Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b171021
pISSN: 1015-8634 / eISSN: 2234-3016

GRADED POST-LIE ALGEBRA STRUCTURES, ROTA-BAXTER OPERATORS AND YANG-BAXTER EQUATIONS ON THE W-ALGEBRA $W(2,2)$

Xiaomin Tang and Yongyue Zhong

Abstract. In this paper, we characterize the graded post-Lie algebra structures on the W-algebra $W(2,2)$. Furthermore, as applications, the homogeneous Rota-Baxter operators on $W(2,2)$ and solutions of the formal classical Yang-Baxter equation on $W(2,2)\ltimes_{\mathrm{ad}^*}W(2,2)^*$ are studied.

1. Introduction and preliminaries

Throughout the paper, denote by $\mathbb{C},\mathbb{Z}$ the sets of complex numbers, integers respectively. For a fixed integer k, let $\mathbb{Z}_{>k}=\{t\in\mathbb{Z}\,|\,t>k\}$, $\mathbb{Z}_{<k}=\{t\in\mathbb{Z}\,|\,t<k\}$, $\mathbb{Z}_{\geqslant k}=\{t\in\mathbb{Z}\,|\,t\geqslant k\}$ and $\mathbb{Z}_{\leqslant k}=\{t\in\mathbb{Z}\,|\,t\leqslant k\}$. In this paper, we aim to determine the graded post-Lie algebra structures on W-algebra $W(2,2)$, and classify some Rota-Baxter operators on $W(2,2)$ and solutions of the formal Yang-Baxter equations on $W(2,2)\ltimes_{\mathrm{ad}^*}W(2,2)^*$. Now we recall some related concepts and facts as follows.

1.1. W-algebra $W(2,2)$

The W-algebra $W(2,2)$ is an infinite-dimensional Lie algebra with the $\mathbb{C}$-basis $\{L_m,H_m|m\in\mathbb{Z}\}$ and the Lie brackets are given by

$$\begin{aligned}[L_m,L_n]&=(m-n)L_{m+n},\\ [L_m,H_n]&=(m-n)H_{m+n},\\ [H_m,H_n]&=0,\ \forall m,n\in\mathbb{Z}.\end{aligned}$$

A class of central extensions of $W(2,2)$ first introduced by [28] in their recent work on the classification of some simple vertex operator algebras, and then

Received November 21, 2017; Revised August 24, 2018; Accepted September 7, 2018.
2010 *Mathematics Subject Classification.* 17A30, 17B68.
Key words and phrases. W-algebra $W(2,2)$, post-Lie algebra, Rota-Baxter operator, Yang-Baxter equation.
This work is supported in part by NNSFC (Grant No. 11771069 and No. 11701158), NSF of Heilongjiang Province (Grant No. A2015007) and Graduate Innovation Project of Heilongjiang University (Grant No. YJSCX2018-058HLJU).

some scholars studied the theory on structures and representations of $W(2,2)$ or its central extensions, see [7,12,15,19,26] and so forth.

1.2. Post-Lie algebra

Post-Lie algebras were introduced around 2007 by B. Vallette [25], who found the structure in a purely operadic manner as the Koszul dual of a commutative trialgebra. Since then, post-Lie algebras have aroused the interest of a great many authors, see [1,4–6,9,10,17,18,23]. It should be pointed out that post-Lie algebras appear in many areas of mathematics and physics including the differential geometry [17], Lie groups [6,17], classical Yang-Baxter equation [1], Hopf algebra, classical r-matrices [11] and Rota-Baxter operators [13]. One of the most important problems in the study of post-Lie algebras is to find the post-Lie algebra structures on the (given) Lie algebras. For the finite-dimensional cases, in [18], the authors determined all post-Lie algebra structures on $sl(2,\mathbb{C})$ of special linear Lie algebra of order 2 and in [23] the authors studied the post-Lie algebra structures on the solvable Lie algebra $t(2,\mathbb{C})$ of the Lie algebra of 2×2 upper triangular matrices. For the infinite-dimensional cases, some classes of post-Lie algebra structures on the Witt algebra are considered by [21], and all commutative post-Lie algebra structures on the W-algebra $W(2,2)$ are given in [22]. We now turn to the definition of post-Lie algebra following reference [25].

Definition 1.1. A post-Lie algebra $(V,\circ,[,])$ is a vector space V over a field k equipped with two k-bilinear products $x\circ y$ and $[x,y]$ satisfying that $(V,[,])$ is a Lie algebra and

$$[x,y]\circ z = x\circ(y\circ z) - y\circ(x\circ z) - \langle x,y\rangle\circ z, \tag{1}$$

$$x\circ[y,z] = [x\circ y, z] + [y, x\circ z] \tag{2}$$

for all $x,y,z\in V$, where $\langle x,y\rangle = x\circ y - y\circ x$. We also say that $(V,\circ,[,])$ is a post-Lie algebra structure on the Lie algebra $(V,[,])$. If a post-Lie algebra $(V,\circ,[,])$ satisfies $x\circ y = y\circ x$ for all $x,y\in V$, then it is called a commutative post-Lie algebra.

Suppose that $(L,[,])$ is a Lie algebra. Two post-Lie algebras $(L,[,],\circ_1)$ and $(L,[,],\circ_2)$ on the Lie algebra L are called to be isomorphic if there is an automorphism τ of the Lie algebra $(L,[,])$ satisfies

$$\tau(x\circ_1 y) = \tau(x)\circ_2\tau(y), \forall x,y\in L.$$

By Proposition 2.5 of [17], we have the following result.

