

INVERSION OF THE CLASSICAL RADON TRANSFORM ON \mathbb{Z}_p^n

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ABSTRACT. The Radon transform introduced by J. Radon in 1917 is the integral transform which is widely applicable to tomography. Here we study the discrete version of the Radon transform. More precisely, when $\mathcal{C}(\mathbb{Z}_p^n)$ is the set of complex-valued functions on \mathbb{Z}_p^n . We completely determine the subset of $\mathcal{C}(\mathbb{Z}_p^n)$ whose elements can be recovered from its Radon transform on \mathbb{Z}_p^n .

1. Introduction

In 1917, Johan Radon introduced and studied a transform that integrates a function over all hyperplanes; later this transform has become known as the (classical) Radon transform. He gave an explicit inversion formula of how to reconstruct a function f from its integrals. This Radon transform is useful in Computed Axial Tomography (CAT scan), barcode scanners, reflection seismology, and in the solution of hyperbolic partial differential equations.

Perceiving many applications of the Radon transform initiated the research of generalizations of Radon transforms using various sets of integration including sphere [1–3, 7, 19], ellipsoids [13, 18], and cone [11, 12, 14]. See also [10, 15–17] for monographs studying general classes of Radon transforms.

As one of the topics of generalizations of the Radon transform, the Radon-type transform of a continuous function f defined by

$$R_S f(\mathbf{u}) = \int_{\mathbb{R}^n} f(\mathbf{u} - \mathbf{x}) \phi_S(\mathbf{x}) d\mathbf{x}, \quad \mathbf{u} = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n (n \geq 2),$$

is introduced. Here S is a compact subset of \mathbb{R}^n and ϕ_S is the characteristic function of S . Especially, when S is a sphere, we call $R_S f$ the spherical Radon transform [7, 19].

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Let G be a finite group and let $\mathcal{C}(G)$ be the space of complex-valued functions on G . For $f \in \mathcal{C}(G)$ and a subset S of G , we define

$$(1) \quad f_S^*(\mathbf{u}) = \sum_{\mathbf{x} \in G} f(\mathbf{x}) \phi_S(\mathbf{u} - \mathbf{x}), \quad \mathbf{u} \in G.$$

In [6] Diaconis and Graham called f_S^* the Radon transform of f on $G = \mathbb{Z}_2^n$, the group of binary n -tuples. They examined for which subsets S are the Radon transform f_S^* invertible and derived the inversion formula when S satisfies a certain condition giving the injectivity of f_S^* . In [9], Frankl and Graham provided a characterization for when f_S^* on \mathbb{Z}_p^n , p being a prime, determines f uniquely. Fill [8] also discussed the uniqueness and an inversion of f_S^* when $G = \mathbb{Z}_k$, the group of integers modulo k . DeDeo and Velasquez [5] derived the inversion of f_S^* when $G = \mathbb{Z}_k^n$, the group of n -tuples of the integers modulo k . The transforms dealt in [5, 6, 8], however, seem to be closely related to the spherical Radon transform $R_S f$.

Let f be an integrable function on \mathbb{R}^n ($n \geq 2$). We introduce the classical Radon transform Rf with hyperplanes of integration defined by

$$Rf(\mathbf{u}, s) = \int_{\mathbb{R}^{n-1}} f(\mathbf{x}, \mathbf{u} \cdot \mathbf{x} + s) d\mathbf{x} \quad \text{for } (\mathbf{u}, s) = (u_1, u_2, \dots, u_{n-1}, s) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$
¹

The classical Radon transform is a powerful tool for many image processing and machine vision applications (see [4]). However, the continuous version of the Radon transform encounters the difficulties when applying it to discrete images. It is desirable for a discrete version of the Radon transform to facilitate the realization in general image processing application.

