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QUANTITATIVE WEIGHTED BOUNDS FOR THE VECTOR-VALUED SINGULAR INTEGRAL OPERATORS WITH NONSMOOTH KERNELS

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Abstract. Let T be the singular integral operator with nonsmooth kernel which was introduced by Duong and McIntosh, and T_q ($q \in (1, \infty)$) be the vector-valued operator defined by $T_q f(x) = \big(\sum_{k=1}^{\infty} |Tf_k(x)|^q\big)^{1/q}$. In this paper, by proving certain weak type endpoint estimate of $L\log L$ type for the grand maximal operator of T, the author establishes some quantitative weighted bounds for T_q and the corresponding vector-valued maximal singular integral operator.

1. Introduction

We will work on $\mathbb{R}^n$, $n \geq 1$. Let $A_p(\mathbb{R}^n)$ ($p \in [1, \infty)$) be the weight functions class of Muckenhoupt, that is, $w \in A_p(\mathbb{R}^n)$ if w is nonnegative, locally integrable and the $A_p(\mathbb{R}^n)$ constant $[w]_{A_p}$ is finite, where

$$[w]_{A_p} := \sup_Q \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}(x)dx\Big)^{p-1},\ p \in (1, \infty),$$

the supremum is taken over all cubes in $\mathbb{R}^n$, and

$$[w]_{A_1} := \sup_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)}.$$

For properties of $A_p(\mathbb{R}^n)$, we refer the reader to the monograph [8]. In the last several years, there has been significant progress in the study of sharp weighted bounds with A_p weights for the classical operators in Harmonic Analysis. The study was begun by Buckley [1], who proved that if $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$, then the Hardy-Littlewood maximal operator M satisfies

$$\|Mf\|_{L^p(\mathbb{R}^n,\, w)} \lesssim_{n,\, p} [w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(\mathbb{R}^n,\, w)}. \tag{1.1}$$

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Moreover, the estimate (1.1) is sharp since the exponent $1/(p-1)$ can not be replaced by a smaller one. Hytönen and Pérez [13] improved the estimate (1.1), and showed that

$$\|Mf\|_{L^p(\mathbb{R}^n,\,w)} \lesssim_{n,\,p} \left([w]_{A_p}[w^{-\frac{1}{p-1}}]_{A_\infty}\right)^{\frac{1}{p}}\|f\|_{L^p(\mathbb{R}^n,\,w)}, \tag{1.2}$$

where and in the following, for a weight $u \in A_\infty(\mathbb{R}^n) = \cup_{p\ge 1}A_p(\mathbb{R}^n)$, $[u]_{A_\infty}$ is the A_∞ constant of u, defined by

$$[u]_{A_\infty} = \sup_{Q\subset\mathbb{R}^n}\frac{1}{u(Q)}\int_Q M(u\chi_Q)(x)dx,$$

see [25]. It is obvious that (1.2) is more subtle than (1.1).

The sharp dependence of the weighted estimates of singular integral operators in terms of the $A_p(\mathbb{R}^n)$ constant was first considered by Petermichl [22,23], who solved this question for Hilbert transform and Riesz transform. Hytönen [11] proved that for a Calderón-Zygmund operator T and $w \in A_2(\mathbb{R}^n)$,

$$\|Tf\|_{L^2(\mathbb{R}^n,\,w)} \lesssim_n [w]_{A_2}\|f\|_{L^2(\mathbb{R}^n,\,w)}. \tag{1.3}$$

This solved the so-called A_2 conjecture. Combining the estimate (1.3) and the extrapolation theorem in [5], we know that for a Calderón-Zygmund operator T, $p \in (1,\,\infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$\|Tf\|_{L^p(\mathbb{R}^n,\,w)} \lesssim_{n,\,p} [w]_{A_p}^{\max\{1,\,\frac{1}{p-1}\}}\|f\|_{L^p(\mathbb{R}^n,\,w)}. \tag{1.4}$$

In [17], Lerner gave a very simple proof of (1.4) by controlling the Calderón-Zygmund operator using sparse operators. For other recent works about the quantitative weighted bounds for singular integral operators, see [9, 12–14, 18] and the related references therein.

Let T be an $L^2(\mathbb{R}^n)$ bounded linear operator with kernel K in the sense that for all $f \in L^2(\mathbb{R}^n)$ with compact support and a.e. $x \in \mathbb{R}^n\backslash\mathrm{supp}\,f$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,\,y)f(y)dy, \tag{1.5}$$

where K is a locally integrable function on $\mathbb{R}^n \times \mathbb{R}^n\backslash\{(x,\,y)\,:\,x = y\}$. To obtain a weak (1, 1) estimate for certain Riesz transforms, and L^p boundedness with $p \in (1,\,\infty)$ of holomorphic functional calculi of linear elliptic operators on irregular domains, Duong and McIntosh [6] introduced singular integral operators with nonsmooth kernels via the following generalized approximation to the identity.

Definition 1.1. Let h be a positive, bounded and decreasing function such that for some constant $\eta > 0$,

$$\lim_{r\to\infty} r^{n+\eta}h(r) = 0, \tag{1.6}$$

$\{a_t\}_{t>0}$ be a family of functions in $\mathbb{R}^n \times \mathbb{R}^n$ such that for all $x, y \in \mathbb{R}^n$ and $t > 0$,

$$|a_t(x, y)| \le h_t(x, y) = t^{-n/s} h\Big(\frac{|x-y|}{t^{1/s}}\Big), \tag{1.7}$$

where $s > 0$ is a constant. The family of operators $\{A_t\}_{t>0}$ is said to be an approximation to the identity, if for every $t > 0$, A_t can be represented by the kernel a_t in the sense that

$$A_t u(x) = \int_{\mathbb{R}^n} a_t(x, y) u(y) dy$$

for every function $u \in \cup_{p\ge 1} L^p(\mathbb{R}^n)$ and almost everywhere $x \in \mathbb{R}^n$.

Assumption 1.2. There exists an approximation to the identity $\{A_t\}_{t>0}$ such that the composite operator TA_t has an associated kernel K_t in the sense of (1.5), and there exists a positive constant c_1 such that for all $y \in \mathbb{R}^n$ and $t > 0$,

$$\int_{|x-y|\ge c_1 t^{\frac{1}{s}}} |K(x, y) - K_t(x, y)| dx \lesssim 1.$$

An $L^2(\mathbb{R}^n)$ bounded linear operator with kernel K satisfying Assumption 1.2 is called a singular integral operator with nonsmooth kernel, since K does not enjoy smoothness in space variables. Duong and McIntosh [6] proved that if T is an $L^2(\mathbb{R}^n)$ bounded linear operator with kernel K, and satisfies Assumption 1.2, then T is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. To consider the weighted boundedness with $A_p(\mathbb{R}^n)$ for singular integral operators with nonsmooth kernels, Martell [19] introduced the following assumptions.