Proposition 1.2. *Let $(V,\circ,[,])$ be a post-Lie algebra defined by Definition 1.1. Then the following product*

$$\{x,y\} \triangleq \langle x,y\rangle + [x,y], \tag{3}$$

induces a Lie algebra structure on V, where $\langle x,y\rangle = x\circ y - y\circ x$. Furthermore, if two post-Lie algebras $(V,\circ_1,[,])$ and $(V,\circ_2,[,])$ on the same Lie algebra $(V,[,])$

are isomorphic, then the two induced Lie algebras $(V, \{,\}_1)$ *and* $(V, \{,\}_2)$ *are isomorphic.*

Remark 1.3. The left multiplications of the post-Lie algebra $(V, [,], \circ)$ are denoted by $\mathcal{L}(x)$, i.e., we have $\mathcal{L}(x)(y) = x \circ y$ for all $x, y \in V$. By (2), we see that all operators $\mathcal{L}(x)$ are Lie algebra derivations of the Lie algebra $(V, [,])$.

1.3. Rota-Baxter operator

As a matter of fact, the Rota-Baxter operators were originally defined on associative algebras by G. Baxter to solve an analytic formula in probability [2] and then developed by the Rota school [20]. These operators have showed up in many areas in mathematics and mathematical physics (see [8, 13, 14, 24] and the references therein). Now let us recall the definition of Rota-Baxter operator.

Definition 1.4. Let L be a complex Lie algebra. A Rota-Baxter operator of weight $\lambda \in \mathbb{C}$ is a linear map $R : L \to L$ satisfying

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)]) + \lambda R([x, y]), \ \forall x, y \in L. \tag{4}$$

Note that if R is a Rota-Baxter operator of weight $\lambda \neq 0$, then $\lambda^{-1}R$ is a Rota-Baxter operator of weight 1. Therefore, one only needs to consider Rota-Baxter operators of weight 0 and 1.

1.4. Yang-Baxter equation

The Yang-Baxter equation first appeared in theoretical physics and statistical mechanics in the works of Yang [27] and Baxter [3] and it has led to several interesting applications in quantum groups and Hopf algebras, knot theory, tensor categories and integrable systems [16]. Let $\mathfrak{g}$ be a Lie algebra. An element $r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ is called a solution of the classical Yang-Baxter equation (CYBE) on $\mathfrak{g}$ if r satisfies

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \quad \text{in} \ \ U(\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}),$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1, \ r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \ r_{23} = \sum_i 1 \otimes a_i \otimes b_i.$$

For any $r = \sum_i a_i \otimes b_i$, set

$$r^{21} = \sum_i b_i \otimes a_i.$$

It is obvious that r is skew-symmetric if and only if $r = -r^{21}$.

Our results can be briefly summarized as follows: In Section 2, we classify the graded post-Lie algebra structures on the W-algebra $W(2, 2)$, and then we obtain the induced graded Lie algebras. In Section 3, we give the induced Rota-Baxter operators of weight 1 from the post-Lie algebras on $W(2, 2)$. In

Section 4, we give some solutions of the formal classical Yang-Baxter equation on $W(2,2)\ltimes_{\mathrm{ad}^*} W(2,2)^*$.

2. The graded post-Lie algebra structure on the W-algebra $W(2,2)$

Recently the author in [22] proved that any commutative post-Lie algebra structure on the W-algebra $W(2,2)$ is trivial (namely, $x\circ y=0$ for all $x,y\in W(2,2)$). We now will dedicate on the study of the noncommutative cases. Since the W-algebra $W(2,2)$ is graded, we suppose that the post-Lie algebra structure on the W-algebra $W(2,2)$ to be graded. Namely, we mainly consider the post-Lie algebra structure on W-algebra $W(2,2)$ which satisfies

$$L_m\circ L_n=\phi(m,n)L_{m+n}, \tag{5}$$

$$L_m\circ H_n=\varphi(m,n)H_{m+n}, \tag{6}$$

$$H_m\circ L_n=\theta(m,n)H_{m+n}, \tag{7}$$

$$H_m\circ H_n=0 \tag{8}$$

for all $m,n\in\mathbb{Z}$, where ϕ, φ, θ are complex-valued functions on $\mathbb{Z}\times\mathbb{Z}$.

Lemma 2.1 (see [12])**.** *Denote by* $\mathrm{Der}(W(2,2))$ *and by* $\mathrm{Inn}(W(2,2))$ *the space of derivations and the space of inner derivations of* $W(2,2)$ *respectively. Then*

$$\mathrm{Der}(W(2,2))=\mathrm{Inn}(W(2,2))\oplus\mathbb{C}D,$$

where D *is an outer derivation defined by* $D(L_m)=0$, $D(H_m)=H_m$ *for all* $m\in\mathbb{Z}$.

Lemma 2.2. *There exists a graded post-Lie algebra structure on* $W(2,2)$ *satisfying* (5)-(8) *if and only if there are complex-valued functions* f, g *on* $\mathbb{Z}$ *and a complex number* μ *such that*

$$\phi(m,n)=(m-n)f(m), \tag{9}$$

$$\varphi(m,n)=(m-n)f(m)+\delta_{m,0}\mu, \tag{10}$$

$$\theta(m,n)=(m-n)g(m), \tag{11}$$

$$(m-n)(f(m+n)+f(m)f(m+n)+f(n)f(m+n)-f(m)f(n))=0, \tag{12}$$

$$(m-n)(g(m+n)+f(m)g(m+n)+g(n)g(m+n)-f(m)g(n))=0, \tag{13}$$

$$(m-n)(f(m)+f(n)+1)\delta_{m+n,0}\mu=0. \tag{14}$$

Proof. Suppose that there exists a graded post-Lie algebra structure satisfying (5)-(8) on $W(2,2)$. By Remark 1.3, $\mathcal{L}(x)$ is a derivation of $W(2,2)$. It follows by Lemma 2.1 that there are a linear map ψ from $W(2,2)$ into itself and a linear function ρ on $W(2,2)$ such that

