $a\notin\langle ab^p\rangle^G$, so G is not a $\mathcal{P}$-group. If $n\geq 3$, then $\langle b\rangle^G=\langle b,a^{p^{n-1}}\rangle$. Since $a^p\in N_G(\langle b\rangle)$ and $a^p\notin\langle b\rangle^G$, we see G is not a $\mathcal{P}$-group. Therefore $G\cong M_p(2,2)$. Now assume G is type (b). Then $\langle b\rangle^G=\langle b,c\rangle$. Noticing that $a^p\in N_G(\langle b\rangle)$ and $a^p\notin\langle b\rangle^G$, hence G is not a $\mathcal{P}$-group. The proof is complete. □

Lemma 3.2. *Let G be a 2-group of maximal class. Then G is a $\mathcal{P}$-group.*

Proof. Assume the lemma is false and let G be a counterexample of minimal order. Then G has a non-normal subgroup H such that $N_G(H)\nleq H^G$ and by Lemma 3.1, we see $|G|\geq 2^4$. Now write $\overline{G}=G/Z(G)$. Then $\overline{G}$ is also a 2-group of maximal class, and so $\overline{G}$ is a $\mathcal{P}$-group.

If $\overline{H}\ntrianglelefteq\overline{G}$, then $N_{\overline{G}}(\overline{H})\leq\overline{H}^{\overline{G}}=H^G/Z(G)$ and it follows that $N_G(H)\leq H^G$, a contradiction. Now assume $\overline{H}\trianglelefteq\overline{G}$. Then $H^G=HZ(G)\neq H$ and $|H^G:H|=2$. In this case, if $|G:H|=4$, then $N_G(H)=H^G$, a contradiction. If $|G:H|>4$, then $|G:H^G|\geq 4$ and thus $H^G\leq G'$. By Lemma 2.4, G' is cyclic and so H *char* G', which implies $H\trianglelefteq G$, the final contradiction. □

Lemma 3.3. *Let G be a p-group of order at least p^5. If there exists a normal subgroup N of order p such that G/N is a Dedekind group, then G is either a non-$\mathcal{P}$-group or a Dedekind group.*

Proof. Assume G is not a Dedekind group. In the following, we will prove that G is a non-$\mathcal{P}$-group. Write $\overline{G} = G/N$. Then $\overline{G}$ is either abelian or isomorphic to $Q_8 \times C$, where C is an elementary abelian 2-group. Hence $|G'| = p$ or 4.

Firstly, assume $|G'| = p$. If $d(G) = 2$, then G is a non-$\mathcal{P}$-group by [6, Lemma 2.2] and Lemma 3.1. If $d(G) \geq 3$, then G is also a non-$\mathcal{P}$-group by Lemma 2.7. Now assume $|G'| = 4$. Then $\overline{G} \cong Q_8 \times C$ and thus $\exp(G) = 8$ or 4. If $\exp(G) = 8$, then there exist elements $x,\ y \in G$ such that $o(x) = 8$ and $\langle \overline{x}, \overline{y} \mid \overline{x}^4 = 1, \overline{y}^2 = \overline{x}^2, [\overline{x}, \overline{y}] = \overline{x}^2 \rangle \cong Q_8$. Let $N = \langle z \rangle$. From $\overline{y}^2 = \overline{x}^2$, we get $x^2 = y^2 z^k$, where $k = 0$ or 1, and therefore $[x^2, y] = 1$. On the other hand, by $[\overline{x}, \overline{y}] = \overline{x}^2$,we have $[x, y] = x^2 z^i$ with $i = 0$ or 1, and it follows that $[x^2, y] = [x, y]^x [x, y] = [x, y]^2 = x^4$, a contradiction. Hence $\exp(G) = 4$. Since G is non-Dedekind and $\overline{G}$ is Dedekind, there exists an element $u \in G$ such that $\langle u \rangle \ntrianglelefteq G$, $\langle u \rangle^G = \langle u \rangle \times N$ and thus $|\langle u \rangle^G| \bigm| 8$. Let C be the conjugacy class of u. Noticing that $u^g = u[u, g] \in uG'$ with $g \in G$, we see $|C| \leq |uG'| \leq 4$ and thus $|G : C_G(u)| = |C| \bigm| 4$. If $N_G(\langle u \rangle) \leq \langle u \rangle^G$, then since $|G| \geq 2^5$, it is easy to see that $|G| = 2^5$, $o(u) = 4$ and $N_G(\langle u \rangle) = C_G(u) = \langle u \rangle^G$. Hence, for any $h \in G$, $u^h \neq u^3$ and so $|C| \leq 2$ as $C \subseteq \langle u \rangle^G$, which implies $|G| \leq 2^4$, a contradiction. Therefore G is a non-$\mathcal{P}$-group. The proof is complete. □

