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DISTRIBUTION OF THE APPROXIMATION EXPONENTS OF A FAMILY OF POWER SERIES OVER A FINITE FIELD

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Abstract. In this paper, we exhibit the explicit forms of continued fraction expansions of a family of algebraic power series over a finite field and we study their asymptotic distribution of approximation exponents.

1. Introduction

Let p be a prime number and $q = p^s$, where s is a positive integer. We consider the finite field $\mathbb{F}_q$ with q elements. Then we introduce with an indeterminate T, the ring of polynomials $\mathbb{F}_q[T]$ and the field of rational functions $\mathbb{F}_q(T)$. We also consider the absolute value defined on $\mathbb{F}_q(T)$ by $|0| = 0$ and $|P/Q| = |T|^{\deg P - \deg Q}$ for $P, Q \in \mathbb{F}_q[T]$, where $|T|$ is a fixed real number greater than 1. By completing $\mathbb{F}_q(T)$ with this absolute value, we obtain the field $\mathbb{F}_q((T^{-1}))$ of formal power series with coefficients in $\mathbb{F}_q$. Then, if $\alpha \in \mathbb{F}_q((T^{-1}))\backslash\{0\}$, we can write

$$\alpha = \sum_{i \leq i_0} c_i T^i, \quad \text{where } i_0 \in \mathbb{Z},\ c_i \in \mathbb{F}_q,\ c_{i_0} \neq 0,$$

and $|\alpha| = |T|^{i_0}$. Observe the analogy between the classical construction of the field of real numbers and this field of formal power series. The roles of ± 1, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are played by $\mathbb{F}_q^*$, $\mathbb{F}_q[T]$, $\mathbb{F}_q(T)$ and $\mathbb{F}_q((T^{-1}))$ respectively. We study here rational approximation to elements of $\mathbb{F}_q((T^{-1}))$ which are algebraic over $\mathbb{F}_q(T)$. For a presentation in a larger context and for more references, see [10] or [8]. Indeed the finiteness of the base field plays an essential role in many results and this makes the field $\mathbb{F}_q((T^{-1}))$ particularly interesting.

As in the classical context of the real numbers, we have a continued fraction algorithm in $\mathbb{F}_q((T^{-1}))$. If $\alpha \in \mathbb{F}_q((T^{-1}))$, then we can write

$$\alpha = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}} = [a_1, a_2, a_3, \ldots],$$

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where $a_i \in \mathbb{F}_q[T]$. The a_i are called the partial quotients and we have $\deg a_i > 0$ for $i > 1$. This continued fraction expansion is finite if and only if $\alpha \in \mathbb{F}_q[T]$. As in the classical theory, we define recursively the two sequences of polynomials $(P_n)_{n>0}$ and $(Q_n)_{n>0}$ by $P_n = a_n P_{n-1} + P_{n-2}$ and $Q_n = a_n Q_{n-1} + Q_{n-2}$, with the initial conditions $P_1 = a_1, P_2 = a_1 a_2 + 1, Q_1 = 1$ and $Q_2 = a_2$. We have $P_{n+1}Q_n - Q_{n+1}P_n = (-1)^n$, whence P_n and Q_n are coprime polynomials. The rational P_n/Q_n is called a convergent to α and we have $P_n/Q_n = [a_1, a_2, \ldots, a_n]$. Moreover we have for $n \geq 1$ the equality:

$$\alpha = [a_1, a_2, \ldots, a_n, \alpha_{n+1}] = \frac{P_n \alpha_{n+1} + P_{n-1}}{Q_n \alpha_{n+1} + Q_{n-1}}, \tag{1.1}$$

where $\alpha_{n+1} = [a_{n+1}, a_{n+2}, \ldots]$ is called the complete quotient of α.