$$x\circ y=(\mathrm{ad}\psi(x)+\rho(x)D)(y)=[\psi(x),y]+\rho(x)D(y),$$

where D is given by Lemma 2.1. This, together with (5)-(8), gives that

$$L_m\circ L_n=[\psi(L_m),L_n]=\phi(m,n)L_{m+n}, \tag{15}$$

$$L_m \circ H_n = [\psi(L_m), H_n] + \rho(L_m)H_n = \varphi(m,n)H_{m+n}, \tag{16}$$
$$H_m \circ L_n = [\psi(H_m), L_n] = \theta(m,n)H_{m+n}, \tag{17}$$
$$H_m \circ H_n = [\psi(H_m), H_n] + \rho(H_m)H_n = 0. \tag{18}$$

Let

$$\psi(L_m) = \sum_{i\in\mathbb{Z}} a_i^{(m)} L_i + \sum_{i\in\mathbb{Z}} b_i^{(m)} H_i \text{ and } \psi(H_m) = \sum_{i\in\mathbb{Z}} c_i^{(m)} L_i + \sum_{i\in\mathbb{Z}} d_i^{(m)} H_i,$$

where $a_i^{(m)}$, $b_i^{(m)}$, $c_i^{(m)}$, $d_i^{(m)} \in \mathbb{C}$ for all $i \in \mathbb{Z}$. Then we have by (15)-(18) that

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}}(i-n)a_i^{(m)}L_{i+n} + \sum_{i\in\mathbb{Z}}(i-m)b_i^{(m)}H_{i+n} = \phi(m,n)L_{m+n},\\
&\sum_{i\in\mathbb{Z}}(i-n)a_i^{(m)}H_{i+n} + \rho(L_m)H_n = \varphi(m,n)H_{m+n},\\
&\sum_{i\in\mathbb{Z}}(i-n)c_i^{(m)}L_{i+n} - \sum_{i\in\mathbb{Z}}(n-i)d_i^{(m)}H_{i+n} = \theta(m,n)H_{m+n},\\
&\sum_{i\in\mathbb{Z}}(i-n)c_i^{(m)}H_{i+n} + \rho(H_m)H_n = 0.
\end{aligned}$$

It is not difficult to see by the above equations that (9), (10) and (11) are established with

$$f(m) = a_m^{(m)},\ g(m) = d_m^{(m)},\ \mu = \rho(L_0) = \varphi(0,0).$$

By a simple computation, we see that (1) with $(x, y, z) = (L_m, L_n, L_k)$ holds if and only if the following equation holds:

(19)
$$\begin{aligned}
&(m-n)(m+n-k)f(m+n)\\
=\ &(n-k)(m-n-k)f(n)f(m) - (m-k)(n-m-k)f(m)f(n)\\
&- (m-n)(m+n-k)f(m)f(m+n) + (n-m)(n+m-k)f(n)f(m+n).
\end{aligned}$$

The above equation can be viewed as a polynomial equation in k, then we see that (19) holds if and only if (12) holds. Similarly, one can see that (1) with $(x, y, z) = (L_m, H_n, L_k)$ or (H_n, L_m, L_k) holds if and only if the following equation holds:

(20)
$$\begin{aligned}
&(m-n)(m+n-k)g(m+n)\\
=\ &(n-k)\left((m-n-k)f(m) + \delta_{m,0}\mu\right)g(n) - (m-k)(n-m-k)f(m)g(n)\\
&- ((m-n)f(m) + \delta_{m,0}\mu - (n-m)g(n))(n+m-k)g(m+n).
\end{aligned}$$

Viewing (20) as a polynomial equation in k, we see that (20) holds if and only if the coefficients of degrees $0, 1$ and 2, respectively, are the same on both sides of the polynomial equation (20), i.e.,

$$(m-n)(m+n)(g(m+n) + f(m)g(m+n) + g(n)g(m+n) - f(m)g(n))$$

$$= n\delta_{m,0}(ng(n) - (m+n)g(m+n)),$$

$$(n-m)(g(m+n) + f(m)g(m+n) + g(n)g(m+n) - f(m)g(n))$$
$$= \delta_{m,0}\mu(g(m+n) - g(n))$$

and $0 = f(m)g(n) - f(m)g(n)$ hold. Note that $n\delta_{m,0}(ng(n) - (m+n)g(m+n)) = 0$ and $\delta_{m,0}\mu(g(m+n) - g(n)) = 0$. This implies that (20) holds if and only if (13) holds. In a similar way as above, we obtain that (1) with $(x, y, z) = (L_m, L_n, H_k)$ holds if and only if (13) and (14) hold. It has been proved that (9)-(14) hold.

Conversely, suppose that there are $\mu \in \mathbb{C}$ and complex-valued functions f, g on $\mathbb{Z}$ satisfying (9)-(14). It is easy to verify that (2) holds by (9)-(11). We have to prove that (1) holds for all $x, y, z \in W(2, 2)$. We observe that this is obviously right when at least two elements in x, y, z belong to the set $\{H_k, k \in \mathbb{Z}\}$. Next, the discussion in the above paragraph tells us that (1) with $(x, y, z) = (L_m, L_n, L_k)$ holds by (12); (1) with $(x, y, z) = (L_m, H_n, L_k)$ or (H_n, L_m, L_k) holds by (13); and (1) with $(x, y, z) = (L_m, L_n, H_k)$ holds by (13) and (14). The proof is completed. □