Assumption 1.3. There exists an approximation to the identity $\{D_t\}_{t>0}$ such that the composite operator D_tT has an associated kernel K^t in the sense of (1.5), and there exist positive constants c_2 and $\alpha \in (0, 1]$, such that for all $t > 0$ and $x, y \in \mathbb{R}^n$ with $|x - y| \ge c_2 t^{\frac{1}{s}}$,

$$|K(x, y) - K^t(x, y)| \lesssim \frac{t^{\alpha/s}}{|x-y|^{n+\alpha}}.$$

Assumption 1.4. There exists an approximation to the identity $\{A_t\}_{t>0}$ such that the composite operator TA_t has an associated kernel K_t in the sense of (1.5), and there exists a positive constant c_1 and some $\alpha \in (0, 1]$, such that for all $t > 0$ with $|x - y| \ge c_1 t^{\frac{1}{s}}$,

$$|K(x, y) - K_t(x, y)| \lesssim \frac{t^{\alpha/s}}{|x-y|^{n+\alpha}}.$$

Martell [19] proved that if T is an $L^2(\mathbb{R}^n)$ bounded linear operator, satisfies Assumption 1.2 and Assumption 1.3, then for any $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$, T is bounded on $L^p(\mathbb{R}^n, w)$. Moreover, if T satisfies Assumption 1.3 and Assumption (1.4), then for $w \in A_1(\mathbb{R}^n)$, T is bounded from $L^1(\mathbb{R}^n, w)$ to

$L^{1,\infty}(\mathbb{R}^n, w)$. Hu and Yang [10] considered the weighted estimates with general weights for T and the corresponding maximal operator T^* defined by

$$T^*f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|,$$

with

$$T_\epsilon f(x) = \int_{|x-y|>\epsilon} K(x, y) f(y) dy.$$

Now let $q \in (1, \infty)$, and define the vector-valued singular integral operator with nonsmooth kernel by

$$T_q f(x) = |Tf(x)|_q = \Big(\sum_{k=1}^{\infty} |Tf_k(x)|^q \Big)^{1/q},$$

with $f = \{f_k\}$. Also, we define the vector-valued maximal singular integral operator T_q^* by

$$T_q^* f(x) = \Big(\sum_{k=1}^{\infty} |T^* f_k(x)|^q \Big)^{1/q}.$$

Mo and Lu [20] proved that for all $p, q \in (1, \infty)$,

$$\|T_q f\|_{L^p(\mathbb{R}^n)} \lesssim \| |f|_q \|_{L^p(\mathbb{R}^n)}.$$

Le [16] considered the weighted boundedness for T_q and T_q^*, proved that for all $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$\|T_q f\|_{L^p(\mathbb{R}^n, w)} + \|T_q^* f\|_{L^p(\mathbb{R}^n, w)} \lesssim \big\| |f|_q \big\|_{L^p(\mathbb{R}^n, w)},$$

and for $w \in A_1(\mathbb{R}^n)$,

$$\|T_q f\|_{L^{1,\infty}(\mathbb{R}^n, w)} \lesssim \| |f|_q \|_{L^1(\mathbb{R}^n, w)}.$$

The main purpose of this paper is to establish the quantitative weighted bounds for T_q and T_q^*. Our main results can be stated as follows.

Theorem 1.5. *Let T be an $L^2(\mathbb{R}^n)$ bounded linear operator with kernel K in the sense of* (1.5)*. Suppose that T satisfies Assumption 1.3 and Assumption 1.4. Then for $p, q \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$,*

$$\|T_q f\|_{L^p(\mathbb{R}^n, w)} \lesssim_{n,p,q} [w]_{A_p}^{\frac{1}{p}} \big([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}\big) [\sigma]_{A_\infty} \big\| |f|_q \big\|_{L^p(\mathbb{R}^n, w)}. \tag{1.8}$$

Here and in the following, for $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$, $p' = p/(p-1)$, $\sigma = w^{-\frac{1}{p-1}}$. Moreover, if the kernels $\{K^t\}_{t>0}$ in Assumption 1.3 satisfy that for all $t > 0$ and $x, y \in \mathbb{R}^n$ with $|x-y| \le c_2 t^{\frac{1}{s}}$,

$$|K^t(x, y)| \lesssim t^{-\frac{n}{s}}, \tag{1.9}$$

then (1.8) *holds true for T_q^*.*

Theorem 1.6. *Let T be an $L^2(\mathbb{R}^n)$ bounded linear operator with kernel K in the sense of* (1.5)*. Suppose that T satisfies Assumption 1.3 and Assumption 1.4. Then for $w \in A_1(\mathbb{R}^n)$ and $q \in (1, \infty)$,*

$$\|T_q f\|_{L^{1,\infty}(\mathbb{R}^n, w)} \lesssim_{n,q} [w]_{A_1}[w]_{A_\infty} \log^2(\mathrm{e}+[w]_{A_\infty})\||f|_q\|_{L^1(\mathbb{R}^n, w)}, \tag{1.10}$$

and

$$\begin{aligned} w\big(\{x \in \mathbb{R}^n : T_q f(x) > \lambda\}\big) \\ \lesssim_{n,q} [w]_{A_1} \log^2(\mathrm{e}+[w]_{A_\infty}) \int_{\mathbb{R}^n} \frac{|f(x)|_q}{\lambda} \log\Big(\mathrm{e}+\frac{|f(x)|_q}{\lambda}\Big) w(x)dx. \end{aligned} \tag{1.11}$$

Moreover, if the kernels $\{K^t\}_{t>0}$ in Assumption 1.3 satisfy (1.9)*, then the estimate* (1.11) *also holds for T_q^*.*

Remark 1.7. Theorem 1.5 implies that

$$\begin{aligned} &\|T_q f\|_{L^p(\mathbb{R}^n, w)} + \|T_q^* f\|_{L^p(\mathbb{R}^n, w)} \\ &\lesssim_{n,p,q} [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}+\frac{1}{p-1}} \big\||f|_q\big\|_{L^p(\mathbb{R}^n, w)}. \end{aligned} \tag{1.12}$$

Even for the scalar case, the weighted bounds in (1.11) and (1.12) are new. However, we do not know if these bounds are sharp.

Remark 1.8. Let $w \in A_1(\mathbb{R}^n)$. We do not know if the estimates

$$\big\|T_q f\big\|_{L^{1,\infty}(\mathbb{R}^n, w)} \lesssim_{n,q} [w]_{A_1} \log^2(\mathrm{e}+[w]_{A_\infty})\||f|_q\|_{L^1(\mathbb{R}^n, w)}$$

is true under the hypothesis of Theorem 1.6. It should be pointed out that the boundedness of T_q^* in (1.11) is new.

In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \le CB$. Specially, we use $A \lesssim_{n,p} B$ to denote that there exists a positive constant C depending only on n, p such that $A \le CB$. Constant with subscript such as c_1, does not change in different occurrences. For any set $E \subset \mathbb{R}^n$, χ_E denotes its characteristic function. For a cube $Q \subset \mathbb{R}^n$ and $\lambda \in (0, \infty)$, we use $\ell(Q)$ (diam$Q$) to denote the side length (diameter) of Q, and λQ to denote the cube with the same center as Q and whose side length is λ times that of Q. For $x \in \mathbb{R}^n$ and $r > 0$, $B(x, r)$ denotes the ball centered at x and having radius r. For locally integrable function g and a cube $Q \subset \mathbb{R}^n$, $\langle g\rangle_Q$ denotes the mean value of g on Q, that is, $\langle g\rangle_Q = |Q|^{-1}\int_Q g(y)dy$.

2. Endpoint estimates

This section is devoted to some endpoint estimates for the grand maximal operators corresponding to T and T^* in Theorem 1.5. These endpoint estimates play important roles in the proofs of the theorems and are of independent interest. We begin with some preliminary lemmas.