Continued fraction expansions of formal power series over finite fields are well studied because, for example, of their close connection with best diophantine approximations. For an algebraic irrational element of $\mathbb{F}_q((T^{-1}))$, define its diophantine approximation exponent $\nu(\alpha)$ by

$$\nu(\alpha) := \limsup(-\frac{\log|\alpha - P/Q|}{\log|Q|}),$$

where P and Q run over polynomials in $\mathbb{F}_q[T]$. If we know the continued fraction expansion for α, the following equation satisfied by α:

$$|\alpha - P_n/Q_n| = |a_{n+1}|^{-1}|Q_n|^{-2} \quad \text{for } n \geq 0$$

allows us to compute the approximation exponent of α by the following way:

$$\nu(\alpha) = 2 + \limsup(\frac{\deg a_{k+1}}{\sum_{1\leq i\leq k} \deg a_i}). \tag{1.2}$$

According to Schmidt [8], it is also possible to define the approximation spectrum of α. This is the set of the accumulation points of the sequence $\left(2 + \deg a_{k+1} / \sum_{1\leq i\leq k} \deg a_i\right)_{k\geq 1}$. This set is denoted $S(\alpha)$. Then $\nu(\alpha)$ is the upper bound of $S(\alpha)$.

The well-known theorems of Dirichlet and Liouville in the case of real number and their analogues for function fields [6] show that $2 \leq \nu(\alpha) \leq n$, where n is the algebraic degree of α. For the real number case, the well-known theorem of Roth shows that $\nu(\alpha) = 2$, but Mahler [6] showed that $\nu(\beta) = n = q$ for $T\beta^q - T\beta - 1 = 0$, as a direct estimate of approximation by truncation of the series $\sum T^{-q^i}$ shows. This then raises the question of the possibilities of knowing the distribution of the exponents of algebraic power series.

We consider the following algebraic equation with coefficients A, B, C and D in $\mathbb{F}_q[T]$ and $\Delta = AD - BC \neq 0$:

$$x = \frac{Ax^r + B}{Cx^r + D} \tag{1.3}$$

with $r = p^t$, $t \geq 0$. If α is an irrational solution in $\mathbb{F}_q((T^{-1}))$ of such an equation, we say that α is hyperquadratic or of *Class* **I**. After the work of Baum and Sweet [3], we have find that this class preserve special consideration. In fact, Baum and Sweet have given the first example of power series of *Class* **I** with $\nu(\alpha) = 2$. They have also given other example with $\nu(\alpha) > 2$. Few years later, the rational approximation of elements of *Class* **I** has been studied also by Voloch [11] and more deeply by de Mathan [7]. They could show that if the partial quotients in the continued fraction expansion of such elements α are unbounded, then $\nu(\alpha) > 2$. By the work of de Mathan [7], we know moreover that for elements of *Class* **I**, the approximation exponent $\nu(\alpha)$ is a rational number. Most of rational approximation that we know are for formal power series belonging to this class. The reader can consult [2] to find examples of formal power series with approximation exponent explicitly given.

A special elements of *Class* **I** which have been considered and studied by Schmidt [8] and Thakur [9]. These element called of *Class* **IA** have particulary condition, that is, Δ is a constant polynomial. In this case the expansion is completely and explicitly described and it has a very regular pattern. If α is of *Class* **IA** such that $\alpha = [a_1, \ldots, a_t, \ldots, a_n, \ldots]$ where $(a_1, \ldots, a_t)$ an arbitrary t-tuple of non-constant polynomials in $\mathbb{F}_q[T]$, then the sequence of partial quotients for α satisfies $a_{t+i} = \epsilon^{(-1)^i} a_i^q$ which can be obtained by the relation $\alpha^q = \epsilon\alpha_{t+1}$ derived from the equation satisfied by α. Note that in Schmidt's work [8] a finite number of polynomials has been added to the beginning of the expansion. Indeed if we add a finite number of partial quotients at the head of a hyperquadratic expansion, then the resulting expansion is obtained as the image of the first one by a linear fractional transformation and consequently it is still hyperquadratic. Moreover, Shmidt [8] and Thakur [9] proved that given any rational number μ between 2 and $q+1$, there exist elements of *Class* **I** with their approximation exponents equals to μ and with degree of their algebricity at most $q+1$. Thakur [9] had shown that most element of the particulary *Class* **IA** have exponents near 2. Chen [4] has improved this result by studying how the exponents of such elements are asymptotically distributed with respect to their heights.