For complex-valued functions f, g on $\mathbb{Z}$, we denote I, J, M and N by

$$I = \{m \in \mathbb{Z} \,|\, f(m) = 0\}, \quad J = \{m \in \mathbb{Z} \,|\, f(m) = -1\},$$
$$M = \{n \in \mathbb{Z} \,|\, g(n) = 0\}, \quad N = \{n \in \mathbb{Z} \,|\, g(n) = -1\}.$$

Lemma 2.3. *Suppose that f, g are complex-valued functions on $\mathbb{Z}$. Then* (12) *and* (13) *hold if and only if the following statements hold*:

(i) $I \cup J = M \cup N = \mathbb{Z} \setminus \{0\}$;
(ii) $m, n \in I \Rightarrow m + n \in I$ *and* $m, n \in J \Rightarrow m + n \in J$ *for* $m \neq n$;
(iii) $m \in I, n \in M \Rightarrow m + n \in M$, *and* $m \in J, n \in N \Rightarrow m + n \in N$ *for all* $m \neq n$.

Proof. We first prove the "only if" part. Letting $n = 0$ in (12), we have $m(f(m) + f(m)^2) = 0$. Thus, for $m \neq 0$, $f(m) = 0$ or $f(m) = -1$. Similarly, by letting $m = 0$ in (13), it follows that $g(n) = 0$ or $g(n) = -1$ for $n \neq 0$. This proves (i). Now we chose a pair of $m, n \in \mathbb{Z}$ with $m \neq n$, then by (12) and (13) we see that

$$f(m+n) + f(m)f(m+n) + f(n)f(m+n) - f(m)f(n) = 0, \tag{21}$$

$$g(m+n) + f(m)g(m+n) + g(n)g(m+n) - f(m)g(n) = 0. \tag{22}$$

According to (21) and (22), it is easy to verify that (ii) and (iii) hold.

Next, we prove the "if" part. In fact, if $m = n$, then (12) and (13) are obvious. Now we suppose that $m \neq n$. In this case, if $m = 0$ then $n \neq 0$, then we also can obtain (12) and (13) since $f(n), g(n) \in \{0, -1\}$. Finally, we assume that $m \neq n$ with $m, n \neq 0$. By (i), we know $f(m), f(n), g(m), g(n) \in \{0, -1\}$. It is easy to verify that (12) and (13) hold one by one according to values of f, g. □

Lemma 2.4. *Suppose that f, g are complex-valued functions on $\mathbb{Z}$. Then* (12) *and* (13) *hold if and only if f and g meet one of the situations listed in Table 2.*

Proof. The proof of the "if" direction can be directly verified. We now prove the " only if" direction. In view of f satisfies (12), by Theorem 2.4 of [21] we know that f is determined by Table 1. Next, we discuss the cases of $g(1), g(-1), g(2)$

TABLE 1. Values of f satisfying (12), where $a \in \mathbb{C}$.

Cases	$f(n)$
$\mathcal{P}_1$	$f(\mathbb{Z}) = 0$
$\mathcal{P}_2$	$f(\mathbb{Z}) = -1$
$\mathcal{P}_3^a$	$f(\mathbb{Z}_{>0}) = -1$, $f(\mathbb{Z}_{<0}) = 0$ and $f(0) = a$
$\mathcal{P}_4^a$	$f(\mathbb{Z}_{>0}) = 0$, $f(\mathbb{Z}_{<0}) = -1$ and $f(0) = a$
$\mathcal{P}_5$	$f(\mathbb{Z}_{\geqslant 2}) = -1$ and $f(\mathbb{Z}_{\leqslant 1}) = 0$
$\mathcal{P}_6$	$f(\mathbb{Z}_{\geqslant 2}) = 0$ and $f(\mathbb{Z}_{\leqslant 1}) = -1$
$\mathcal{P}_7$	$f(\mathbb{Z}_{\geqslant -1}) = 0$ and $f(\mathbb{Z}_{\leqslant -2}) = -1$
$\mathcal{P}_8$	$f(\mathbb{Z}_{\geqslant -1}) = -1$ and $f(\mathbb{Z}_{\leqslant -2}) = 0$

and $g(-2)$. Lemma 2.3(i) tells us that $g(1)$, $g(-1)$, $g(2)$, $g(-2) \in \{-1, 0\}$, and so that there are $2^4 = 16$ cases for $g(x)$ where $x = \pm 1, \pm 2$. Using Lemma 2.3(ii) and (iii), it follows by a simple discussion that 30 cases listed in Tabular 2 are established. □

Lemma 2.5. *Let $(\mathcal{P}(\phi_i, \varphi_i, \theta_i), \circ_i)$, $i = 1, 2$ be two algebras with the same linear space as $W(2,2)$ and equipped with $\mathbb{C}$-bilinear products $x \circ_i y$ such that*

$$L_m \circ_i L_n = \phi_i(m,n) L_{m+n}, \qquad L_m \circ_i H_n = \varphi_i(m,n) H_{m+n},$$
$$H_m \circ_i L_n = \theta_i(m,n) H_{m+n}, \qquad H_m \circ_i H_n = 0$$

for all $m, n \in \mathbb{Z}$, where ϕ_i, φ_i, θ_i, $i = 1, 2$ are complex-valued functions on $\mathbb{Z}\times\mathbb{Z}$. Furthermore, let $\tau : \mathcal{P}(\phi_1, \varphi_1, \theta_1) \to \mathcal{P}(\phi_2, \varphi_2, \theta_2)$ be a linear map determined by $\tau(L_m) = -L_{-m}$, $\tau(H_m) = -H_{-m}$ for all $m \in \mathbb{Z}$. In addition, suppose that $(\mathcal{P}(\phi_1, \varphi_1, \theta_1), [,], \circ_1)$ is a post-Lie algebra. Then $(\mathcal{P}(\phi_2, \varphi_2, \theta_2), [,], \circ_2)$ is a post-Lie algebra and τ is a isomorphism from $\mathcal{P}(\phi_1, \varphi_1, \theta_1)$ to $\mathcal{P}(\phi_2, \varphi_2, \theta_2)$ if and only if

$$\begin{cases} \phi_2(m,n) = -\phi_1(-m,-n), \\ \varphi_2(m,n) = -\varphi_1(-m,-n), \\ \theta_2(m,n) = -\theta_1(-m,-n). \end{cases} \tag{23}$$