Lemma 3.4. *Let G be a non-abelian p-group of order p^4. Then G is a $\mathcal{P}$-group if and only if G is isomorphic to one of the following groups*:

(1) *Maximal class 2-groups of order p^4*; (2) $Q_8 \times C_2$; (3) $M_p(2, 2)$.

Proof. By Lemma 3.1 and Lemma 3.2, the sufficiency holds. We now prove the necessity. Since G is non-abelian of order p^4, we have $|G'| = p$ or p^2.

Assume $|G'| = p$. If $d(G) = 2$, then G is a minimal non-abelian p-group by [6, Lemma 2.2] and it follows from Lemma 3.1 that $G \cong M_p(2, 2)$. If $d(G) \geq 3$, then G is a Dedekind p-group by Lemma 2.7 and therefore $G \cong Q_8 \times C_2$. Now assume $|G'| = p^2$. From $G/C_G(G') \lesssim \mathrm{Aut}\,(G')$, we get that G has an abelian maximal subgroup A, and so G is a p-group of maximal class by [1, §1, Exercise 4]. Hence for $i = 1, 2$, G has unique normal subgroup of order p^i. If $p = 2$, then G is a 2-group of maximal class. If $p > 2$, then $G' \cong C_p^2$ by Lemma 2.3, and therefore G' has a subgroup H such that $|H| = p$ and $H \ntrianglelefteq G$. Thus $H^G = G'$. Noticing that $A \leq N_G(H)$, we see $N_G(H) \nleq H^G$. This show that G is not a $\mathcal{P}$-group. The proof is complete. □

Lemma 3.5. *If G is a group of one of the following types, then G is a $\mathcal{P}$-group.*

(1) $\langle a, b \mid a^{2^2} = b^{2^n} = 1, [b, a] = b^{-2} \rangle$, *where* $n \geq 2$;

(2) $\langle a, b \mid a^{2^2} = b^{2^n} = 1, [b, a] = b^{2^{n-1}-2} \rangle$, *where* $n \geq 3$.

Moreover, groups of different types, or of same type but with different values of parameters, are not isomorphic.

Proof. Let $G_i = \langle a_i, b_i\rangle$ be a group of type (i) with $i \in \{1, 2\}$. Then

(1) $G_1 = \langle a_1, b_1 \mid a_1^{2^2} = b_1^{2^n} = 1, [b_1, a_1] = b_1^{-2}\rangle$, where $n \geq 2$;
(2) $G_2 = \langle a_2, b_2 \mid a_2^{2^2} = b_2^{2^n} = 1, [b_2, a_2] = b_2^{2^{n-1}-2}\rangle$, where $n \geq 3$.

Clearly, $C_{\langle a_i\rangle}(b_i) = \langle a_i^2\rangle$, $C_{\langle b_i\rangle}(a_i) = \langle b_i^{2^{n-1}}\rangle$. If $a_i^k b_i^j \in Z(G_i)$, then $1 = [a_i^k b_i^j, b_i] = [a_i^k, b_i]$, and so $a_i^k \in \langle a_i^2\rangle$. Similarly we have $b_i^j \in \langle b_i^{2^{n-1}}\rangle$. This shows that $Z(G_i) = \langle a_i^2\rangle \times \langle b_i^{2^{n-1}}\rangle$, and for any integers s, t, it follows that

$$(a_i^{2s} b_i^t)^2 = b_i^{2t};$$
$$(a_1^{2s+1} b_1^t)^2 = (a_1 b_1^t)^2 = a_1^2 (b_1^{a_1})^t b_1^t = a_1^2;$$
$$(a_2^{2s+1} b_2^t)^2 = (a_2 b_2^t)^2 = a_2^2 (b_2^{a_2})^t b_2^t = a_2^2 b_2^{t2^{n-1}}.$$