Relying to the present our work, we will study the distribution of the approximation exponents of a other family of formal power series, belonging to the *Class* **I**$\backslash\{$**IA**$\}$, with respect to the heights of its elements. The regularity of their continued fraction expansions allows us to compute the value of their diophantine approximation exponents. This value will be given in Theorem 2.2. Furthermore, we will describe how the exponents of such elements are asymptotically distributed with respect to their heights. Precisely, in Theorem

2.5 we will prove that most of these elements have exponents very close to 2 as for elements of *Class* **IA**. We begin our results by Theorem 2.1, in which we will describe explicitly the continued fraction expansion of an element of this family. For this, we have to introduce a technical lemma. Let $P_n/Q_n \in \mathbb{F}_q(T)$ such that $P_n/Q_n := [a_1, a_2, \dots, a_n]$. For all $x \in \mathbb{F}_q(T)$, we will note

$$\big[[a_1, a_2, \dots, a_n], x\big] := \frac{P_n}{Q_n} + \frac{1}{x}.$$

Lemma 1.1. *Let $a_1, \dots, a_n, x \in \mathbb{F}_q(T)$. We have the following equality:*

$$\big[[a_1, a_2, \dots, a_n], x\big] = [a_1, a_2, \dots, a_n, y], \quad \textit{where } y = (-1)^{n-1}Q_n^{-2}x - Q_{n-1}Q_n^{-1}.$$

The proof of this lemma can be found in Lasjaunias's article [5] page 336.

2. Main results

Theorem 2.1. *Let $r = p^t$ with an integer $t \geq 1$ and n be a fixed positive integer. Let $a_1, a_2, \dots, a_n$ be n polynomials of $\mathbb{F}_q[T]$ with $\deg a_i = d_i \geq 1$. Let P be a nonzero polynomial of $\mathbb{F}_q[T]$ such that P divides a_i^{r-1} for $1 \leq i \leq n$. Let $\epsilon \in \mathbb{F}_q^*$. If α is the infinite continued fraction expansion $\alpha = [a_1, a_2, \dots, a_n, \alpha_{n+1}]$ in $\mathbb{F}_q((T^{-1}))$ satisfying an equation of the form*

$$\alpha^r = P\alpha_{n+1} + \epsilon, \tag{2.4}$$

then the sequence of partial quotients $(a_i)_{i\geq n+1}$ of α is defined recursively for all $k \geq 1$ by

$$a_{n+2k} = (-1)^k \epsilon^{-1} P$$

$$a_{n+2k-1} = \begin{cases} a_k^r/P & \text{if } k \text{ is odd;} \\ -\epsilon^2 a_k^r/P & \text{if } k \text{ is even.} \end{cases}$$

Proof. We have $\alpha = a_1 + \alpha_2^{-1}$. Then the equation (2.4) gives that

$$\frac{a_1^r - \epsilon}{P} + P^{-1}\alpha_2^{-r} = \alpha_{n+1},$$

or

$$[(a_1^r - \epsilon)/P, P\alpha_2^r] = \alpha_{n+1}. \tag{2.5}$$

Since P divides a_1^{r-1}, we have $(a_1^r - \epsilon)/P = [a_1^r P^{-1}, -\epsilon^{-1}P]$. Hence from Lemma (1.1) the equality (2.5) becomes

$$\alpha_{n+1} = \Big[[a_1^r P^{-1}, -\epsilon^{-1}P], P\alpha_2^r\Big] = \Big[a_1^r P^{-1}, -\epsilon^{-1}P, -\epsilon^2 \frac{\alpha_2^r}{P} + \frac{\epsilon}{P}\Big].$$

So it follows that

$$a_{n+1} = a_1^r/P \quad \text{and} \quad a_{n+2} = -\epsilon^{-1}P \tag{2.6}$$

and

$$\alpha_{n+3} = -\epsilon^2 \frac{\alpha_2^r}{P} + \frac{\epsilon}{P}. \tag{2.7}$$

We know that $\alpha_2 = a_2 + \alpha_3^{-1}$. Then the equation (2.7) gives that

$$\alpha_{n+3} = -\epsilon^2 \frac{a_2^r}{P} - \frac{\epsilon^2}{P\alpha_3^r} + \frac{\epsilon}{P}.$$