Lemma 2.1. *Let $q,\, p_0 \in (1,\, \infty)$, $\varrho \in [0,\, \infty)$ and S be a sublinear operator. Suppose that*

$$\||Sf|_q\|_{L^{p_0}(\mathbb{R}^n)} \lesssim \||f|_q\|_{L^{p_0}(\mathbb{R}^n)},$$

and for all $\lambda > 0$,

$$\big|\{x \in \mathbb{R}^n : |Sf(x)|_q > \lambda\}\big| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|_q}{\lambda} \log^{\varrho}\Big(\mathrm{e} + \frac{|f(x)|_q}{\lambda}\Big)dx.$$

Then for cubes $Q_2 \subset Q_1 \subset \mathbb{R}^n$,

$$\frac{1}{|Q_1|}\int_{Q_1} |S(f\chi_{Q_2})(x)|_q dx \lesssim \big\||f|_q\big\|_{L(\log L)^{\varrho+1},\, Q_2},$$

here and in the following, for $f = \{f_k\}$ and a cube Q, $f\chi_Q = \{f_k\chi_Q\}$, and for $\beta \in [0,\, \infty)$,

$$\|g\|_{L(\log L)^{\beta},\, Q} = \inf\Big\{\lambda > 0 : \frac{1}{|Q|}\int_Q \frac{|g(y)|}{\lambda} \log^{\beta}\Big(\mathrm{e} + \frac{|g(y)|}{\lambda}\Big)dy \le 1\Big\}.$$

Proof. Lemma 2.1 is a generalization of Lemma 3.1 in [10]. Their proofs are very similar. By homogeneity, we may assume that $\big\||f|_q\big\|_{L(\log L)^{\varrho+1},\, Q_2} = 1$, which implies that

$$\int_{Q_2} |f(x)|_q \log^{\varrho+1}\big(\mathrm{e} + |f(x)|_q\big)dx \le |Q_2|.$$

For each fixed $\lambda > 0$, set $\Omega_\lambda = \big\{x \in \mathbb{R}^n : |f(x)|_q > \lambda^{\frac{p_0-1}{2p_0}}\big\}$. Decompose f_k as

$$f_k(x) = f_k(x)\chi_{\Omega_\lambda}(x) + f_k(x)\chi_{\mathbb{R}^n\setminus\Omega_\lambda}(x) = f_k^1(x) + f_k^2(x).$$

Set

$$f^1 = \{f_k^1\},\ f^2 = \{f_k^2\};\ f^1\chi_{Q_2} = \{f_k^1\chi_{Q_2}\},\ f^2\chi_{Q_2} = \{f_k^2\chi_{Q_2}\}.$$

It is obvious that $\||f^2|_q\|_{L^\infty(\mathbb{R}^n)} \le \lambda^{\frac{p_0-1}{2p_0}}$. A trivial computation leads to that

$$\begin{aligned}
&\int_1^\infty \big|\{x \in \mathbb{R}^n : |S(f^2\chi_{Q_2})(x)|_q > \lambda/2\}\big|d\lambda\\
\lesssim\ &\int_1^\infty \int_{Q_2} |f^2(x)|_q^{p_0} dx\lambda^{-p_0}d\lambda\\
\lesssim\ &\int_{Q_2} |f^2(x)|_q dx \int_1^\infty \lambda^{-p_0+\frac{(p_0-1)^2}{2p_0}} d\lambda \lesssim |Q_2|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_1^\infty \big|\{x \in \mathbb{R}^n : |S(f^1\chi_{Q_2})(x)|_q > \lambda/2\}\big|d\lambda\\
\lesssim\ &\int_1^\infty \int_{Q_2} |f^1(x)|_q \log^{\varrho}\big(\mathrm{e} + |f^1(x)|_q\big)dx\lambda^{-1}d\lambda\\
\lesssim\ &\int_{Q_2} |f^1(x)|_q \log^{\varrho}\big(\mathrm{e} + |f^1(x)|_q\big)\int_1^{|f(x)|_q^{\frac{2p_0}{p_0-1}}} \frac{1}{\lambda}d\lambda dx
\end{aligned}$$

$$\lesssim \int_{Q_2} |f(x)|_q \log^{\varrho+1}\big(\mathrm{e}+|f(x)|_q\big)dx.$$

Combining the estimates above then yields

$$\begin{aligned}&\int_0^\infty \big|\{x\in Q_1:\,|S(f\chi_{Q_2})(x)|_q>\lambda\}\big|d\lambda\\ \lesssim&\int_0^1 \big|\{x\in Q_1:\,|S(f\chi_{Q_2})(x)|_q>\lambda\}\big|d\lambda\\ &+\int_1^\infty \big|\{x\in\mathbb{R}^n:\,|S(f^1\chi_{Q_2})(x)|_q>\lambda/2\}\big|d\lambda\\ &+\int_1^\infty \big|\{x\in\mathbb{R}^n:\,|S(f^2\chi_{Q_2})(x)|_q>\lambda/2\}\big|d\lambda\lesssim|Q_1|.\end{aligned}$$

This completes the proof of Lemma 2.1. □

Recall that the standard dyadic grid in $\mathbb{R}^n$ consists of all cubes of the form

$$2^{-k}([0,\,1)^n+j),\ k\in\mathbb{Z},\ \ j\in\mathbb{Z}^n.$$

Denote the standard grid by $\mathcal{D}$. For a fixed cube Q, denote by $\mathcal{D}(Q)$ the set of dyadic cubes with respect to Q, that is, the cubes from $\mathcal{D}(Q)$ are formed by repeating subdivision of Q and each of descendants into 2^n congruent subcubes.

As usual, by a general dyadic grid $\mathscr{D}$, we mean a collection of cubes with the following properties: (i) for any cube $Q\in\mathscr{D}$, its side length $\ell(Q)$ is of the form 2^k for some $k\in\mathbb{Z}$; (ii) for any cubes $Q_1,\,Q_2\in\mathscr{D}$, $Q_1\cap Q_2\in\{Q_1,\,Q_2,\,\emptyset\}$; (iii) for each $k\in\mathbb{Z}$, the cubes of side length 2^k in $\mathscr{D}$form a partition of $\mathbb{R}^n$. By the one-third trick, (see [12, Lemma 2.5]), there exist dyadic grids $\mathscr{D}_1,\,\dots,\,\mathscr{D}_{3^n}$, such that for each cube $Q\subset\mathbb{R}^n$, there exists a cube $I\in\mathscr{D}_j$ for some j, $Q\subset I$ and $\ell(Q)\approx\ell(I)$.

Let $\{D_t\}_{t>0}$ be an approximation to the identity. Associated with $\{D_t\}_{t>0}$, define the sharp maximal operator $M^\sharp_D$ by

$$M^\sharp_D g(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|g(y)-D_{t_Q}g(y)|dy,\ g\in\bigcup_{p\in[1,\,\infty]}L^p(\mathbb{R}^n),$$

here, $t_Q=\{\ell(Q)\}^s$, $\ell(Q)$ is the side length of Q and s is the constant appeared in (1.7), the supremum is taken over all cubes in $\mathbb{R}^n$. This operator was introduced by Martell [19] and plays an important role in the weighted estimates for singular integral operators with nonsmooth kernels. Let $q\in(1,\,\infty)$, $f=\{f_k\}\subset L^{p_0}(\mathbb{R}^n)$ for some $p_0\in[1,\,\infty]$, define the sharp maximal function of f by

$$M^\sharp_{D,\,q}(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-D_{t_Q}f(y)|_q dy;$$

see [20].

Lemma 2.2. *Let Φ be an increasing function on $[0,\infty)$ satisfying that*

$$\Phi(2t)\le C\Phi(t),\ \ t\in[0,\infty).$$

$\{D_t\}_{t>0}$ be an approximation to the identity as in Definition 1.1. Let $f=\{f_k\}$ be a sequence of functions such that for any $R>0$,

$$\sup_{0<\lambda<R}\Phi(\lambda)|\{x\in\mathbb{R}^n:\,M(|f|_q)(x)>\lambda\}|<\infty.$$

Then

$$\sup_{\lambda>0}\Phi(\lambda)|\{x\in\mathbb{R}^n:M(|f|_q)(x)>\lambda\}|\lesssim\sup_{\lambda>0}\Phi(\lambda)|\{x\in\mathbb{R}^n:M^{\sharp}_{D,\,q}(f)(x)>\lambda\}|.$$

Proof. Let $\lambda>0$, $\{f_k\}\subset L^1(\mathbb{R}^n)$ with compact supports, $Q\subset\mathbb{R}^n$ be a cube such that there exists $x_0\in Q$ with $M(|f|_q)(x_0)<\lambda$. It was proved in [16] that, for every $\zeta\in(0,1)$, we can find $\gamma>0$ (independent of λ, Q, f, x_0), such that

$$|\{x\in Q:\,M(|f|_q)(x)>A\lambda,\,M^{\sharp}_{D,\,q}(f)(x)\le\gamma\lambda\}|\le\zeta|Q|,$$

where $A>1$ is a fixed constant which only depends on the approximation to the identity $\{D_t\}_{t>0}$. This, via the argument used in the proof of the Fefferman-Stein inequality (see [8, pp. 150–151]), leads to our desired conclusion immediately. □

Lemma 2.3. *Let T be an $L^2(\mathbb{R}^n)$ bounded linear operator with kernel K in the sense of* (1.5)*. Suppose that T satisfies Assumption 1.3 and Assumption 1.4. Then for any $q\in(1,\infty)$ and $\lambda>0$,*

$$|\{x\in\mathbb{R}^n:\,|Tf(x)|_q>\lambda\}|\lesssim\lambda^{-1}\||f|_q\|_{L^1(\mathbb{R}^n)}.$$

For the proof of Lemma 2.3, see [20, Theorem 2.3].