Hence

$$Z(G_i) = \langle a_i^2\rangle \times \langle b_i^{2^{n-1}}\rangle = \Omega_1(G_i).$$

In addition, we have $G_i' = \langle b_i^2\rangle$, and then

$$|G_i/\langle a_i^2\rangle : (G_i/\langle a_i^2\rangle)'| = |G_i/\langle a_i^2 b_i^{2^{n-1}}\rangle : (G_i/\langle a_i^2 b_i^{2^{n-1}}\rangle)'| = 4.$$

By Lemma 2.4, $G_i/\langle a_i^2\rangle$ and $G_i/\langle a_i^2 b_i^{2^{n-1}}\rangle$ are all 2-groups of maximal class, and therefore $G_i/\langle a_i^2\rangle$ and $G_i/\langle a_i^2 b_i^{2^{n-1}}\rangle$ are all $\mathcal{P}$-groups by Lemma 3.2. For convenience, write $\overline{G_i} = G_i/\langle b_i^{2^{n-1}}\rangle$ in the following.

Firstly, we prove that G_1 is a $\mathcal{P}$-group. Suppose that G_1 is a counterexample of minimal order. Then $n \geq 3$ by Lemma 3.1. Noticing that $\overline{G_1}$ has the same type as G_1, we see that $\overline{G_1}$ is a $\mathcal{P}$-group. Thus for any subgroup M of order 2, G_1/M is a $\mathcal{P}$-group. Let H be any non-normal subgroup of G. Choose a subgroup N of H of order 2. Then $H/N \ntrianglelefteq G_1/N$ and so $N_{G_1/N}(H/N) \leq (H/N)^{G_1/N}$. Therefore $N_{G_1}(H) \leq H^{G_1}$, a contradiction.

Next, we prove G_2 is a $\mathcal{P}$-group. Since $\overline{G_2} = \langle \overline{a_2}, \overline{b_2} \mid \overline{a_2}^{2^2} = \overline{b_2}^{2^{n-1}} = \overline{1}, [\overline{b_2}, \overline{a_2}] = \overline{b_2}^{-2}\rangle$ is of the same type as G_1, $\overline{G_2}$ is a $\mathcal{P}$-group. Thus G_2/L is a $\mathcal{P}$-group for any subgroup L of order 2. If $H \ntrianglelefteq G_2$, then $N_{G_2}(H) \leq H^{G_2}$ by the same way as above and G_2 is a $\mathcal{P}$-group.

Clearly, $\mho_{\{1\}}(G_1) = \{a_1^2, b_1^{2e}\}$ and $\mho_{\{1\}}(G_2) = \{a_2^2 b_2^{l2^{n-1}}, b_2^{2f}\}$, where $0 \leq l \leq 1$, $0 \leq e \leq 2^{n-1} - 1$ and $0 \leq f \leq 2^{n-1} - 1$. Therefore groups of different types, or of same type but with different values of parameters, are not isomorphic. The proof is complete. □

Lemma 3.6. *Let G be a $\mathcal{P}$-group. If there exists a subgroup $N \leq Z(G)$ such that $|N| = 2$ and $G/N \cong \langle a, b \mid a^{2^2} = b^{2^{n-1}} = 1, [b, a] = b^{-2}\rangle$ with $n \geq 3$, then G is isomorphic to one of the following groups*:

(1) $\langle a, b \mid a^{2^2} = b^{2^n} = 1, [b, a] = b^{-2}\rangle$, *where* $n \geq 3$;
(2) $\langle a, b \mid a^{2^2} = b^{2^n} = 1, [b, a] = b^{2^{n-1}-2}\rangle$, *where* $n \geq 3$.