Thus, from Lemma 1.1 and with the same method as previous we get:

$$a_{n+3} = -\epsilon^2 \frac{a_2^r}{P} \quad \text{and} \quad a_{n+4} = \epsilon^{-1} P. \tag{2.8}$$

Moreover,

$$\alpha_{n+5} = \frac{P\alpha_3^r}{\epsilon^2(\epsilon^{-1}P)^2} - \frac{\epsilon}{P} = \frac{\alpha_3^r}{P} - \frac{\epsilon}{P}$$

or equivalently

$$\alpha_3^r = P\alpha_{n+5} + \epsilon. \tag{2.9}$$

Hence, by (2.6) and (2.8), the initial conditions, i.e., $k = 1$, stated in the theorem for the sequence of partial quotients are satisfied. Since (2.9) has the same shape as (2.4) and observing that P divides a_i^{r-1} for $n+1 \le i \le n+4$, the proof of the theorem follows by induction. □

Throughout the paper we are dealing with finite sequences (or words), consequently we recall the following notation on sequences in $\mathbb{F}_q[T]$. Let $B = a_1, a_2, \ldots, a_n$ be such a finite sequence, then we set $|B| = n$ for the length of the word B. If we have two words B_1 and B_2, then B_1, B_2 denotes the word obtained by concatenation. As usual, we denote by $[B] \in \mathbb{F}_q[T]$ the finite continued fraction $a_1 + 1/(a_2 + 1/(\cdots))$. We let $D(B)$ be the sum $\sum_{1 \le i \le n} \deg(a_i)$.

Theorem 2.2. *The approximation exponent of the formal power series defined by Theorem 2.1 is equal to*

$$\nu(\alpha) = 2 + \max(m_1, \ldots, m_n),$$

where for all $1 \le j \le n$

$$m_j = \frac{r(r-1)d_j - \deg P}{r^2 \sum_{k=1}^{j-1} d_k + r \sum_{k=j}^{n} d_k}.$$

Proof. It follow from the last Theorem that the continued fraction expansion of α have a regular pattern. By the method described in the proof, every known partial quotient of α give birth to a pair of partial quotients by the transformation given by (2.4) or (2.9). For example, for an input a_1 we get an output $a_1^r/P, -\epsilon^{-1}P$. We introduce now the following bloc of partial quotients:

For all $1 \le l \le n$, $B_{(l,1)} = a_l$, $B_{(l,2)} = a_l^r/P, -\epsilon^{-1}P$ and for all $i \ge 2$, $B_{(l,i)}$ is constructed from $B_{(l,i-1)}$ as the image of its partial quotients, one by one, by the transformation given by (2.4) and (2.9) respectively. So we get

$$B_{(1,1)} = a_1 \longrightarrow B_{(1,2)} = a_1^r/P, -\epsilon^{-1}P \longrightarrow B_{(1,3)} = a_1^{r^2} P^{-r-1}, -\epsilon^{-1}P, \epsilon P^{r-1},$$

$$\epsilon^{-1}P \longrightarrow B_{(1,4)} = a_1^{r^3}P^{-r^2-r-1}, -\epsilon^{-1}P, \epsilon P^{r-1}, \epsilon^{-1}P, -\epsilon P^{r^2-r-1}, \epsilon^{-1}P,$$
$$-\epsilon P^{r-1}, \epsilon^{-1}P \longrightarrow \cdots \longrightarrow B_{(1,i)} = a_1^{r^{i-1}}P^{-r^{i-2}-\cdots-r-1}, -\epsilon^{-1}P,$$
$$\ldots, -\epsilon P^{r^2-r-1}, \epsilon^{-1}P, -\epsilon P^{r-1}, \epsilon^{-1}P \longrightarrow \cdots .$$

The continued fraction expansion of α can be described as follow:

(2.10)
$$\alpha = [\underbrace{B_{(1,1)}, B_{(2,1)}, \ldots, B_{(n,1)}}_{C_1}, \underbrace{B_{(1,2)}, B_{(2,2)}, \ldots, B_{(n,2)}}_{C_2}, \ldots, \underbrace{B_{(1,i)}, B_{(2,i)}, \ldots, B_{(n,i)}}_{C_i} \ldots].$$