For $\beta\in[0,\infty)$, let $M_{L(\log L)^\beta}$ be the maximal operator defined by

$$M_{L(\log L)^\beta}g(x)=\sup_{Q\ni x}\|g\|_{L(\log L)^\beta,\,Q}.$$

For simplicity, we denote $M_{L(\log L)^1}$ by $M_{L\log L}$. It is well known (see [21]) that for any $\lambda>0$,

$$|\{x\in\mathbb{R}^n:\,M_{L(\log L)^\beta}g(x)>\lambda\}|\lesssim\int_{\mathbb{R}^n}\frac{|g(x)|}{\lambda}\log^\beta\Big(\mathrm{e}+\frac{|g(x)|}{\lambda}\Big)\mathrm{d}x. \tag{2.1}$$

Lemma 2.4. *Let T be the singular integral operator in Theorem 1.6. Then for each $N\in\mathbb{N}$ and functions $f=\{f_k\}_{k=1}^N\subset L^{p_0}(\mathbb{R}^n)$ for some $p_0\in[1,\infty)$,*

$$M^{\sharp}_{D,\,q}(Tf)(x)\lesssim M_{L\log L}(|f|_q)(x).$$

Proof. Without loss of generality, we may assume that $c_2=2$. Let $x\in\mathbb{R}^n$, B be a ball containing x and $t_B=r_B^s$. Write

$$\frac{1}{|B|}\int_B|Tf_k(y)-D_{t_B}Tf_k(y)|_qdy\le E_1+\mathrm{E}_2+\mathrm{E}_3,$$

with

$$\mathrm{E}_1 = \frac{1}{|B|}\int_B |T(f\chi_{4B})(y)|_q dy,$$

$$\mathrm{E}_2 = \frac{1}{|B|}\int_B |D_{t_B}T(f\chi_{4B})(y)|_q dy,$$

and

$$\mathrm{E}_3 = \frac{1}{|B|}\int_B |T(f\chi_{\mathbb{R}^n\setminus 4B})(y) - D_{t_B}T(f\chi_{\mathbb{R}^n\setminus 4B})(y)|_q dy.$$

Recall that T is bounded on $L^q(\mathbb{R}^n)$. Thus by Lemma 2.1 and Lemma 2.3,

$$\mathrm{E}_1 \lesssim \big\||f|_q\big\|_{L\log L,\,4B} \lesssim M_{L\log L}(|f|_q)(x).$$

On the other hand, it follows from Minkowski's inequality that

$$|D_{t_B}T(f\chi_{4B})(y)|_q \lesssim \int_{\mathbb{R}^n} |h_{t_B}(y,\,z)||T(f\chi_{4B})(z)|_q dz.$$

Let

$$\mathrm{F}_0 = \int_{16B} |h_{t_B}(y,\,z)||T(f\chi_{4B})(z)|_q dz$$

and for $j \in \mathbb{N}$,

$$\mathrm{F}_j = \int_{2^{j+5}B\setminus 2^{j+4}B} |h_{t_B}(y,\,z)||T(f\chi_{4B})(z)|_q dz.$$

By the estimate (1.7) and Lemma 2.1, we know that

$$\mathrm{F}_0 \le \big\||f|_q\big\|_{L\log L,\,4B},$$

and

$$\mathrm{F}_j \le \frac{1}{|B|}h(2^j)\int_{2^{j+5}B} |T(f\chi_{4B})(z)|_q dz \lesssim 2^{-\delta j}\big\||f|_q\big\|_{L\log L,\,4B}.$$

This, in turn gives us that

$$\mathrm{E}_2 \lesssim \big\||f|_q\big\|_{L\log L,\,4B}.$$

Finally, another application of Minkowski's inequality yields

$$\begin{aligned}&|Tf(\chi_{\mathbb{R}^n\setminus 4B})(y) - D_{t_B}T(f\chi_{\mathbb{R}^n\setminus 4B}(y)|_q\\ &\le \int_{\mathbb{R}^n\setminus 4B} |K(y,\,z) - K^{t_B}(y,\,z)||f\chi_{\mathbb{R}^n\setminus 4B}(z)|_q dz.\end{aligned}$$

This, via Assumption 1.3, tells us that for each $y \in B$,

$$\big|T(f\chi_{\mathbb{R}^n\setminus 4B})(y) - D_{t_B}T(f\chi_{\mathbb{R}^n\setminus 4B}(y)\big|_q \lesssim M(|f|_q)(x),$$

which implies that

$$\mathrm{E}_3 \lesssim M(|f|_q)(x).$$

Combining the estimates for E_1, E_2 and E_3 then leads to our desired conclusion. □

Let $\mathscr{D}$ be a dyadic grid. Associated with $\mathscr{D}$, define the maximal operator $M_{\mathscr{D}}$ by

$$M_{\mathscr{D}}g(x)=\sup_{Q\ni x,\,Q\in\mathscr{D}}\langle|g|\rangle_Q.$$

Also, we define the sharp maximal function $M^{\sharp}_{\mathscr{D}}$ as

$$M^{\sharp}_{\mathscr{D}}g(x)=\sup_{Q\ni x,\,Q\in\mathscr{D}}\inf_{c\in\mathbb{C}}\langle|g-c|\rangle.$$

For $\delta\in(0,\,1)$, let

$$M_{\mathscr{D},\,\delta}g(x)=\big[M_{\mathscr{D}}(|g|^{\delta})(x)\big]^{1/\delta}\ \text{and}\ M^{\sharp}_{\mathscr{D},\,\delta}g(x)=\big[M^{\sharp}_{\mathscr{D}}(|g|^{\delta})(x)\big]^{1/\delta}.$$

Repeating the argument in [24, p. 153], we can verify that if Φ is an increasing function on $[0,\,\infty)$ which satisfies that

$$\Phi(2t)\le C\Phi(t),\ t\in[0,\,\infty),$$

then

$$\sup_{\lambda>0}\Phi(\lambda)|\{x\in\mathbb{R}^n:|g(x)|>\lambda\}|\lesssim\sup_{\lambda>0}\Phi(\lambda)|\{x\in\mathbb{R}^n:M^{\sharp}_{\mathscr{D},\delta}g(x)>\lambda\}|,\tag{2.2}$$

provided that $\sup_{\lambda>0}\Phi(\lambda)|\{x\in\mathbb{R}^n:\,M_{\mathscr{D},\,\delta}g(x)>\lambda\}|<\infty$.