The discrete version of the Radon transform of a complex-valued function f on $V = \mathbb{Z}_p^{n-1} \times \mathbb{Z}_p$ is a linear operator from $\mathcal{C}(V)$, the set of complex-valued functions to itself defined by

$$Rf(\mathbf{x}, y) = \sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1}} f(\mathbf{u}, \mathbf{u} \cdot \mathbf{x} + y).$$

Notice that Rf can not be covered by f_S^* defined in (1). Let us define the dual operator $R^\#$ of the Radon transform R by

$$R^\# f(\mathbf{x}, y) = \sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1}} f(\mathbf{u}, y - \mathbf{u} \cdot \mathbf{x}) = Rf(-\mathbf{x}, y).$$

Then by simple computation, we have

$$\sum_{s \in \mathbb{Z}_p} \sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1}} Rf(\mathbf{u}, s) g(\mathbf{u}, s) = \sum_{\mathbf{x} \in \mathbb{Z}_p^{n-1}} \sum_{y \in \mathbb{Z}_p} f(\mathbf{x}, y) R^\# g(\mathbf{x}, y).$$

¹To tell the true, the classical Radon transform is usually defined by

$$\mathcal{R}f(\boldsymbol{\theta}, s) = \int_{\boldsymbol{\theta}^\perp} f(s\boldsymbol{\theta} + \boldsymbol{\tau}) d\boldsymbol{\tau} \quad \text{for } (\boldsymbol{\theta}, s) \in S^{n-1} \times \mathbb{R}.$$

In view of the integration area, both are the same. Boris Rubin calls $\mathcal{R}f$ the transversal Radon transform in [17].

The aim of this paper is to find out completely the subset of $\mathcal{C}(V)$ whose elements can be reconstructed from Rf .

We are now in a position to state the main result and the proof is given in the next section.

Theorem 1. *Let R be the Radon transform from $\mathcal{C}(V)$ to itself. Then f is in $\text{Im}(R)$ generated by $\{\zeta_p^{(\mathbf{u},s)} : (\mathbf{u}, s) \text{ is } (0,0) \text{ or in } \mathbb{Z}_p^{n-1} \times \mathbb{Z}_p^*\}$ if and only if*

$$f(\mathbf{x}, y) = (p^{-n} - p^{-1}) \sum_{s \in \mathbb{Z}_p} Rf(0, s) + p^{1-n} R^\# Rf(\mathbf{x}, y) \text{ for any } (\mathbf{x}, y) \in V.$$

Next section is devoted to the proof of Theorem 1. To prove Theorem, we start with the analogue of the Fourier slice theorem because it plays a critical role to find the inversion formula in the continuous version. Then we describe the image and kernel of the Radon transform.

1.1. Preliminaries

The inner product of f and g in $\mathcal{C}(V)$ is defined by

$$\langle f, g \rangle = \sum_{(\mathbf{x}, y) \in V} f(\mathbf{x}, y) \overline{g(\mathbf{x}, y)},$$

where the bar denotes the complex conjugation. It is known that $\{\zeta_p^{(\mathbf{u},s)} : (\mathbf{u}, s) \in \mathbb{Z}_p^n\}$ is an orthogonal basis for $\mathcal{C}(V)$, where $\zeta_p = e^{\frac{2\pi i}{p}}$ is a primitive p -th root of unity and $\zeta_p^{(\mathbf{u},s)}(\mathbf{x}, y) = \zeta_p^{(\mathbf{u},s) \cdot (\mathbf{x}, y)}$ with the dot being the usual inner product.

For a complex-valued function f on V , the Fourier transform of f with respect to (\mathbf{u}, s) in V and its inversion are defined by

$$\mathcal{F}f(\mathbf{x}, y) = \sum_{(\mathbf{u}, s) \in V} \zeta_p^{(\mathbf{u},s) \cdot (\mathbf{x}, y)} f(\mathbf{u}, s), \quad f(\mathbf{u}, s) = p^{-n} \sum_{(\mathbf{x}, y) \in V} \zeta_p^{-(\mathbf{x}, y) \cdot (\mathbf{u}, s)} \mathcal{F}f(\mathbf{x}, y).$$

The partial Fourier transform of f with respect to the last variable in \mathbb{Z}_p is defined by $\mathcal{F}_s f(\mathbf{x}, y) = \sum_{s \in \mathbb{Z}_p} f(\mathbf{x}, s) \zeta_p^{sy}$.

2. Proof of Theorem 1

We start by proving an analogue of the Fourier slice theorem.