It is clear that $|B(l,i)| = 2^{i-1}$ and $D(B(l,i)) = r^{i-1}\deg a_l$ and the degree of the first quotient partial of $B(l,i)$ is $r^{i-1}d_l - (1 + r + \cdots + r^{i-2})\deg P$. Then $D(C_i) = \sum_{l=1}^{n} r^{i-1}\deg a_l$. Furthermore we have

$$\begin{aligned}
\lim_i \frac{r^{i-1}d_l - (1 + r + \cdots + r^{i-2})\deg P}{\sum_{j=1}^{l-1} D(B_{(j,i)}) + \sum_{j=1}^{i-1} D(C_j)} &= \lim_i \frac{(r-1)r^{i-1}d_l + (r^{i-2}-1)\deg P}{\sum_{j=1}^{l-1} r^{i-1}d_j + \sum_{j=1}^{i-1} D(C_j)} \\
&= \lim_i \frac{(r-1)r^{i-1}d_l + (r^{i-2}-1)\deg P}{r^i \sum_{j=1}^{l-1} d_j + r^{i-1}\sum_{j=l}^{n} d_j} \\
&= \frac{r(r-1)d_l - \deg P}{r^2 \sum\limits_{j=1}^{l-1} d_j + r\sum\limits_{j=l}^{n} d_j}.
\end{aligned}$$

So, according to (1.2), the approximation exponent of α is:

$$\nu(\alpha) = 2 + \max\left\{\left(\frac{r(r-1)d_l - \deg P}{r^2 \sum\limits_{j=1}^{l-1} d_j + r\sum\limits_{j=l}^{n} d_j}\right)_{1\le l\le n}\right\}.$$

□

We will now prove that most of elements satisfying the equation (2.4) have an approximation exponent close to 2. For this, we need to introduce the following definitions.

Definition 2.3. Let t be a positive integer and $r = p^t$. We define:

$$I = \left\{\alpha \in \mathbb{F}_q((T^{-1})) \,\middle|\, \alpha = \frac{A\alpha^r + B}{C\alpha^r + D}, A, B, C, D \in \mathbb{F}_q[T] \text{ and } AD - BC \neq 0\right\}.$$

A real algebraic element α is of *Class* **I** if $\alpha \in I$ for some $t \in \mathbb{N}$.

Let t be a fixed positive integer and $\alpha \in I$. We define

$$H(\alpha) = \max(\deg A, \deg B, \deg C, \deg D),$$

which is called the height of α. We remark that $H(\alpha)$ is well defined for $\alpha \in I$.

For an integer $n \geq 1$, we denote by I^e the subset of continued fraction expansion $\alpha \in I$ defined by

$$\alpha = [a_1, a_2, \ldots, a_n, \alpha_{n+1}] \text{ and } \alpha^r = P\alpha_{n+1} + \epsilon,$$

where $\epsilon \in \mathbb{F}_q^*$ and P is a nonzero polynomial of $\mathbb{F}_q[T]$ such that P divides a_i^{r-1} for $1 \leq i \leq n$.

For any $d \in \mathbb{N}$, we let

$$I_d^e = \{\alpha \in I^e / H(\alpha) \leq d\}.$$

If $\alpha \in I^e$, then $\alpha = [a_1, \ldots, a_n, \alpha_{n+1}]$ where α and α_{n+1} satisfy (2.4). Put $P_n/Q_n = [a_1, a_2, \ldots, a_n]$, then from equality (1.1) we deduce easily that $\alpha_{n+1} = \frac{-Q_{n-1}\alpha + P_{n-1}}{Q_n\alpha - P_n}$. Then the equation (2.4) is equivalent to:

$$Q_n\alpha^{r+1} - P_n\alpha^r - (\epsilon Q_n - PQ_{n-1})\alpha + \epsilon P_n - PP_{n-1} = 0. \tag{2.11}$$

Note that α is the unique root of strictly positive degree satisfying this equation (see [5], Theorem 1, page 332). For this, if we put $d_i = \deg a_i$ for $1 \leq i \leq n$ and $\deg P \leq d_n$, we obtain that the height of an element $\alpha \in I^e$ is:

$$H(\alpha) = d_1 + \cdots + d_n.$$

Before giving our next result, we need to introduce the following lemma.