Proof. Clearly, τ is a Lie automorphism of the W-algebra $W(2,2)$. Suppose that $(\mathcal{P}(\phi_2, \varphi_2, \theta_2), [,], \circ_2)$ is a post-Lie algebra and τ is a post-Lie isomorphism from $\mathcal{P}(\phi_1, \varphi_1, \theta_1)$ to $\mathcal{P}(\phi_2, \varphi_2, \theta_2)$. Then from

$$\tau(L_m \circ_1 L_n) = -\phi_1(m,n) L_{-(m+n)},$$

$$\tau(L_m \circ_1 H_n) = -\varphi_1(m,n)H_{-(m+n)},$$
$$\tau(H_m \circ_1 L_n) = -\theta_1(m,n)H_{-(m+n)}$$

and

$$\tau(L_m) \circ_2 \tau(L_n) = \phi_2(-m,-n)L_{-(m+n)},$$

TABLE 2. Values of f and g satisfying (12) and (13), where $a, b \in \mathbb{C}$.

Cases	$f(n)$ from Table 1	$g(n)$
$\mathcal{W}_1^{\mathcal{P}_1}$	$\mathcal{P}_1$	$g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_1}$	$\mathcal{P}_1$	$g(\mathbb{Z}) = -1$
$\mathcal{W}_1^{\mathcal{P}_2}$	$\mathcal{P}_2$	$g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_2}$	$\mathcal{P}_2$	$g(\mathbb{Z}) = -1$
$\mathcal{W}_1^{\mathcal{P}_3^a}$	$\mathcal{P}_3^a$	$g(\mathbb{Z}) = 0,$
$\mathcal{W}_2^{\mathcal{P}_3^a}$	$\mathcal{P}_3^a$	$g(\mathbb{Z}) = -1$
$\mathcal{W}_3^{\mathcal{P}_3^{a,b}}$	$\mathcal{P}_3^a$	$g(\mathbb{Z}_{>0}) = -1, g(\mathbb{Z}_{<0}) = 0, g(0) = b$
$\mathcal{W}_4^{\mathcal{P}_3^a}$	$\mathcal{P}_3^a$	$g(\mathbb{Z}_{\geqslant 2}) = -1,\ g(\mathbb{Z}_{\leqslant 1}) = 0$
$\mathcal{W}_5^{\mathcal{P}_3^a}$	$\mathcal{P}_3^a$	$g(\mathbb{Z}_{\geqslant -1}) = -1,\ g(\mathbb{Z}_{\leqslant -2}) = 0$
$\mathcal{W}_1^{\mathcal{P}_4^a}$	$\mathcal{P}_4^a$	$g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_4^a}$	$\mathcal{P}_4^a$	$g(\mathbb{Z}) = -1$
$\mathcal{W}_3^{\mathcal{P}_4^{a,b}}$	$\mathcal{P}_4^a$	$g(\mathbb{Z}_{>0}) = 0, g(\mathbb{Z}_{<0}) = -1, g(0) = b$
$\mathcal{W}_4^{\mathcal{P}_4^a}$	$\mathcal{P}_4^a$	$g(\mathbb{Z}_{\geqslant -1}) = 0,\ g(\mathbb{Z}_{\leqslant -2}) = -1$
$\mathcal{W}_5^{\mathcal{P}_4^a}$	$\mathcal{P}_4^a$	$g(\mathbb{Z}_{\geqslant 2}) = 0,\ g(\mathbb{Z}_{\leqslant 1}) = -1$
$\mathcal{W}_1^{\mathcal{P}_5}$	$\mathcal{P}_5$	$g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_5}$	$\mathcal{P}_5$	$g(\mathbb{Z}) = -1$
$\mathcal{W}_3^{\mathcal{P}_5}$	$\mathcal{P}_5$	$g(\mathbb{Z}_{\geqslant 2}) = -1, g(\mathbb{Z}_{\leqslant 1}) = 0$
$\mathcal{W}_4^{\mathcal{P}_5}$	$\mathcal{P}_5$	$g(\mathbb{Z}_{>0}) = -1, g(\mathbb{Z}_{\leqslant 0}) = 0$
$\mathcal{W}_1^{\mathcal{P}_6}$	$\mathcal{P}_6$	$g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_6}$	$\mathcal{P}_6$	$g(\mathbb{Z}) = -1$
$\mathcal{W}_3^{\mathcal{P}_6}$	$\mathcal{P}_6$	$g(\mathbb{Z}_{\geqslant 2}) = 0, g(\mathbb{Z}_{\leqslant 1}) = -1$
$\mathcal{W}_4^{\mathcal{P}_6}$	$\mathcal{P}_6$	$g(\mathbb{Z}_{>0}) = 0, g(\mathbb{Z}_{\leqslant 0}) = -1$
$\mathcal{W}_1^{\mathcal{P}_7}$	$\mathcal{P}_7$	$g(\mathbb{Z}) = 0$
$\mathcal{W}_2^{\mathcal{P}_7}$	$\mathcal{P}_7$	$g(\mathbb{Z}) = -1$
$\mathcal{W}_3^{\mathcal{P}_7}$	$\mathcal{P}_7$	$g(\mathbb{Z}_{\geqslant -1}) = 0, g(\mathbb{Z}_{\leqslant -2}) = -1$
$\mathcal{W}_4^{\mathcal{P}_7}$	$\mathcal{P}_7$	$g(\mathbb{Z}_{\geqslant 0}) = 0,\ g(\mathbb{Z}_{<0}) = -1$
$\mathcal{W}_1^{\mathcal{P}_8}$	$\mathcal{P}_8$	$g(\mathbb{Z}) = 0,$
$\mathcal{W}_2^{\mathcal{P}_8}$	$\mathcal{P}_8$	$g(\mathbb{Z}) = -1,$
$\mathcal{W}_3^{\mathcal{P}_8}$	$\mathcal{P}_8$	$g(\mathbb{Z}_{\geqslant -1}) = -1,\ g(\mathbb{Z}_{\leqslant -2}) = 0,$
$\mathcal{W}_4^{\mathcal{P}_8}$	$\mathcal{P}_8$	$g(\mathbb{Z}_{\geqslant 0}) = -1,\ g(\mathbb{Z}_{<0}) = 0.$