Lemma 2. *Let R be the Radon transform from $\mathcal{C}(V)$ to itself and f in $\mathcal{C}(V)$. Then we have*

$$\mathcal{F}_s(Rf)(\mathbf{x}, y) = \mathcal{F}f(-y\mathbf{x}, y)$$

for any (\mathbf{x}, y) in V .

Proof. We see that

$$\mathcal{F}_s(Rf)(\mathbf{x}, y) = \sum_{t \in \mathbb{Z}_p} (Rf)(\mathbf{x}, t) \zeta_p^{ty} = \sum_{t \in \mathbb{Z}_p} \sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1}} f(\mathbf{u}, \mathbf{x} \cdot \mathbf{u} + t) \zeta_p^{ty}$$

$$\begin{aligned}
&= \sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1}} \zeta_p^{-y\mathbf{u} \cdot \mathbf{x}} \sum_{t \in \mathbb{Z}_p} f(\mathbf{u}, \mathbf{x} \cdot \mathbf{u} + t) \zeta_p^{y(t+\mathbf{u} \cdot \mathbf{x})} \\
&= \sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1}} \zeta_p^{-y\mathbf{x} \cdot \mathbf{u}} \sum_{s \in \mathbb{Z}_p} f(\mathbf{u}, s) \zeta_p^{ys} = \mathcal{F}f(-y\mathbf{x}, y).
\end{aligned}$$

□

We present basic properties of the Fourier and Radon transforms on V .

Lemma 3. *Let f be in $\mathcal{C}(V)$. Then we have*

- (i) $\sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1}} \mathcal{F}f(\mathbf{u}, 0) \zeta_p^{-\mathbf{x} \cdot \mathbf{u}} = p^{n-1} \sum_{\bar{y} \in \mathbb{Z}_p} f(\mathbf{x}, \bar{y})$ and $\mathcal{F}f(0, 0) = \sum_{(\bar{x}, \bar{y}) \in V} f(\bar{x}, \bar{y})$,
- (ii) $\sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1}} \mathcal{F}_s(Rf)(\mathbf{u}, 0) = p^{n-1} \mathcal{F}f(0, 0) = p^{n-1} \sum_{(\bar{x}, \bar{y}) \in V} f(\bar{x}, \bar{y})$,
- (iii) $R^\# Rf(\mathbf{x}, y) = p^{-1} \sum_{(\mathbf{u}, s) \in V} \zeta_p^{-s(-\mathbf{u}, 1) \cdot (\mathbf{x}, y)} \mathcal{F}_s(Rf)(\mathbf{u}, s)$.

Proof. The result (i) is straightforward because of the definition of $\mathcal{F}f$ and the result (ii) follows from using $\mathcal{F}f(0, 0) = \mathcal{F}_s(Rf)(\mathbf{u}, 0)$ for any $\mathbf{u} \in \mathbb{Z}_p^{n-1}$ by Lemma 2. For (iii), we see that

$$\begin{aligned}
R^\# Rf(\mathbf{x}, y) &= R^\# \sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1}} f(\mathbf{u}, \mathbf{u} \cdot \mathbf{x} + y) \\
&= \sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1}} R^\# f(\mathbf{u}, \mathbf{u} \cdot \mathbf{x} + y) \\
&= \sum_{\mathbf{u}, \mathbf{v} \in \mathbb{Z}_p^{n-1}} f(\mathbf{v}, \mathbf{u} \cdot \mathbf{v} + y - \mathbf{u} \cdot \mathbf{x}) \\
&= \sum_{\mathbf{u}, \mathbf{v} \in \mathbb{Z}_p^{n-1}} \sum_{t \in \mathbb{Z}_p} f(\mathbf{v}, \mathbf{u} \cdot \mathbf{v} + t) \delta_{y-\mathbf{u} \cdot \mathbf{x}, t} \\
&= p^{-1} \sum_{\mathbf{u}, \mathbf{v} \in \mathbb{Z}_p^{n-1}} \sum_{t \in \mathbb{Z}_p} f(\mathbf{v}, \mathbf{u} \cdot \mathbf{v} + t) \sum_{s \in \mathbb{Z}_p} \zeta_p^{s(\mathbf{u} \cdot \mathbf{x} - y + t)} \\
&= p^{-1} \sum_{(\mathbf{u}, s) \in V} \zeta_p^{s(\mathbf{u} \cdot \mathbf{x} - y)} \sum_{t \in \mathbb{Z}_p} \sum_{\mathbf{v} \in \mathbb{Z}_p^{n-1}} f(\mathbf{v}, \mathbf{u} \cdot \mathbf{v} + t) \zeta_p^{st} \\
&= p^{-1} \sum_{(\mathbf{u}, s) \in V} \zeta_p^{s(\mathbf{u} \cdot \mathbf{x} - y)} \mathcal{F}_s(Rf)(\mathbf{u}, s),
\end{aligned}$$