Lemma 2.4 (cf. [1])**.** *For $t, s, k_1, \ldots, k_t \in \mathbb{N}$, let*

$$C(t, s) := |\{(d_1, \ldots, d_t) \in \mathbb{N}^t \mid d_1 + \cdots + d_t = s\}|$$

and

$$C(k_1, k_2, \ldots, k_t, t, s) := |\{(d_1, \ldots, d_t) \in \mathbb{N}^t \mid d_1 + \cdots + d_t = s; d_i \geq k_i\}|.$$

By the theory of binomial coefficient,

$$C(t, s) = \binom{s-1}{t-1}$$

and

$$C(k_1, k_2, \ldots, k_t, t, s) = \binom{s + t - (k_1 + \cdots + k_t) - 1}{t-1}.$$

Theorem 2.5. *The approximation exponent of the formal power series defined by Theorem 2.1 such that $\deg P \leq d_n$ is near 2.*

Before giving the proof of Theorem 2.5, we should note that for $n = 1$, the value of the approximation exponent of α defined by Theorem 2.1 is $\nu(\alpha) = r + 1 - (\deg P/rd_1)$. By an easy calculation, we can see that $\nu(\alpha) > r + (1/r)$. So we exclude this case from this theorem.

Proof of Theorem 2.5. For $n \in \mathbb{N}^*$, let:

$$P_n = \Big\{(a_1, a_2, \ldots, a_n) \,|\, a_i \in \mathbb{F}_q[T] \backslash \mathbb{F}_q \text{ for } i = 1, \ldots, n\Big\},$$

$$P_n(d) = \left\{(a_1, a_2, \ldots, a_n) \in P_n \mid \sum_{i=1}^{n} d_i \le d\right\}.$$

Let $T(d)$ the cardinal of $P_n(d)$. Without loss of generality, we consider $T(d)$ as the cardinal of (I_d^e). In fact, for a given polynomial P we can construct infinitely n-uplet of nonzero polynomials $(a_1, a_2, \ldots, a_n)$ such that P divides a_i^{r-1} for $1 \le i \le n$.

We can replace counting the number of elements of $\alpha \in I_d^e$ by counting the number of elements in

$$T = \{(n, a_1, \ldots, a_n) \mid n \in \mathbb{N}, a_i \in \mathbb{F}_q[T] \text{ with } \sum_{i=1}^{n} \deg a_i \le d\}.$$

For fixed number of base terms n and fixed positive integers $d_1, \ldots, d_n$, the number of polynomials $a_i \in \mathbb{F}_q[T]$ with $\deg a_i = d_i$ is $q^{d_i+1} - q^{d_i}$, hence we have for $d_1, \ldots, d_n \in \mathbb{N}$, the numbers of $\alpha \in T(d)$ such that $\alpha \in I_d^e$ is

$$(q^{d_1+1} - q^{d_1}) \cdots (q^{d_n+1} - q^{d_n}) = (q-1)^n q^{d_1+\cdots+d_n} = (q-1)^n q^{H(\alpha)}. \tag{2.12}$$

On the other hand, for fixed $n, D \in \mathbb{N}$ and from Lemma 2.4, the numbers of the set

$$N(n, D) := \{(d_1, \ldots, d_n) \in \mathbb{N}^n \mid d_1 + \cdots + d_n = D\}$$

is

$$|N(n, D)| = C(n, D) = \begin{pmatrix} D-1 \\ n-1 \end{pmatrix}. \tag{2.13}$$

Combining (2.12) and (2.13), we conclude that for fixed $n, D \in \mathbb{N}$, there are

$$(q-1)^n q^D \begin{pmatrix} D-1 \\ n-1 \end{pmatrix}$$

elements α such that $\alpha \in I_d^e$ and $H(\alpha) = D$.