Proof. Suppose that $N = \langle x\rangle$ and let $G/N = \langle \overline{a}, \overline{b} \mid \overline{a}^{2^2} = \overline{b}^{2^{n-1}} = 1, [\overline{b}, \overline{a}] = \overline{b}^{-2}\rangle$ with $n \geq 3$. Then there exist integers i, j, $k \in \{0, 1\}$ such that

$$G = \langle a, b, x \mid a^{2^2} = x^i, b^{2^{n-1}} = x^j, x^2 = 1, [b, a] = b^{-2}x^k, [x, a] = [x, b] = 1\rangle.$$

By Lemma 2.8, $[b, a^2] = 1$. If $a^{2^2} = x$, $b^{2^{n-1}} = x^j$, $[b, a] = b^{-2}x$, then, replacing b with a^2b, we get $a^{2^2} = x$, $b^{2^{n-1}} = x^j$, $[b, a] = b^{-2}$. Hence, G is one of the following groups:

(a) $\langle a, b \mid a^{2^3} = 1, b^{2^{n-1}} = a^{2^2}, [b, a] = b^{-2}\rangle$, which is isomorphic to $\langle a, b \mid a^{2^3} = 1, b^{2^{n-1}} = a^{2^2}, [b, a] = b^{2^{n-1}-2}\rangle$;

(b) $\langle a, b \mid a^{2^3} = b^{2^{n-1}} = 1, [b, a] = b^{-2}\rangle$, which is isomorphic to $\langle a, b \mid a^{2^3} = 1, b^{2^{n-1}} = 1, [b, a] = a^4b^{-2}, [a^4, b] = 1\rangle$;

(c) $\langle a, b, x \mid a^{2^2} = b^{2^{n-1}} = x^2 = 1, [b, a] = b^{-2}x, [x, a] = [x, b] = 1\rangle$;

(d) $\langle a, b, x \mid a^{2^2} = b^{2^{n-1}} = x^2 = 1, [b, a] = b^{-2}, [x, a] = [x, b] = 1\rangle$;

(e) $\langle a, b \mid a^{2^2} = b^{2^n} = 1, [b, a] = b^{-2}\rangle$;

(f) $\langle a, b \mid a^{2^2} = b^{2^n} = 1, [b, a] = b^{2^{n-1}-2}\rangle$.

We will prove the groups (a), (b), (c) and (d) all are not $\mathcal{P}$-groups. In fact, if G is (a), then $\langle a^2b^{2^{n-2}}\rangle \ntrianglelefteq G$, $\langle a^2b^{2^{n-2}}\rangle^G = \langle a^2b^{2^{n-2}}, a^4\rangle$ and $b \in N_G(\langle a^2b^{2^{n-2}}\rangle) - \langle a^2b^{2^{n-2}}\rangle^G$; if G is (b), then $\langle a^2b\rangle \ntrianglelefteq G$, $\langle a^2b\rangle^G = \langle a^2b, a^4\rangle$ and $a^2 \in N_G(\langle a^2b\rangle) - \langle a^2b\rangle^G$; if G is (c), then $\langle a\rangle \ntrianglelefteq G$, $\langle a\rangle^G = \langle a, b^2x\rangle$ and $x \in N_G(\langle a\rangle) - \langle a\rangle^G$; if G is (d), then $\langle a\rangle \ntrianglelefteq G$, $\langle a\rangle^G = \langle a, b^2\rangle$ and $x \in N_G(\langle a\rangle) - \langle a\rangle^G$. Hence, G can only be (e) or (f). The proof is complete. □

Lemma 3.7. *Let G be a $\mathcal{P}$-group of order $2^{n+3} \geq 2^6$. Then for any subgroup $N \leq Z(G)$ with $|N| = 2$, $G/N \ncong \langle a, b \mid a^{2^2} = b^{2^n} = 1, [b, a] = b^{2^{n-1}-2}\rangle$.*