Lemma 2.5. *Under the assumption of Theorem 1.6, for bounded functions $f=\{f_k\}$ with compact supports and each $\lambda>0$,*

$$\big|\{x\in\mathbb{R}^n:\,|MTf(x)|_q>\lambda\}\big|\lesssim\int_{\mathbb{R}^n}\frac{|f(x)|_q}{\lambda}\log\Big(\mathrm{e}+\frac{|f(x)|_q}{\lambda}\Big)dx.$$

Proof. By the well known one-third trick (see [12, Lemma 2.5]), we only need to prove that, for each dyadic grid $\mathscr{D}$, the inequality

$$\big|\big\{x\in\mathbb{R}^n:|M_{\mathscr{D}}(Tf)(x)|_q>1\big\}\big|\lesssim\int_{\mathbb{R}^n}|f(x)|_q\log\big(1+|f(x)|_q\big)dx\tag{2.3}$$

for bounded functions $f=\{f_k\}_{1\le k\le N}$ $(N\in\mathbb{N})$ with compact supports. As in the proof of Lemma 8.1 in [4], we can verify that for each cube $Q\in\mathscr{D}$, $\delta\in(0,\,1)$,

$$\begin{aligned}\inf_{c\in\mathbb{C}}\Big(\frac{1}{|Q|}\int_Q\big||M_{\mathscr{D}}f(y)|_q-c\big|^{\delta}dy\Big)^{\frac{1}{\delta}}&\lesssim\Big(\frac{1}{|Q|}\int_Q|M_{\mathscr{D}}(f\chi_Q)(y)|_q^{\delta}dy\Big)^{\frac{1}{\delta}}\\&\lesssim\langle|f\chi_Q|_q\rangle_Q,\end{aligned}$$

where in the last inequality, we invoked the fact that for each $\lambda>0$,

$$|\{x\in\mathbb{R}^n:\,|Mf(x)|_q>\lambda\}|\lesssim\lambda^{-1}\int_{\mathbb{R}^n}|f(x)|_qdx;$$

see [7]. This, in turn, implies that

$$M^{\sharp}_{\mathscr{D},\,\delta}\big(|M_{\mathscr{D}}f|_q\big)(x)\lesssim M_{\mathscr{D}}\big(|f|_q\big)(x).\tag{2.4}$$

Now let $\Phi(t) = t\log^{-1}(\mathrm{e}+t^{-1})$. It follows from (2.2), (2.4), Lemma 2.2, Lemma 2.4 and (2.1) that

$$\begin{aligned}
&\big|\{x\in\mathbb{R}^n:\,|M_{\mathscr{D}}Tf(x)|_q>1\}\big|\\
&\lesssim \sup_{t>0}\Phi(t)\big|\{x\in\mathbb{R}^n:\,M^{\sharp}_{\mathscr{D},\,\delta}\big(|M_{\mathscr{D}}Tf|_q\big)(x)>t\}\big|\\
&\lesssim \sup_{t>0}\Phi(t)\big|\{x\in\mathbb{R}^n:\,M\big(|Tf|_q\big)(x)>\lambda\}\big|\\
&\lesssim \sup_{t>0}\Phi(t)|\{x\in\mathbb{R}^n:\,M^{\sharp}_{D}(Tf)(x)>t\}|\\
&\lesssim \sup_{t>0}\Phi(t)\big|\{x\in\mathbb{R}^n:\,M_{L\log L}(|f|_q)(x)>t\}\big|\\
&\lesssim \int_{\mathbb{R}^n}|f(x)|_q\log(\mathrm{e}+|f(x)|_q)dx.
\end{aligned}$$

This establishes (2.3) and completes the proof of Lemma 2.5. □

We are now ready to establish the main result in this section. As in [17], for a sublinear operator U, we define the associated grand maximal operator $\mathcal{M}_U$ by

$$\mathcal{M}_U g(x) = \sup_{Q\ni x}\operatorname*{ess\,sup}_{\xi\in Q}|U(g\chi_{\mathbb{R}^n\setminus 3Q})(\xi)|,$$

where the supremum is taken over all cubes $Q\subset\mathbb{R}^n$ containing x.

Theorem 2.6. *Let $q\in(1,\infty)$, T be an $L^2(\mathbb{R}^n)$ bounded linear operator with kernel K as in* (1.5)*. Suppose that T satisfies Assumption 1.3 and Assumption 1.4. Then for each $f=\{f_k\}$ and each $\lambda>0$,*

$$\big|\{x\in\mathbb{R}^n:\,|\mathcal{M}_T f(x)|_q>\lambda\}\big|\lesssim\int_{\mathbb{R}^n}\frac{|f(x)|_q}{\lambda}\log\big(\mathrm{e}+\frac{|f(x)|_q}{\lambda}\big)dx. \tag{2.5}$$

If we further assume that the kernels $\{K^t\}_{t>0}$ in Assumption 1.3 also satisfy (1.9)*, then* (2.5) *is also true for T^*.*

Proof. As it was proved in [9], the maximal operator $M_{L\log L}$ satisfies that

$$\big|\big\{x\in\mathbb{R}^n:|M_{L\log L}f(x)|_q>\lambda\big\}\big|\lesssim\int_{\mathbb{R}^n}\frac{|f(x)|_q}{\lambda}\log\Big(\mathrm{e}+\frac{|f(x)|_q}{\lambda}\Big)dx.$$

Thus, by Lemma 2.5, our proof is now reduced to proving that the inequalities

$$\mathcal{M}_T g(x)\lesssim MTg(x)+M_{L\log L}g(x), \tag{2.6}$$

and

$$\mathcal{M}_{T^*} g(x)\lesssim MTg(x)+M_{L\log L}g(x) \tag{2.7}$$

hold. Without loss of generality, we assume that $c_2>1$.

Let $Q\subset\mathbb{R}^n$ be a cube and $x,\,\xi\in Q$. Set $t_Q=\big(\frac{1}{c_2\sqrt{n}}\ell(Q)\big)^s$ and write

$$\begin{aligned}
T(g\chi_{\mathbb{R}^n\setminus 3Q})(\xi) &= D_{t_Q}Tg(\xi)-D_{t_Q}T(g\chi_{3Q})(\xi)\\
&\quad+\Big(T(g\chi_{\mathbb{R}^n\setminus 3Q})(\xi)-D_{t_Q}T(g\chi_{\mathbb{R}^n\setminus 3Q})(\xi)\Big).
\end{aligned}$$

A trivial computation involving (1.6) leads to that

$$\begin{aligned}|D_{t_Q}Tg(\xi)| &\lesssim |Q|^{-1}\sum_{j=1}^{\infty}\int_{2^jnt_Q^{\frac{1}{s}}<|\xi-y|\le 2^{j+1}nt_Q^{\frac{1}{s}}} h\Big(\frac{|\xi-y|}{t_Q^{\frac{1}{s}}}\Big)|Tg(y)|dy\\ &\quad+|Q|^{-1}\int_{|\xi-y|\le 2nt_Q^{\frac{1}{s}}}|Tg(y)|dy\\ &\lesssim |Q|^{-1}\sum_{j=1}^{\infty}\int_{2^{j-1}nt_Q^{\frac{1}{s}}<|x-y|\le 2^{j+2}nt_Q^{\frac{1}{s}}} h\Big(\frac{|\xi-y|}{2t_Q^{\frac{1}{s}}}\Big)|Tg(y)|dy\\ &\quad+|Q|^{-1}\int_{|x-y|\le 3nt_Q^{\frac{1}{s}}}|Tg(y)|dy\\ &\lesssim MTg(x).\end{aligned}$$

On the other hand, it follows from Lemma 2.1 that

$$\begin{aligned}|D_{t_Q}T(g\chi_{3Q})(\xi)| &\lesssim \frac{1}{|Q|}\sum_{j=1}^{\infty}\int_{2^{j-1}nt_Q^{\frac{1}{s}}<|x-y|\le 2^{j+2}nt_Q^{\frac{1}{s}}} h\Big(\frac{|\xi-y|}{2t_Q^{\frac{1}{s}}}\Big)|T(g\chi_{3Q})(y)|dy\\ &\quad+|Q|^{-1}\int_{|x-y|\le 3nt_Q^{\frac{1}{s}}}|T(g\chi_{3Q})(y)|dy\\ &\lesssim M_{L\log L}g(x).\end{aligned}$$

Finally, Assumption 1.3 tells us that

$$\begin{aligned}\Big|T(g\chi_{\mathbb{R}^n\setminus 3Q})(\xi)-D_{t_Q}T(g\chi_{\mathbb{R}^n\setminus 3Q})(\xi)\Big| &\lesssim \int_{\mathbb{R}^n\setminus 3Q}\big|K(\xi,\,y)-K^{t_Q}(\xi,\,y)\big||g(y)|dy\\ &\lesssim t_Q^{\frac{\alpha}{s}}\int_{\mathbb{R}^n\setminus 3Q}\frac{1}{|\xi-y|^{n+\alpha}}|g(y)|dy\\ &\lesssim Mg(x).\end{aligned}$$

Combining the estimates above leads to (2.6).