Hence by considering all $D \le d$ and $n \le D$, we have

$$\begin{aligned} T(d) &= \sum_{D=1}^{d} \sum_{n=1}^{D} (q-1)^n q^D \begin{pmatrix} D-1 \\ n-1 \end{pmatrix} \\ &= (q-1) \sum_{D=1}^{d} q^D (1 + q - 1)^{D-1} \\ &= (q-1) \sum_{D=1}^{d} q^{2D-1} \\ &= \frac{q^{2d} - 1}{q(q+1)}. \end{aligned}$$

On the other hand, let $s_m(d) = \left\{\alpha \in I_d^e \mid H(\alpha) \le d, m < \nu(\alpha) \le r-1\right\}$ and $S_m(d)$ the cardinality of $s_m(d)$. Then we can replace counting the number of elements of $\alpha \in s_m(d)$ by counting the number of elements in

$$S = \{(n, a_1, \dots, a_n) \in T \mid 2 + \max\{m_1, \dots, m_n\} > m\},$$

We have the condition on d_i's as follows:

$$\begin{aligned} &\nu(\alpha) > m \\ \Leftrightarrow\ & 2 + \max\{m_1, \dots, m_n\} > m \\ \Leftrightarrow\ & m_j = \frac{r(r-1)d_j - \deg P}{r^2(d_1 + \cdots + d_{j-1}) + r(d_j + \cdots + d_n)} > m - 2 \text{ for some } j \\ \Leftrightarrow\ & d_j > \frac{(m-2)D}{r-1} + (m-2)(d_1 + \cdots + d_{j-1}) + \frac{\deg P}{r(r-1)} \text{ for some } j, \end{aligned}$$

where $D = d_1 + \cdots + d_n = H(\alpha)$. Note that $[\cdot]$ is the floor function.

Since $\frac{(m-2)H(\alpha)}{r-1} + (m-2)(d_1 + \cdots + d_{j-1}) + \frac{\deg P}{r(r-1)} \geq [\frac{(m-2)H(\alpha)}{r-1}] + (m-2)(j-1)$ and to get an upper bounded of $S_m(d)$ we consider a bigger set

$$\{d_j > [\frac{(m-2)H(\alpha)}{r-1}] + (m-2)(j-1)\}.$$

To ease the notation, we define

$$f(D, j) := [\frac{(m-2)D}{r-1}] + (m-2)(j-1).$$

From Lemma 2.4 and for $n, D, j \in \mathbb{N}$, the number of elements in the set

$$N_{n,D,j} := \{(d_1, \dots, d_n) \in \mathbb{N}^n \mid\ d_1 + \cdots + d_k = D, d_j > f(D, j)\}$$

is

$$\begin{aligned} |N_{n,D,j}| &= C(k_1, \dots, k_n, n, D) \\ &= \binom{D - [\frac{(m-2)D}{r-1}] - (m-2)(j-1) - 1}{n-1}, \end{aligned}$$

where $k_i = 1$ if $i \neq j$ and $k_j = [\frac{(m-2)H(\alpha)}{r-1}] + (m-2)(j-1) + 1$. Combine the result above and the same argument in the proof of $T(d)$, we conclude that for $n, D \in \mathbb{N}$ with $D \leq d$, the number of elements $\alpha \in s_m(d)$ such that α has n base terms and

$$H(\alpha) = D,\ \nu(\alpha) > m$$

is less than

$$q^D (q-1)^n \sum_{j=1}^{n} \binom{D - [\frac{(m-2)D}{r-1}] - (m-2)(j-1) - 1}{n-1}.$$

Hence by considering all $D \leq d$, we have

$$S_m(d) \leq \sum_{D=1}^{d} \sum_{n=1}^{D} q^D (q-1)^n \sum_{j=1}^{n} \binom{D - [\frac{(m-2)D}{r-1}] - (m-2)(j-1) - 1}{n-1}$$