Proof. Assume the conclusion is false. Then G has a normal subgroup $\langle x\rangle$ of order 2 such that $G/\langle x\rangle = \langle \overline{a}, \overline{b} \mid \overline{a}^{2^2} = \overline{b}^{2^n} = 1, [\overline{b}, \overline{a}] = \overline{b}^{2^{n-1}-2}\rangle$ with $n \geq 3$. From which we see that there exist integers i, j, $k \in \{0, 1\}$ such that

$$(*)\quad G = \langle a, b, x \mid a^{2^2} = x^i, b^{2^n} = x^j, x^2 = 1, [b, a] = b^{2^{n-1}-2}x^k, [x, a] = [x, b] = 1\rangle.$$

By Lemma 2.8, we have $[b, a^2] = b^{2^n}$, and then

$$b^{a^2} = b^{2^n+1}\ ,\ [b^2, a^2] = 1\ ,\ b^{a^3} = b^{2^{n-1}-1}b^{2^n}x^k.$$

Also $(ab)^2 = a^2(a^{-1}ba)b = a^2(b^{2^{n-1}-1}x^k)b = a^2b^{2^{n-1}}x^k$.

If $a^{2^2} = x$, $b^{2^n} = x$, $[b, a] = b^{2^{n-1}-2}x^k$, then, replacing a with ab, we get $a^{2^2} = 1$, $b^{2^n} = x$, $[b, a] = b^{2^{n-1}-2}x^k$. If $a^{2^2} = 1$, $b^{2^n} = x$, $[b, a] = b^{2^{n-1}-2}x$, then replacing a with a^{-1}, we get $a^{2^2} = 1$, $b^{2^n} = x$, $[b, a] = b^{2^{n-1}-2}$. The argument shows that groups with relations expressed in $(*)$ are isomorphic to each other whenever $j = 1$. On the other hand, if $a^{2^2} = x$, $b^{2^n} = 1$, $[b, a] = b^{2^{n-1}-2}x$, then $[b, a^2] = 1$, and by replacing b with a^2b, we get $a^{2^2} = x$, $b^{2^n} = 1$, $[b, a] = b^{2^{n-1}-2}$. This indicates that the group in $(*)$ with $i = 1, j = 0, k = 1$ is isomorphic to the group in $(*)$ with $i=1, j=0, k=0$. Hence, G is one of the following groups:

(a) $\langle a, b \mid a^{2^2} = b^{2^{n+1}} = 1, [b, a] = b^{2^{n-1}-2}\rangle$;
(b) $\langle a, b \mid a^{2^3} = b^{2^n} = 1, [b, a] = b^{2^{n-1}-2}\rangle$;
(c) $\langle a, b, x \mid a^{2^2} = b^{2^n} = x^2 = 1, [b, a] = b^{2^{n-1}-2}x, [x, a] = [x, b] = 1\rangle$;
(d) $\langle a, b, x \mid a^{2^2} = b^{2^n} = x^2 = 1, [b, a] = b^{2^{n-1}-2}, [x, a] = [x, b] = 1\rangle$.

It is easy to check that all above listed groups are not $\mathcal{P}$-groups. In fact, if G is (a), then $\langle a^2\rangle \ntrianglelefteq G$, $\langle a^2\rangle^G = \langle a^2, b^{2^n}\rangle$ and $a \in N_G(\langle a^2\rangle) - \langle a^2\rangle^G$; if G is (b), then $\langle a^2b^{2^{n-2}}\rangle \ntrianglelefteq G$, $\langle a^2b^{2^{n-2}}\rangle^G = \langle a^2b^{2^{n-2}}, a^4\rangle$ and $b \in N_G(\langle a^2b^{2^{n-2}}\rangle) - \langle a^2b^{2^{n-2}}\rangle^G$; if G is (c), then $\langle a\rangle \ntrianglelefteq G$, $\langle a\rangle^G = \langle a, b^{2^{n-1}-2}x\rangle$ and $x \in N_G(\langle a\rangle) - \langle a\rangle^G$; if G is (d), then $\langle a\rangle \ntrianglelefteq G$, $\langle a\rangle^G = \langle a, b^2\rangle$ and $x \in N_G(\langle a\rangle) - \langle a\rangle^G$. The proof is complete. □