$$\tau(L_m)\circ_2\tau(H_n)=\varphi_2(-m,-n)H_{-(m+n)},$$
$$\tau(H_m)\circ_2\tau(L_n)=\theta_2(-m,-n)H_{-(m+n)}$$

we see that (23) holds. Conversely, suppose that (23) holds. Then, by using Lemma 2.2 and $(\mathcal{P}(\phi_1,\varphi_1,\theta_1),[,],\circ_1)$ is a post-Lie algebra, we know that there are complex-valued functions f_1, g_1 on $\mathbb{Z}$ and a complex number μ_1 such that

$$\phi_1(m,n)=(m-n)f_1(m), \tag{24}$$
$$\varphi_1(m,n)=(m-n)f_1(m)+\delta_{m,0}\mu_1, \tag{25}$$
$$\theta_1(m,n)=(m-n)g_1(m), \tag{26}$$

(27)
$$(m-n)(f_1(m+n)+f_1(m)f_1(m+n)+f_1(n)f_1(m+n)-f_1(m)f_1(n))=0,$$
(28)
$$(n-m)(g_1(m+n)+f_1(m)g_1(m+n)+g_1(n)g_1(m+n)-f_1(m)g_1(n))=0,$$
(29)
$$(m-n)(f_1(m)+f_1(n)+1)\delta_{m+n,0}\mu_1=0$$

for all $m,n\in\mathbb{Z}$. It follows by (24), (25), (26) and (23) that

$$\phi_2(m,n)=-\phi_1(-m,-n)=-(n-m)f_1(-m)=(m-n)f_2(m), \tag{30}$$
$$\begin{aligned}\varphi_2(m,n)&=-\varphi_1(-m,-n)=-(n-m)f_1(-m)-\delta_{m,0}\mu_1\\&=(m-n)f_2(m)+\delta_{m,0}\mu_2,\end{aligned} \tag{31}$$
$$\theta_2(m,n)=-\theta_1(-m,-n)=-(n-m)g_1(-m)=(m-n)g_2(m), \tag{32}$$

where f_2, g_2 are complex-valued functions on $\mathbb{Z}$ and μ_2 is a complex number determined by $f_2(m)=f_1(-m)$, $g_2(m)=g_1(-m)$ and $\mu_2=-\mu_1$.

Furthermore, by (27), (28) and (29) with $f_2(m)=f_1(-m)$, $\mu_2=-\mu_1$ we obtain

(33)
$$(m-n)(f_2(m+n)+f_2(m)f_2(m+n)+f_2(n)f_2(m+n)-f_2(m)f_2(n))=0,$$
(34)
$$(n-m)(g_2(m+n)+f_2(m)g_2(m+n)+f_2(n)g_2(m+n)-f_2(m)g_2(n))=0,$$
(35)
$$(m-n)(f_2(m)+f_2(n)+1)\delta_{m+n,0}\mu_2=0.$$

In view of (30)-(35), it follows by Lemma 2.2 that $\mathcal{P}(\phi_2,\varphi_2,\theta_2)$ is a post-Lie algebra. The remainder is to prove that τ is a isomorphism between post-Lie algebras. But one has

$$\tau(L_m\circ_1 L_n)=-\phi_1(m,n)L_{-(m+n)}=\phi_2(-m,-n)L_{-(m+n)}=\tau(L_m)\circ_2\tau(L_n),$$
$$\tau(L_m\circ_1 H_n)=-\varphi_1(m,n)H_{-(m+n)}=\varphi_2(-m,-n)H_{-(m+n)}=\tau(L_m)\circ_2\tau(H_n),$$
$$\tau(H_m\circ_1 L_n)=-\theta_1(m,n)H_{-(m+n)}=\theta_2(-m,-n)H_{-(m+n)}=\tau(H_m)\circ_2\tau(L_n),$$

and $\tau(H_m\circ_1 H_n)=0=\tau(H_m)\circ_2\tau(H_n)$, which completes the proof. □

We now can prove the main theorem of this paper as follows.