It remains to prove (2.7). Let $x,\,\xi\in Q$. Observe that $\mathrm{supp}\chi_{\mathbb{R}^n\setminus 3Q}\subset\{y: |y-x|\ge \ell(Q)\}$ and

$$T^*(g\chi_{\mathbb{R}^n\setminus 3Q})(\xi)\le |T(g\chi_{\mathbb{R}^n\setminus 3Q})(\xi)|+\sup_{\epsilon\ge\ell(Q)}|T_\epsilon(g\chi_{\mathbb{R}^n\setminus 3Q})(\xi)|. \tag{2.8}$$

Now let $\epsilon\ge\ell(Q)$. Write

$$\begin{aligned}T_\epsilon(g\chi_{\mathbb{R}^n\setminus 3Q})(\xi) &= D_{(\epsilon/c_2)^s}Tg(\xi)-D_{(\epsilon/c_2)^s}T(g\chi_{3Q})(\xi)\\ &\quad+\Big(T_\epsilon(g\chi_{\mathbb{R}^n\setminus 3Q})(\xi)-D_{\epsilon^s}T(g\chi_{\mathbb{R}^n\setminus 3Q})(\xi)\Big).\end{aligned}$$

Invoking the argument for $\mathcal{M}_T$, we can verify that

$$|D_{(\epsilon/c_2)^s}Tg(\xi)|\lesssim MTg(x)$$

and

$$|D_{(\epsilon/c_2)^s}T(g\chi_{3Q})(\xi)|\lesssim M_{L\log L}g(x).$$

As in [6, p. 249], write

$$\begin{aligned}&T_\epsilon(g\chi_{\mathbb{R}^n\setminus 3Q})(\xi)-D_{\epsilon^s}T(g\chi_{\mathbb{R}^n\setminus 3Q})(\xi)\\ &=\int_{|\xi-y|\le\epsilon}K^{(\epsilon/c_2)^s}(\xi,\,y)g(y)\chi_{\mathbb{R}^n\setminus 3Q}(y)dy\\ &\quad+\int_{|\xi-y|>\epsilon}\big(K(\xi,\,y)-K^{(\epsilon/c_2)^s}(\xi,\,y)\big)g(y)\chi_{\mathbb{R}^n\setminus 3Q}(y)dy.\end{aligned}$$

The fact that $K^{(\epsilon/c_2)^s}$ satisfies the size condition (1.9), implies that

$$\Big|\int_{|\xi-y|\le\epsilon}K^{(\epsilon/c_2)^s}(\xi,\,y)g(y)dy\Big|\lesssim\epsilon^{-n}\int_{|\xi-y|<\epsilon}|g(y)|dy\lesssim Mg(x).$$

On the other hand, by Assumption 1.3, we obtain that

$$\Big|\int_{|\xi-y|>\epsilon}\big(K(\xi,\,y)-K^{(\epsilon/c_2)^s}(\xi,\,y)\big)g(y)\chi_{\mathbb{R}^n\setminus 3Q}(y)dy\Big|\lesssim Mg(x).$$

Therefore,

$$\sup_{\epsilon\ge\ell(Q)}|T_\epsilon(g\chi_{\mathbb{R}^n\setminus 3Q})(\xi)|\lesssim MTg(x)+M_{L\log L}g(x),$$

which, via the estimates (2.6) and (2.8), shows that

$$\mathcal{M}_{T^*}g(x)\lesssim MTg(x)+M_{L\log L}g(x).$$

This completes the proof of Theorem 2.6. □

3. Proof of theorems

Let $\eta\in(0,\,1)$ and $\mathcal{S}$ be a family of cubes. We say that $\mathcal{S}$ is η-sparse, if for each fixed $Q\in\mathcal{S}$, there exists a measurable subset $E_Q\subset Q$, such that $|E_Q|\ge\eta|Q|$ and E_Q's are pairwise disjoint. Associated with the sparse family $\mathcal{S}$ and constant $\beta\in[0,\,\infty)$, we define the sparse operator $\mathcal{A}_{\mathcal{S},\,L(\log L)^\beta}$ by

$$\mathcal{A}_{\mathcal{S},\,L(\log L)^\beta}f(x)=\sum_{Q\in\mathcal{S}}\|f\|_{L(\log L)^\beta,\,Q}\chi_Q(x).$$

We denote $\mathcal{A}_{\mathcal{S},\,L(\log L)^1}$ by $\mathcal{A}_{\mathcal{S},\,L\log L}$.

To prove Theorem 1.5 and Theorem 1.6, we will employ the following lemmas.

Lemma 3.1. *Let $q\in(1,\,\infty)$ and $\beta\in[0,\,\infty)$, U be a sublinear operator and $\mathcal{M}_U$ the corresponding grand maximal operator. Suppose that U is bounded on $L^q(\mathbb{R}^n)$, and satisfies the endpoint estimate that, for any $\lambda>0$,*

$$\big|\{y\in\mathbb{R}^n:\,|\mathcal{M}_Uf(y)|_q>\lambda\}\big|\lesssim\int_{\mathbb{R}^n}\frac{|f(y)|_q}{\lambda}\log^\beta\Big(\mathrm{e}+\frac{|f(y)|_q}{\lambda}\Big)dy.$$

Then for $N\in\mathbb{N}$ and bounded functions $f=\{f_k\}_{1\le k\le N}$ with compact supports, there exists a $\frac{1}{2}\frac{1}{3^n}$-sparse family $\mathcal{S}$ such that for a.e. *$x\in\mathbb{R}^n$,*

$$|Uf(x)|_q\lesssim\mathcal{A}_{\mathcal{S},\,L(\log L)^\beta}\big(|f|_q\big)(x).$$

For the proof of Lemma 3.1, see [9].