Lemma 3.8. *Let p be an odd prime. Then there is no non-abelian $\mathcal{P}$-group of order at least p^5.*

Proof. By Lemmas 2.6 and 3.3, we only need to prove there exists no non-abelian $\mathcal{P}$-group of order p^5. If exists, let G be a non-abelian $\mathcal{P}$-group of order p^5. Hence there is an element $x \in Z(G)$ with $o(x) = p$ such that $G/\langle x\rangle$ is a non-abelian $\mathcal{P}$-group by Lemmas 2.6 and 3.3. Thus, by Lemma 3.4, $G/\langle x\rangle \cong M_p(2, 2)$ and so $|G'| = p$ or p^2. If $|G'| = p$, then by Lemma 2.7, $d(G) = 2$ and so G is a minimal non-abelian group by [6, Lemma 2.2], in contradiction to Lemma 3.1. Now assume $|G'| = p^2$ and write $\overline{G} = G/\langle x\rangle = \langle \overline{a}, \overline{b} \mid \overline{a}^{p^2} = \overline{b}^{p^2} = 1, [\overline{a}, \overline{b}] = \overline{a}^p\rangle$. Then $a^{p^2} \neq 1$, which implies $G' \cong C_{p^2}$ and by [3, Chapter VIII, Lemma 1.1(b)], we have $\langle [a, b^p]\rangle = \langle x\rangle$. Let $A = \langle a, b^p\rangle$. Then $A \cong M_p(3, 1)$ by [6, Lemma 2.2] and Lemma 2.2. Hence there exists an element $\alpha \in A\backslash Z(A)$ such that $o(\alpha) = p$, and so $\langle \alpha\rangle \ntrianglelefteq G$. Since $1 = [\alpha^p, g] = [\alpha, g]^p = [\alpha, g^p]$ for any $g \in G$ by [3, Chapter VIII, Lemma 1.1(b)] once more, we see $\langle \alpha\rangle^G = \langle \alpha, x\rangle$ and $a^p \in C_G(\langle \alpha\rangle)$. Noticing that $o(a^p) = p^2$, we see $a^p \notin \langle \alpha\rangle^G$ and so $N_G(\langle \alpha\rangle) \nleq \langle \alpha\rangle^G$, which implies that G is not a $\mathcal{P}$-group, a contradiction. The proof is complete. □

Proof of Theorem 1.2. The sufficiency follows from Lemmas 3.2 and 3.5. In the following, we will prove the necessity.

Let G be a $\mathcal{P}$-group. Without loss of generality, we may assume that G is non-Dedekind. If $|G| = 2^3$, then $G \cong D_8$ which is of maximal class. If $|G| = 2^4$, then by Lemma 3.4, G is either of maximal class or isomorphic to $M_2(2, 2)$. Now assume $|G| \geq 2^5$. Choose a subgroup $N \trianglelefteq G$ such that $N \leq G'$ and $|N| = 2$. By Lemma 2.6, $\overline{G} = G/N$ is a $\mathcal{P}$-group, and so $\overline{G}$ is of one of the types (1) to (4) listed in Theorem 1.2 by induction. It follows from Lemmas 3.3 and 3.7 that $\overline{G}$ can not be (1) and (4). If $\overline{G}$ is (2), then $C_2^2 \cong \overline{G}/\overline{G}' \cong G/G'$ by Lemma 2.4, and therefore G is also (2). If $\overline{G}$ is (3), then by Lemma 3.6, G is (2) or (3). □

Proof of Theorem 1.3. The sufficiency follows from Lemma 3.1. Conversely, let G be a non-abelian $\mathcal{P}$-group, where p is an odd prime. By Lemma 3.8, $|G| \leq p^4$. If $|G| = p^4$, then $G \cong M_p(2,2)$ by Lemma 3.4. If $|G| = p^3$, then G is isomorphic to either $M_p(2,1)$ or $M_p(1,1,1)$ by Lemma 2.2. □

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