where $\delta_{s,t}$ is the Kronecker delta function. □

In the following proposition, we describe the image and kernel of the Radon transform which play a crucial role in proving our main result.

Proposition 4. *Let R be the Radon transform from $\mathcal{C}(V)$ to itself. Then we have*

- (i) $\text{Im}(R) = \left\{ f \in \mathcal{C}(V) : \sum_{y \in \mathbb{Z}_p} f(\mathbf{x}, y) \text{ is a constant for any } \mathbf{x} \in \mathbb{Z}_p^{n-1} \right\}$
and

- (ii) $\text{Ker}(R) = \{f(\mathbf{x}, y) - p^{1-n} \sum_{\bar{\mathbf{x}} \in \mathbb{Z}_p^{n-1}} f(\bar{\mathbf{x}}, y) : f(\mathbf{x}, y) \in \mathcal{C}(V) \text{ does not depend on } y \in \mathbb{Z}_p\}$.

Proof. (i) Let $A = \left\{g \in \mathcal{C}(V) : \sum_{s \in \mathbb{Z}_p} g(\mathbf{u}, s) \text{ is a constant for any } \mathbf{u} \in \mathbb{Z}_p^{n-1}\right\}$.

Since

$$\begin{aligned} \sum_{s \in \mathbb{Z}_p} Rf(\mathbf{u}, s) &= \sum_{s \in \mathbb{Z}_p} \sum_{\mathbf{x} \in \mathbb{Z}_p^{n-1}} f(\mathbf{x}, \mathbf{x} \cdot \mathbf{u} + s) \\ &= \sum_{\mathbf{x} \in \mathbb{Z}_p^{n-1}} \sum_{s \in \mathbb{Z}_p} f(\mathbf{x}, \mathbf{x} \cdot \mathbf{u} + s) = \sum_{\mathbf{x} \in \mathbb{Z}_p^{n-1}} \sum_{s \in \mathbb{Z}_p} f(\mathbf{x}, s), \end{aligned}$$

we have $\text{Im}(R) \subseteq A$. Let $g \in A$ and $\mathbf{u} \in \mathbb{Z}_p^{n-1}$ be fixed. We claim that $\sum_{\mathbf{v} \in \mathbb{Z}_p^{n-1} \setminus \{\mathbf{u}\}} \sum_{\mathbf{x} \in \mathbb{Z}_p^{n-1}} g(\mathbf{v}, (\mathbf{u} - \mathbf{v}) \cdot \mathbf{x} + s)$ is a constant on any s in \mathbb{Z}_p . This term can be written as