Lemma 3.2. *Let $\beta \in [0, \infty)$, $\mathcal{S}$ be a sparse family and $\beta \in [0, \infty)$, $\mathcal{A}_{\mathcal{S},\, L(\log L)^{\beta}}$ be the associated sparse operator. Then for $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$,*

$$\|\mathcal{A}_{\mathcal{S},\, L(\log L)^{\beta}} g\|_{L^p(\mathbb{R}^n,w)} \lesssim [w]_{A_p}^{\frac{1}{p}}\big([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}\big)[\sigma]_{A_\infty}^{\beta}\|g\|_{L^p(\mathbb{R}^n,\, w)}.$$

Proof. Lemma 3.2 was indicated by Lemma 2.1 in [9], and can be proved by the argument used in the proof of Theorem 2.1 in [3]. In fact, by the one-third trick, we may assume that $\mathcal{S} \subset \mathscr{D}$ for some dyadic grid $\mathscr{D}$. As it was pointed out in the proof of Theorem 2.1 in [3], for each cube $Q \subset \mathscr{D}$,

$$\|g\sigma\|_{L(\log L)^{\beta},\, Q} \lesssim [\sigma]_{A_\infty}^{\beta}\langle \sigma M_{\sigma,\,\varrho}^{\mathscr{D}} g\rangle_Q, \tag{3.1}$$

here, $\varrho = (1+p)/2$, $M_{\sigma,\,\varrho}^{\mathscr{D}}$ is the maximal operator defined by

$$M_{\sigma,\,\varrho}^{\mathscr{D}} g(x) = \sup_{I \ni x,\, I \in \mathscr{D}} \Big(\frac{1}{\sigma(I)}\int_I |g(y)|^{\varrho}\sigma(y)\mathrm{d}y\Big)^{\frac{1}{\varrho}}.$$

On the other hand, it follows from Theorem 2.3 in [15] that for $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$\Big\|\sum_{Q\in\mathcal{S}}\langle |v|\sigma\rangle_Q \chi_Q\Big\|_{L^p(\mathbb{R}^n,w)} \lesssim [w]_{A_p}^{\frac{1}{p}}\big([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}\big)\|v\|_{L^p(\mathbb{R}^n,\,\sigma)}. \tag{3.2}$$

Combining the inequalities (3.1) and (3.2) leads to that

$$\begin{aligned}\|\mathcal{A}_{\mathcal{S},\, L(\log L)^{\beta}}(g\sigma)\|_{L^p(\mathbb{R}^n,w)} &\lesssim [w]_{A_p}^{\frac{1}{p}}\big([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}\big)[\sigma]_{A_\infty}^{\beta}\|M_{\sigma,\varrho}^{\mathscr{D}} g\|_{L^p(\mathbb{R}^n,\sigma)}\\ &\lesssim [w]_{A_p}^{\frac{1}{p}}\big([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}\big)[\sigma]_{A_\infty}^{\beta}\|g\|_{L^p(\mathbb{R}^n,\,\sigma)},\end{aligned}$$

since $M_{\sigma,\varrho}^{\mathscr{D}}$ is bounded on $L^p(\mathbb{R}^n,\, \sigma)$ with bound independent of σ. This completes the proof of Lemma 3.2. □

Lemma 3.3. *Let $\beta \in [0, \infty)$, $\mathcal{S}$ be a sparse family and $\mathcal{A}_{\mathcal{S},\, L(\log L)^{\beta}}$ be the corresponding sparse operator. Then for $p \in (1, \infty)$, $\epsilon \in (0, 1]$ and weight u,*

$$\|\mathcal{A}_{\mathcal{S},\, L(\log L)^{\beta}} g\|_{L^p(\mathbb{R}^n,\, u)} \lesssim p'^{1+\beta}p^2\big(\frac{1}{\epsilon}\big)^{\frac{1}{p'}}\|g\|_{L^p(\mathbb{R}^n,\, M_{L(\log L)^{p-1+\epsilon}}u)}.$$

Moreover, for any $\lambda > 0$,

$$\begin{aligned}&u(\{x \in \mathbb{R}^n : \mathcal{A}_{\mathcal{S},\, L(\log L)^{\beta}} g(x) > \lambda\})\\ &\lesssim \frac{1}{\epsilon^{1+\beta}}\int_{\mathbb{R}^n}\frac{|g(x)|}{\lambda}\log^{\beta}\Big(\mathrm{e} + \frac{|g(x)|}{\lambda}\Big)M_{L(\log L)^{\epsilon}}u(x)\mathrm{d}x.\end{aligned}$$

Lemma 3.3 is a combination of Lemma 4.1 and Lemma 4.2 in [9].

Let u be a weight, $\epsilon \in (0, 1)$ and T be the operator in Theorem 1.5. By Theorem 2.6, Lemma 3.1 and Lemma 3.3, we know that for each $p \in (1, \infty)$,

$$\begin{aligned}&\|T_q f\|_{L^p(\mathbb{R}^n,\, u)} + \|T_q^* f\|_{L^p(\mathbb{R}^n,\, u)}\\ &\lesssim p'^2 p^2\big(\frac{1}{\epsilon}\big)^{\frac{1}{p'}}\||f|_q\|_{L^p(\mathbb{R}^n,\, M_{L(\log L)^{p-1+\epsilon}}u)}.\end{aligned} \tag{3.3}$$

Proof of Theorem 1.5. Let $q \in (1, \infty)$. Under the hypothesis of Theorem 1.5, we know that T and T^* are bounded on $L^q(\mathbb{R}^n)$, see [6]. The conclusion of Theorem 1.5 now follows from Theorem 2.6, Lemma 3.1 and Lemma 3.2 directly. □

Proof of Theorem 1.6. By Theorem 2.6, Lemma 3.1 and Lemma 3.3, we know that for each weight w and $\epsilon \in (0, 1)$,

$$\begin{aligned} &w\big(\{x \in \mathbb{R}^n : T_q f(x) > \lambda\}\big) + w\big(\{x \in \mathbb{R}^n : T_q^* f(x) > \lambda\}\big) \\ &\lesssim \frac{1}{\epsilon^2} \int_{\mathbb{R}^n} \frac{|f(x)|_q}{\lambda} \log\Big(\mathrm{e} + \frac{|f(x)|_q}{\lambda}\Big) M_{L(\log L)^{\epsilon}} w(x)dx. \end{aligned} \tag{3.4}$$

The estimate (1.11) now follows if we apply the argument used in the proof of [14, Corollary 1.4], see also the proof [18, Corollay 1.3].

We now prove (1.10). As in the proof of [14, Corollary 1.4], it suffices to show that for each weight w and $\epsilon \in (0, 1)$,

$$w(\{x \in \mathbb{R}^n : T_q f(x) > \lambda\}) \lesssim \frac{1}{\lambda \epsilon^2} \int_{\mathbb{R}^n} |f(x)|_q M_{L(\log L)^{1+\epsilon}} w(x)dx. \tag{3.5}$$

We assume that $c_1 = 2$. For $\lambda > 0$ and $f = \{f_k\}$, applying the Calderón-Zygmund decomposition to $|f|_q$ at level λ, we obtain a sequence of cubes $\{Q_l\}$ with disjoint interiors, such that

$$\lambda < \frac{1}{|Q_l|} \int_{Q_l} |f(x)|_q dx \lesssim \lambda,$$

and $|f(x)|_q \lesssim \lambda$ for a.e. $x \in \mathbb{R}^n \setminus \cup_l Q_l$. For each fixed k, set

$$f_k^1(x) = f_k(x) \chi_{\mathbb{R}^n \setminus (\cup_l Q_l)}(x),$$

$$f_k^2(x) = \sum_l A_{t_{Q_l}} b_{k,\,l}(x),\ \ f_k^3(x) = \sum_l \big(b_{k,\,l}(x) - A_{t_{Q_l}} b_{k,\,l}(x)\big) \chi_{Q_l}(x),$$

with $b_{k,\,l}(y) = f_k(y)\chi_{Q_l}(y)$, $t_{Q_l} = \{\ell(Q_l)\}^s$. Set $f^j(x) = \{f_k^j(x)\}$ with $j = 1, 2, 3$. By the fact that $\big\||f^1|_q\big\|_{L^\infty(\mathbb{R}^n)} \lesssim \lambda$, we deduce from (3.3) that

$$\begin{aligned} w(\{x \in \mathbb{R}^n : |Tf^1(x)|_q > \lambda\}) &\lesssim \frac{1}{\lambda^2 \epsilon} \int_{\mathbb{R}^n} |f^1(x)|_q^2 M_{L(\log L)^{1+\epsilon}} w(x)dx \\ &\lesssim \frac{1}{\lambda \epsilon} \int_{\mathbb{R}^n} |f(x)|_q M_{L(\log L)^{1+\epsilon}} w(x)dx. \end{aligned} \tag{3.6}$$