$$\begin{aligned}
&\le \sum_{n=1}^{D} q^D (q-1) \sum_{j=1}^{D} \sum_{n=1}^{D} (q-1)^{n-1} \binom{D - [\frac{(m-2)D}{r-1}] - (m-2)(j-1) - 1}{n-1} \\
&\le \sum_{D=1}^{d} q^D (q-1) \sum_{j=1}^{D} q^{D - [\frac{(m-2)D}{r-1}] - (m-2)(j-1) - 1} \\
&= (q-1) \sum_{D=1}^{d} q^{2D - [\frac{(m-2)D}{r-1}] - 1} \sum_{j=1}^{D} q^{-(m-2)(j-1)} \\
&\le (q-1) \sum_{D=1}^{d} 2q^{2D - [\frac{(m-2)D}{r-1}] - 1} \\
&\le (q-1) \sum_{D=1}^{d} 2q^{2D - \frac{(m-2)D}{r-1}} \\
&= \frac{2(q-1)q^{2-\frac{m-2}{r-1}} (q^{(2-\frac{m-2}{r-1})d} - 1)}{q^{2-\frac{m-2}{r-1}} - 1} \\
&\le 4(q-1)q^{(2-\frac{m-2}{r-1})d}.
\end{aligned}$$

On the other hand, by the fact that $s_m(d)$ contains a subset

$$\{\alpha \in s_m(d) \,|\, d_1 > \frac{(m-2)D}{r-1}\} = \{\alpha \in s_m(d) \,|\, d_1 \ge [\frac{(m-2)D}{r-1}] + 1\}.$$

Hence we have

$$\begin{aligned}
S_m(d) &\ge \sum_{D=1}^{d} \sum_{n=1}^{D} q^D (q-1)^n C([\frac{(m-2)D}{r-1}] + 1, 1, \ldots, 1, n, D) \\
&= (q-1) \sum_{D=1}^{d} \sum_{n=1}^{D} q^D (q-1)^{n-1} \binom{D - [\frac{(m-2)D}{r-1}] - 1}{n-1}.
\end{aligned}$$

Observe that if

$$D \ge \frac{1}{1 - \frac{m-2}{r-1}},$$

then

$$D \ge \frac{(m-2)D}{r-1} + 1 \ge [\frac{(m-2)D}{r-1}] + 1.$$

Hence

$$\begin{aligned}
S_m(d) &\geq (q-1) \sum_{D \geq \frac{1}{1-\frac{m-2}{r-1}}}^{d} \sum_{n=1}^{D} q^D (q-1)^{n-1} \binom{D - [\frac{(m-2)D}{r-1}] - 1}{n-1} \\
&= (q-1) \sum_{D \geq \frac{1}{1-\frac{m-2}{r-1}}}^{d} q^D q^{D-[\frac{(m-2)D}{r-1}]-1} \\
&\geq (q-1) \sum_{D \geq \frac{1}{1-\frac{m-2}{r-1}}}^{d} q^{2D-\frac{(m-2)D}{r-1}-1}.
\end{aligned}$$

Since

$$\lim_{d\to\infty} \frac{\sum_{D<\frac{1}{1-\frac{m-2}{r-1}}} q^{2D-\frac{(m-2)D}{r-1}-1}}{\sum_{D=1}^{d} q^{2D-\frac{(m-2)D}{r-1}-1}} = 0$$

for d large enough, we have

$$\begin{aligned}
S_m(d) &\geq \frac{1}{2}(q-1) \sum_{D=1}^{d} q^{2D-\frac{(m-2)D}{r-1}-1} \\
&= \frac{(q-1)q^{2-\frac{m-2}{r-1}}(q^{(2-\frac{m-2}{r-1})d}-1)}{2q(q^{2-\frac{m-2}{r-1}}-1)} \\
&\geq \frac{1}{2q}(q^{(2-\frac{m-2}{r-1})d}-1).
\end{aligned}$$

Then we prove that

$$\frac{1}{2q}(q^{(2-\frac{m-2}{r-1})d}-1) \leq S_m(d) \leq 4(q-1)q^{(2-\frac{m-2}{r-1})d}.$$

So we obtain that

$$\lim_{d\longrightarrow\infty} \frac{|I^e_{d,m}|}{|I^e_d|} = \lim_{d\longrightarrow\infty} \frac{S_m(d)}{T(d)} = 0,$$

which equivalent to

$$\lim_{d\longrightarrow\infty} \frac{|\{\alpha \in I^e_d | \nu(\alpha) \leq m|\}}{|I^e_d|} = 1.$$

So we conclude that for any given real number $\varepsilon > 0$, "most" element in I^e have approximation exponent bounded by $2+\varepsilon$. □

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