$$\begin{aligned} &\sum_{\substack{\mathbf{v} \in \mathbb{Z}_p^{n-1} \setminus \{\mathbf{u}\} \\ v_{n-1} \neq u_{n-1}}} \sum_{(x_1, \dots, x_{n-2}) \in \mathbb{Z}_p^{n-2}} \sum_{x_{n-1} \in \mathbb{Z}_p} g\left(\mathbf{v}, \sum_{j=1}^{n-2} (u_j - v_j) \cdot x_j + x_{n-1}(u_{n-1} - v_{n-1}) + s\right) \\ &+ \sum_{\substack{\mathbf{v} \in \mathbb{Z}_p^{n-1} \setminus \{\mathbf{u}\} \\ v_{n-1} = u_{n-1}}} \sum_{\tilde{\mathbf{x}}_m \in \mathbb{Z}_p^{n-2}} \sum_{x_m \in \mathbb{Z}_p} g\left(\mathbf{v}, \sum_{j \in \{1, 2, \dots, n-1\}, j \neq m} (u_j - v_j) \cdot x_j + x_m(u_m - v_m) + s\right) \\ &= \sum_{\substack{\mathbf{v} \in \mathbb{Z}_p^{n-1} \setminus \{\mathbf{u}\} \\ v_{n-1} \neq u_{n-1}}} \sum_{(x_1, \dots, x_{n-2}) \in \mathbb{Z}_p^{n-2}} \sum_{x_{n-1} \in \mathbb{Z}_p} g(\mathbf{v}, x_{n-1}) + \sum_{\substack{\mathbf{v} \in \mathbb{Z}_p^{n-1} \setminus \{\mathbf{u}\} \\ v_{n-1} = u_{n-1}}} \sum_{\tilde{\mathbf{x}}_m \in \mathbb{Z}_p^{n-2}} \sum_{x_m \in \mathbb{Z}_p} g(\mathbf{v}, x_m), \end{aligned}$$

where $m \in \{1, 2, \dots, n-2\}$ is chosen such that $v_m \neq u_m$ and $\tilde{\mathbf{x}}_m = (x_1, \dots, x_{m-1}, x_{m+1}, \dots, x_{n-1})$. Since g is in A , the term should be a constant.

Now we can set

$$(2) \quad C = p^{1-n} \sum_{\mathbf{v} \in \mathbb{Z}_p^{n-1} \setminus \{\mathbf{u}\}} \sum_{\mathbf{x} \in \mathbb{Z}_p^{n-1}} g(\mathbf{v}, (\mathbf{u} - \mathbf{v}) \cdot \mathbf{x} + s)$$

and set $f = p^{1-n}(R^\#g - C)$. It is now sufficient to prove that $Rf = g$. We see that

$$\begin{aligned} Rf(\mathbf{u}, s) &= p^{1-n} RR^\#g(\mathbf{u}, s) - C \\ &= p^{1-n} \sum_{\mathbf{x} \in \mathbb{Z}_p^{n-1}} \sum_{\mathbf{v} \in \mathbb{Z}_p^{n-1}} g(\mathbf{v}, (\mathbf{u} \cdot \mathbf{x} + s) - \mathbf{v} \cdot \mathbf{x}) - C \\ &= p^{1-n} \sum_{\mathbf{v} \in \mathbb{Z}_p^{n-1}} \sum_{\mathbf{x} \in \mathbb{Z}_p^{n-1}} g(\mathbf{v}, (\mathbf{u} - \mathbf{v}) \cdot \mathbf{x} + s) - C \\ &= p^{1-n} \sum_{\mathbf{x} \in \mathbb{Z}_p^{n-1}} g(\mathbf{u}, s) + p^{1-n} \sum_{\mathbf{v} \in \mathbb{Z}_p^{n-1} \setminus \{\mathbf{u}\}} \sum_{\mathbf{x} \in \mathbb{Z}_p^{n-1}} g(\mathbf{v}, (\mathbf{u} - \mathbf{v}) \cdot \mathbf{x} + s) - C \\ &= g(\mathbf{u}, s) \end{aligned}$$

by the definition (2) of C . Thus $A \subseteq \text{Im}(R)$ and the proof is complete.

(ii) Let $B = \{f(\mathbf{x}, y) - p^{1-n} \sum_{\bar{\mathbf{x}} \in \mathbb{Z}_p^{n-1}} f(\bar{\mathbf{x}}, y) : f(\mathbf{x}, y) \text{ does not depend on } y \in \mathbb{Z}_p\}$ be the subspace of $\mathcal{C}(V)$. We start with the inverse Fourier transform:

$$\begin{aligned} f(\mathbf{x}, y) &= p^{-n} \sum_{(\mathbf{u}, s) \in V} \zeta_p^{-(\mathbf{u}, s) \cdot (\mathbf{x}, y)} \mathcal{F}f(\mathbf{u}, s) \\ &= p^{-n} \left[\begin{aligned} &\sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1}} \mathcal{F}f(\mathbf{u}, 0) \zeta_p^{-\mathbf{x} \cdot \mathbf{u}} - p^{n-1} \mathcal{F}f(0, 0) + p^{n-1} \mathcal{F}f(0, 0) \\ &+ \sum_{s \in \mathbb{Z}_p^*} \sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1}} \zeta_p^{-(\mathbf{u}, s) \cdot (\mathbf{x}, y)} \mathcal{F}f(\mathbf{u}, s) \end{aligned} \right] \\ &= p^{-n} \left[\begin{aligned} &p^{n-1} \sum_{\bar{y} \in \mathbb{Z}_p} f(\mathbf{x}, \bar{y}) - p^{n-1} \sum_{(\bar{\mathbf{x}}, \bar{y}) \in V} f(\bar{\mathbf{x}}, \bar{y}) + \sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1}} \mathcal{F}_s(Rf)(\mathbf{u}, 0) \\ &+ \sum_{(\mathbf{u}, s) \in \mathbb{Z}_p^{n-1} \times \mathbb{Z}_p^*} \zeta_p^{-s(-\mathbf{u}, 1) \cdot (\mathbf{x}, y)} \mathcal{F}f(-s\mathbf{u}, s) \end{aligned} \right], \end{aligned}$$

where we used Lemma 3 in the last equality. Applying Lemma 2 to the last term right above, we obtain

$$\begin{aligned} f(\mathbf{x}, y) &= p^{-1} \left[\sum_{\bar{y} \in \mathbb{Z}_p} f(\mathbf{x}, \bar{y}) - \sum_{(\bar{\mathbf{x}}, \bar{y}) \in V} f(\bar{\mathbf{x}}, \bar{y}) \right] \\ &\quad + p^{-n} \sum_{(\mathbf{u}, s) \in V} \zeta_p^{-s(-\mathbf{u}, 1) \cdot (\mathbf{x}, y)} \mathcal{F}_s(Rf)(\mathbf{u}, s). \end{aligned}$$

By Lemma 3(iii), we obtain

$$(3) \quad f(\mathbf{x}, y) = p^{-1} \left[\sum_{\bar{y} \in \mathbb{Z}_p} f(\mathbf{x}, \bar{y}) - \sum_{(\bar{\mathbf{x}}, \bar{y}) \in V} f(\bar{\mathbf{x}}, \bar{y}) \right] + p^{1-n} R^\# Rf(\mathbf{x}, y).$$

We see that

$$\sum_{\bar{y} \in \mathbb{Z}_p} f(\mathbf{x}, \bar{y}) - p^{1-n} \sum_{\bar{\mathbf{x}} \in \mathbb{Z}_p^{n-1}} \sum_{\bar{y} \in \mathbb{Z}_p} f(\bar{\mathbf{x}}, \bar{y})$$

is in B because $\sum_{\bar{y} \in \mathbb{Z}_p} f(\mathbf{x}, \bar{y})$ does not depend on y . It then follows from $\sum_{\bar{y} \in \mathbb{Z}_p} Rf(0, \bar{y}) = \sum_{(\bar{\mathbf{x}}, \bar{y}) \in V} f(\bar{\mathbf{x}}, \bar{y})$ that (3) can be written as

$$\begin{aligned} f(\mathbf{x}, y) + B &= (p^{-n} - p^{-1}) \sum_{(\bar{\mathbf{x}}, \bar{y}) \in V} f(\bar{\mathbf{x}}, \bar{y}) + p^{1-n} R^\# Rf(\mathbf{x}, y) + B \\ (4) \quad &= (p^{-n} - p^{-1}) \sum_{\bar{y} \in \mathbb{Z}_p} Rf(0, \bar{y}) + p^{1-n} R^\# Rf(\mathbf{x}, y) + B \quad \text{in } \mathcal{C}(V)/B. \end{aligned}$$