To estimate $|Tf^3|_q$, we set $\Omega = \cup_l 4nQ_l$ and $b^l(y) = \{b_{k,\,l}(y)\}$. Obviously,

$$|b^l(y)|_q = |f(y)|_q \chi_{Q_l}(y).$$

For each k and $x \in \mathbb{R}^n \setminus \Omega$, write

$$\big|Tf_k^3(x)\big| \le \sum_l \int_{\mathbb{R}^n} \big|K(x,y) - K_{A_{t_{Q_l}}}(x,y)\big| |b_{k,l}(y)| dy.$$

Applying Minkowski's inequality twice, we obtain

$$|Tf^3(x)|_q \le \sum_l \int_{\mathbb{R}^n} \big|K(x,y) - K_{A_{t_{Q_l}}}(x,y)|b^l(y)|_q dy.$$

Therefore,

$$\begin{aligned}(3.7)\qquad & w\big(\{x \in \mathbb{R}^n\backslash\Omega:\, |Tf^3(x)|_q > \lambda/3\}\big)\\ &\lesssim \lambda^{-1}\sum_l \int_{\mathbb{R}^n}\int_{\mathbb{R}^n\backslash 4nQ_l} \big|K(x,y) - K_{A_{t_{Q_l}}}(x,y)w(x)dx|b^l(y)|_q dy\\ &\lesssim \lambda^{-1}\sum_l \int_{Q_l} |b^l(y)|_q Mw(y)dy \lesssim \lambda^{-1}\int_{\mathbb{R}^n} |f(x)|_q Mw(x)dx.\end{aligned}$$

It remains to estimate $|Tf^2|_q$. Let $\widetilde{w}(x) = w(x)\chi_{\mathbb{R}^n\backslash\Omega}(x)$. A trivial computation shows that

$$(3.8)\qquad w(\Omega) \lesssim \frac{1}{\lambda\epsilon}\int_{\mathbb{R}^n} |f(y)|_q Mw(y)dy.$$

For each fixed l, a straightforward computation involving Minkowski's inequality gives us that for $v = \{v_k\}$,

$$\sum_k \Big|\int_{\mathbb{R}^n} A_{t_{Q_l}} b_{k,\,l}(y)v_k(y)dy\Big| \lesssim \int_{\mathbb{R}^n} |v(y)|_{q'} \int_{Q_l} h_{t_{Q_l}}(y,\, z)|b^l(z)|_q dzdy.$$

Applying the argument in [6, p. 241], we know that for some $\theta \in (0,\, 1)$,

$$\int_{Q_l} h_{t_{Q_l}}(y,z)|b^l(z)|_q dz \lesssim \big\||b^l|_q\big\|_{L^1(\mathbb{R}^n)} \inf_{z\in Q_l} h_{\theta t_{Q_l}}(y,z) \lesssim \lambda \int_{Q_l} h_{\theta t_{Q_l}}(y,z)dz.$$

Therefore,

$$\begin{aligned}\sum_k \Big|\int_{\mathbb{R}^n} A_{t_{Q_l}} b_{k,\,l}(y)v_k(y)dy\Big| &\lesssim \lambda \int_{Q_l}\int_{\mathbb{R}^n} h_{\theta t_{Q_l}}(y,z)|v(y)|_{q'}dydz\\ &\lesssim \lambda \int_{Q_l} M(|v|_{q'})(z)dz.\end{aligned}$$

Recall that for each l and $\gamma \in [0,\, \infty)$,

$$\inf_{y\in Q_l} M_{L(\log L)^\gamma}\widetilde{w}(y) \approx \sup_{y\in Q_l} M_{L(\log L)^\gamma}\widetilde{w}(y).$$

It then follows that

$$\begin{aligned}\int_{\cup_l Q_l} M_{L(\log L)^\gamma}\widetilde{w}(y)dy &\lesssim \sum_l |Q_l| \inf_{y\in Q_l} M_{L(\log L)^\gamma}\widetilde{w}(y)\\ &\lesssim \lambda^{-1}\int_{\mathbb{R}^n} |f(y)|_q M_{L(\log L)^\gamma}\widetilde{w}(y)dy.\end{aligned}$$

Let $p_1 = 1 + \epsilon/4$. For $v = \{v_k\}$ with $|v|_{q'} \in L^{p_1'}\big(\mathbb{R}^n, (M_{L(\log L)^{\epsilon/2}}\widetilde{w})^{1-p_1'}\big)$, we have that

$$\sum_k \sum_l \int_{\mathbb{R}^n} \big|v_k(y) A_{t_{Q_l}} b_{k,\,l}(y)\big| dy \lesssim \lambda \sum_l \int_{Q_l} M\big(|v|_{q'}\big)(z)dz$$
$$\lesssim \lambda \Big(\int_{\cup_j Q_j} \big\{ M\big(|v|_{q'}\big)(y)\big\}^{p_1'} \big(M_{L(\log L)^{1+\epsilon}}\widetilde{w}(y)\big)^{1-p_1'} dy\Big)^{\frac{1}{p_1'}}$$
$$\times \Big(\int_{\cup_j Q_j} M_{L(\log L)^{1+\epsilon}}\widetilde{w}(y)dy\Big)^{\frac{1}{p_1}}$$
$$\lesssim \lambda^{\frac{p_1-1}{p_1}} \Big(\int_{\mathbb{R}^n} |v(y)|_{q'}^{p_1'} (M_{L(\log L)^{\epsilon/2}}\widetilde{w})^{1-p_1'}(y)dy\Big)^{\frac{1}{p_1'}}$$
$$\times \Big(\int_{\mathbb{R}^n} |f(y)|_q M_{L(\log L)^{1+\epsilon}}\widetilde{w}(y)dy\Big)^{\frac{1}{p_1}},$$

where the last inequality follows from the fact that for any $\epsilon \in (0,\,1)$ and weight u,

$$\big\|Mh\big\|_{L^{p_1'}(\mathbb{R}^n,\,(M_{L(\log L)^{p_1-1+\epsilon/4}}u(y))^{1-p_1'})} \lesssim_n p_1^2 \big(\frac{1}{\epsilon}\big)^{\frac{1}{p_1'}} \|h\|_{L^{p_1'}(\mathbb{R}^n,\,u^{1-p_1'})},$$

see [14, p. 618–619], and the fact that for any weight u,

$$M_{L(\log L)^{\epsilon/2}}(M_{L(\log L)^{\epsilon/2}}u)(x) \approx M_{L(\log L)^{1+\epsilon}}u(x),$$

see [2]. Therefore, we have that

$$\int_{\mathbb{R}^n} |f^2(x)|_q^{p_1} M_{L(\log L)^{\epsilon/2}}\widetilde{w}(x)dx \lesssim \lambda^{p_1-1} \int_{\mathbb{R}^n} |f(x)|_q M_{L(\log L)^{1+\epsilon}} w(x)dx.$$

This, along with the estimate (3.3), tells us that

$$w\big(\{x \in \mathbb{R}^n \backslash \Omega : |Tf^2(x)|_q > \frac{\lambda}{3}\}\big) \tag{3.9}$$
$$\lesssim \frac{1}{\epsilon^{2p_1}} \frac{1}{\lambda^{p_1}} \int_{\mathbb{R}^n} |f^2(x)|_q^{p_1} M_{L(\log L)^{\epsilon/2}}\widetilde{w}(x)dx$$
$$\lesssim \frac{1}{\lambda\epsilon^2} \int_{\mathbb{R}^n} |f(x)|_q M_{L(\log L)^{1+\epsilon}} w(x)dx.$$

Combining the estimates (3.6)–(3.9) yields (3.5) and completes the proof of Theorem 1.6. □

Remark 3.4. The inequalities (3.3), (3.4) and (3.5) extend and improve the results about the weight estimates with general weight for T and T^* established in [10].

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