Theorem 2.6. *A graded post-Lie algebra structure on $W(2,2)$ satisfying (5)-(8) must be one of the following types (in every case $H_m \circ H_n = 0$) for all $m, n \in \mathbb{Z}$,*

$(\mathcal{W}_1^{\mathcal{P}_1})$: $L_m \circ_1^{\mathcal{P}_1} L_n = 0$, $L_m \circ_1^{\mathcal{P}_1} H_n = 0$, $H_m \circ_1^{\mathcal{P}_1} L_n = 0$;
$(\mathcal{W}_2^{\mathcal{P}_1})$: $L_m \circ_2^{\mathcal{P}_1} L_n = 0$, $L_m \circ_2^{\mathcal{P}_1} H_n = 0$, $H_m \circ_2^{\mathcal{P}_1} L_n = (n-m)H_{m+n}$;
$(\mathcal{W}_1^{\mathcal{P}_2})$: $L_m \circ_1^{\mathcal{P}_2} L_n = (n-m)L_{m+n}$, $L_m \circ_1^{\mathcal{P}_2} H_n = (n-m)H_{m+n}$, $H_m \circ_1^{\mathcal{P}_2} L_n = 0$;
$(\mathcal{W}_2^{\mathcal{P}_2})$: $L_m \circ_2^{\mathcal{P}_2} L_n = (n-m)L_{m+n}$, $L_m \circ_2^{\mathcal{P}_2} H_n = (n-m)H_{m+n}$, $H_m \circ_2^{\mathcal{P}_2} L_n = (n-m)H_{m+n}$;
$(\mathcal{W}_{i,\mu}^{\mathcal{P}_3})$: $i = 1, 2, \ldots, 5$,

$$L_m \circ_{i,\mu}^{\mathcal{P}_3^a} L_n = \begin{cases} (n-m)L_{m+n}, & m > 0, \\ -naL_n, & m = 0, \\ 0, & m < 0; \end{cases}$$

$$L_m \circ_{i,\mu}^{\mathcal{P}_3^a} H_n = \begin{cases} (n-m)H_{m+n}, & m > 0, \\ (-na+\mu)H_n, & m = 0, \\ 0, & m < 0; \end{cases}$$

$$\begin{aligned} H_m \circ_{i,\mu}^{\mathcal{P}_3^{a,b}} L_n = {} & \delta_{i,2}(n-m)H_{m+n} \\ & + \delta_{i,3} \begin{cases} (n-m)H_{m+n}, & m > 0, \\ -nbH_n, & m = 0, \\ 0, & m < 0; \end{cases} \\ & + \delta_{i,4} \begin{cases} (n-m)H_{m+n}, & m \geqslant 2, \\ 0, & m \leqslant 1; \end{cases} \\ & + \delta_{i,5} \begin{cases} (n-m)H_{m+n}, & m \geqslant -1, \\ 0, & m \leqslant -2; \end{cases} \end{aligned}$$

$(\mathcal{W}_{i,\mu}^{\mathcal{P}_4^a})$: $i = 1, 2, \ldots, 5$,

$$L_m \circ_{i,\mu}^{\mathcal{P}_4^a} L_n = \begin{cases} (n-m)L_{m+n}, & m < 0, \\ -naL_n, & m = 0, \\ 0, & m > 0; \end{cases}$$

$$L_m \circ_{i,\mu}^{\mathcal{P}_4^a} H_n = \begin{cases} (n-m)H_{m+n}, & m < 0, \\ (-na+\mu)H_n, & m = 0, \\ 0, & m > 0; \end{cases}$$

$$\begin{aligned} H_m \circ_{i,\mu}^{\mathcal{P}_4^{a,b}} L_n = {} & \delta_{i,2}(n-m)H_{n+m} \\ & + \delta_{i,3} \begin{cases} (n-m)H_{m+n}, & m < 0, \\ -nbH_n, & m = 0 \\ 0, & m > 0; \end{cases} \end{aligned}$$

$$+\,\delta_{i,4}\begin{cases}(n-m)H_{m+n}, & m\leqslant -2,\\ 0, & m\geqslant -1;\end{cases}$$
$$+\,\delta_{i,5}\begin{cases}(n-m)H_{m+n}, & m\leqslant 1,\\ 0, & m\geqslant 2;\end{cases}$$

$(\mathcal{W}_j^{\mathcal{P}_5})$: $j=1,\ldots,4$,

$$L_m\circ_j^{\mathcal{P}_5} L_n=\begin{cases}(n-m)L_{m+n}, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}$$
$$L_m\circ_j^{\mathcal{P}_5} H_n=\begin{cases}(n-m)H_{m+n}, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}$$
$$\begin{aligned}H_m\circ_j^{\mathcal{P}_5} L_n=&\,\delta_{j,2}(n-m)H_{m+n}\\ &+\delta_{j,3}\begin{cases}(n-m)H_{m+n}, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}\\ &+\delta_{j,4}\begin{cases}(n-m)H_{m+n}, & m>0,\\ 0, & m\leqslant 0;\end{cases}\end{aligned}$$

$(\mathcal{W}_j^{\mathcal{P}_6})$: $j=1,\ldots,4$,

$$L_m\circ_j^{\mathcal{P}_6} L_n=\begin{cases}(n-m)L_{m+n}, & m\leqslant 1,\\ 0, & m\geqslant 2;\end{cases}$$
$$L_m\circ_j^{\mathcal{P}_6} H_n=\begin{cases}(n-m)H_{m+n}, & m\leqslant 1,\\ 0, & m\geqslant 2;\end{cases}$$
$$\begin{aligned}H_m\circ_j^{\mathcal{P}_6} L_n=&\,\delta_{j,2}(n-m)H_{m+n}\\ &+\delta_{j,3}\begin{cases}(n-m)H_{m+n}, & m\leqslant 1,\\ 0, & m\geqslant 2;\end{cases}\\ &+\delta_{j,4}\begin{cases}(n-m)H_{m+n}, & m\leqslant 0,\\ 0, & m>0;\end{cases}\end{aligned}$$