Now let $f \in \text{Ker}(R)$. Then $Rf = 0$. By (4), we have $f + B = B$. This implies that $\text{Ker}(R) \subseteq B$. Conversely, let $g(\mathbf{x}, y) = f(\mathbf{x}, y) - p^{1-n} \sum_{\bar{\mathbf{x}} \in \mathbb{Z}_p^{n-1}} f(\bar{\mathbf{x}}, y)$ be in B , where $f(\mathbf{x}, y)$ does not depend on y in \mathbb{Z}_p . Then $Rf(\mathbf{u}, s) = \sum_{\mathbf{x} \in \mathbb{Z}_p^{n-1}} f(\mathbf{x}, y)$ is a constant, and so $Rg(\mathbf{u}, s) = \sum_{\mathbf{x} \in \mathbb{Z}_p^{n-1}} f(\mathbf{x}, y) - \sum_{\bar{\mathbf{x}} \in \mathbb{Z}_p^{n-1}} f(\bar{\mathbf{x}}, y) = 0$. Thus g is in $\text{Ker}(R)$ and so $B \subseteq \text{Ker}(R)$. The proof is completed. \square

We notice that $\mathcal{C}(V)/\text{Ker}(R)$ and $\text{Im}(R)$ are isomorphic via $f + \text{Ker}(R) \mapsto Rf$ (by the first isomorphism theorem).

Corollary 5. *Let R be the Radon transform from $\mathcal{C}(V)$ to itself. Then we have*

- (i) $\text{Im}(R)$ is generated by $\{\zeta_p^{(\mathbf{u},s)} : (\mathbf{u}, s) \text{ is } (0, 0) \text{ or in } \mathbb{Z}_p^{n-1} \times \mathbb{Z}_p^*\}$ and
- (ii) $\text{Ker}(R)$ is generated by $\{\zeta_p^{(\mathbf{u},0)} - p^{1-n} \sum_{\mathbf{v} \in \mathbb{Z}_p^{n-1}} \zeta_p^{(\mathbf{v},0)} : 0 \neq \mathbf{u} \in \mathbb{Z}_p^{n-1}\}$.

Proof. One can easily verify that $\{\zeta_p^{(\mathbf{u},s)}(\mathbf{x}, y) : (\mathbf{u}, s) \text{ is } (0, 0) \text{ or in } \mathbb{Z}_p^{n-1} \times \mathbb{Z}_p^*\}$ is a subset of $\text{Im}(R)$. It follows from Proposition 4(i) that the dimension of $\text{Im}(R)$ is at least $p^{n-1}(p-1) + 1$. We claim that

$$(5) \quad \left\{ \zeta_p^{(\mathbf{u},0)}(\mathbf{x}, y) - p^{1-n} \sum_{\bar{\mathbf{x}} \in \mathbb{Z}_p^{n-1}} \zeta_p^{(\mathbf{u},0)}(\bar{\mathbf{x}}, y) : \mathbf{u} \in \mathbb{Z}_p^{n-1} \setminus \{\mathbf{0}\} \right\},$$

where $\mathbf{0}$ is the zero-vector in \mathbb{Z}_p^{n-1} , is the set of linearly independent functions. Assume that

$$\sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1} \setminus \{\mathbf{0}\}} a_{\mathbf{u}} \left(\zeta_p^{(\mathbf{u},0)}(\mathbf{x}, y) - p^{1-n} \sum_{\bar{\mathbf{x}} \in \mathbb{Z}_p^{n-1}} \zeta_p^{(\mathbf{u},0)}(\bar{\mathbf{x}}, y) \right) = 0,$$

where $a_{\mathbf{u}} \in \mathbb{C}$. Then we have

$$0 = \sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1} \setminus \{\mathbf{0}\}} a_{\mathbf{u}} \left(\zeta_p^{\mathbf{u} \cdot \mathbf{x}} - p^{1-n} \sum_{\bar{\mathbf{x}} \in \mathbb{Z}_p^{n-1}} \zeta_p^{\bar{\mathbf{x}} \cdot \mathbf{u}} \right) = \sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1} \setminus \{\mathbf{0}\}} a_{\mathbf{u}} \zeta_p^{\mathbf{u} \cdot \mathbf{x}}$$

for any \mathbf{x} in \mathbb{Z}_p^{n-1} , and so $\sum_{\mathbf{u} \in \mathbb{Z}_p^{n-1} \setminus \{\mathbf{0}\}} a_{\mathbf{u}} \zeta_p^{\mathbf{u} \cdot \mathbf{x}} = 0$ for any \mathbf{x} in \mathbb{Z}_p^{n-1} . By defining $a_0 = 0$, we obtain that $a_{\mathbf{u}} = 0$ for any \mathbf{u} in \mathbb{Z}_p^{n-1} . This proves our claim. It is obvious that the set in (5) is a subset of $\text{Ker}(R)$ because $\zeta_p^{(\mathbf{u},0)}(\mathbf{x}, y) = \zeta_p^{\mathbf{u} \cdot \mathbf{x}}$ does not depend on y . It follows from Proposition 4(ii) that the dimension of $\text{Ker}(R)$ is at least $p^{n-1} - 1$. By the isomorphism of $\mathcal{C}(V)/\text{Ker}(R)$ and $\text{Im}(R)$, we obtain that $p^n - \dim(\text{Ker}(R)) = \dim(\text{Im}(R)) \geq p^{n-1}(p-1) + 1$, or $\dim(\text{Ker}(R)) \leq p^{n-1} - 1$. Thus $\dim(\text{Ker}(R)) = p^{n-1} - 1$ and $\dim(\text{Im}(R)) = p^{n-1}(p-1) + 1$. The proof is completed. \square

We are ready to prove our main result.

Proof of Theorem 1. (Sufficiency) Let f be in $\text{Im}(R)$. Then $\sum_{y \in \mathbb{Z}_p} f(\mathbf{x}, y)$ is a constant for any \mathbf{x} in \mathbb{Z}_p^{n-1} by Proposition 4(i), and so $\sum_{\bar{y} \in \mathbb{Z}_p} f(\mathbf{x}, \bar{y}) = p^{1-n} \sum_{(\bar{\mathbf{x}}, \bar{y}) \in V} f(\bar{\mathbf{x}}, \bar{y})$. It then follows from (3) that

$$\begin{aligned} f(\mathbf{x}, y) &= p^{-1} \left[\sum_{\bar{y} \in \mathbb{Z}_p} f(\mathbf{x}, \bar{y}) - \sum_{(\bar{\mathbf{x}}, \bar{y}) \in V} f(\bar{\mathbf{x}}, \bar{y}) \right] + p^{1-n} R^\# R f(\mathbf{x}, y) \\ &= (p^{-n} - p^{-1}) \sum_{(\bar{\mathbf{x}}, \bar{y}) \in V} f(\bar{\mathbf{x}}, \bar{y}) + p^{1-n} R^\# R f(\mathbf{x}, y). \end{aligned}$$

The result follows from $\sum_{\bar{y} \in \mathbb{Z}_p} Rf(0, \bar{y}) = \sum_{(\bar{\mathbf{x}}, \bar{y}) \in V} f(\bar{\mathbf{x}}, \bar{y})$.

(Necessity) Let $f(\mathbf{x}, y) = (p^{-n} - p^{-1}) \sum_{s \in \mathbb{Z}_p} Rf(0, s) + p^{1-n} R^\# Rf(\mathbf{x}, y)$. Recall from the proof of Proposition 4(i) that $\sum_{y \in \mathbb{Z}_p} Rf(\mathbf{x}, y)$ is a constant for any \mathbf{x} in \mathbb{Z}_p^{n-1} , and so is $\sum_{y \in \mathbb{Z}_p} R^\# Rf(\mathbf{x}, y)$ because $R^\# f(\mathbf{x}, y) = Rf(-\mathbf{x}, y)$. By Proposition 4, it is sufficient to show that $\sum_{y \in \mathbb{Z}_p} f(\mathbf{x}, y)$ is a constant for any \mathbf{x} in \mathbb{Z}_p^{n-1} . We now have that

$$\sum_{y \in \mathbb{Z}_p} f(\mathbf{x}, y) = p(p^{-n} - p^{-1}) \sum_{y \in \mathbb{Z}_p} Rf(0, y) + p^{1-n} R^\# \left(\sum_{y \in \mathbb{Z}_p} Rf(\mathbf{x}, y) \right)$$

is a constant for any \mathbf{x} in \mathbb{Z}_p^{n-1} and the proof is completed. \square

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