$(\mathcal{W}_j^{\mathcal{P}_7})$: $j=1,\ldots,4$,

$$L_m\circ_j^{\mathcal{P}_7} L_n=\begin{cases}(n-m)L_{m+n}, & m\leqslant -2,\\ 0, & m\geqslant -1;\end{cases}$$
$$L_m\circ_j^{\mathcal{P}_7} H_n=\begin{cases}(n-m)H_{m+n}, & m\leqslant -2,\\ 0, & m\geqslant -1;\end{cases}$$
$$\begin{aligned}H_m\circ_j^{\mathcal{P}_7} L_n=&\,\delta_{j,2}(n-m)H_{m+n}\\ &+\delta_{j,3}\begin{cases}(n-m)H_{m+n}, & m\leqslant -2,\\ 0, & m\geqslant -1;\end{cases}\end{aligned}$$

$$+\,\delta_{j,4}\begin{cases}(n-m)H_{m+n}, & m<0,\\ 0, & m\geqslant 0;\end{cases}$$

$(\mathcal{W}_j^{\mathcal{P}_8}):\ j=1,\ldots,4,$

$$L_m \circ_j^{\mathcal{P}_8} L_n = \begin{cases}(n-m)L_{m+n}, & m\geqslant -1,\\ 0, & m\leqslant -2;\end{cases}$$

$$L_m \circ_j^{\mathcal{P}_8} H_n = \begin{cases}(n-m)H_{m+n}, & m\geqslant -1,\\ 0, & m\leqslant -2;\end{cases}$$

$$H_m \circ_j^{\mathcal{P}_8} L_n = \delta_{j,2}(n-m)H_{m+n} + \delta_{j,3}\begin{cases}(n-m)H_{m+n}, & m\geqslant -1,\\ 0, & m\leqslant -2;\end{cases} + \delta_{j,4}\begin{cases}(n-m)H_{m+n}, & m\geqslant 0,\\ 0, & m< 0;\end{cases}$$

where $a, b, \mu \in \mathbb{C}$. Conversely, the above types are all the graded post-Lie algebra structure satisfying (5)-(8) on $W(2,2)$. Furthermore, the post-Lie algebras $\mathcal{W}_i^{\mathcal{P}_3^a}$, $\mathcal{W}_j^{\mathcal{P}_5}$, $\mathcal{W}_j^{\mathcal{P}_6}$ and $\mathcal{W}_{i,\mu}^{\mathcal{P}_4^a}$ are isomorphic to the post-Lie algebras $\mathcal{W}_i^{\mathcal{P}_4^a}$, $\mathcal{W}_j^{\mathcal{P}_7}$, $\mathcal{W}_j^{\mathcal{P}_8}$ and $\mathcal{W}_{i,\mu}^{\mathcal{P}_3^a}$, $i \in \{1,2,3,4,5\}$ and $j \in \{1,2,3,4\}$, respectively, and other post-Lie algebras are not mutually isomorphic.

Proof. Suppose that $(W, [,], \circ)$ is a post-Lie algebra structure satisfying (5)-(8) on $W(2,2)$. By Lemma 2.2, there are complex-valued functions f, g on $\mathbb{Z}$ and $\mu \in \mathbb{C}$ such that (9)-(14) hold. Below two cases of μ are discussed.

Case (I) $\mu = 0$. In this case, f and g satisfy (12) and (13) but (14) is disappeared due to $\mu = 0$. By Lemma 2.4, the 30 cases of f, g listed in Table 2 are established. Thus, by (9)-(11) with $\mu = 0$, we know that the graded post-Lie algebra structure on $W(2,2)$ algebra must be one of the above 30 types. They are exactly the 30 forms described in the theorem but the cases of $\mathcal{W}_{i,\mu}^{\mathcal{P}_k}$, $k = 3, 4, i = 1, 2, \ldots, 5$, should with condition $\mu = 0$.

Case (II) $\mu \neq 0$. Because f and g satisfy (12) and (13), it follows by Lemma 2.4 that the 30 cases of f, g listed in Table 2 can happen. In view of (14), we obtain

$$f(m) + f(-m) = -1 \text{ for all } m \neq 0.$$

This, together with a simple checking, yields the only 10 cases as $\mathcal{W}_{i,\mu}^{\mathcal{P}_k}$, $k = 3, 4, i = 1, 2, \ldots, 5$, with $\mu \neq 0$ are right. Thus, by (9)-(11) with $\mu \neq 0$, we get the corresponding post-Lie algebra structures.

Clearly, they are all graded post-Lie algebra structures on the $W(2,2)$ algebra. Finally, by Lemma 2.5 we know that the post-Lie algebras $\mathcal{W}_{i,\mu}^{\mathcal{P}_3^a}$, $\mathcal{W}_j^{\mathcal{P}_5}$ and $\mathcal{W}_j^{\mathcal{P}_6}$ are isomorphic to the post-Lie algebras $\mathcal{W}_{i,\mu}^{\mathcal{P}_4^a}$, $\mathcal{W}_j^{\mathcal{P}_7}$ and $\mathcal{W}_j^{\mathcal{P}_8}$ respectively, and the other post-Lie algebras are not mutually isomorphic. □

Remark 2.7. Theorem 2.6 tells us that, up to isomorphism, there are 17 types of graded post-Lie algebra structures satisfying (5)-(8) on the $W(2,2)$ algebra, that is $\mathcal{W}_k^{\mathcal{P}_1}$, $\mathcal{W}_k^{\mathcal{P}_2}$, $\mathcal{W}_{i,\mu}^{\mathcal{P}_3^a}$, $\mathcal{W}_j^{\mathcal{P}_5}$ and $\mathcal{W}_j^{\mathcal{P}_6}$ where $k \in \{1,2\}$, $i \in \{1,2,3,4,5\}$ and $j \in \{1,2,3,4\}$.

From Theorem 2.6 and Proposition 1.2 we can give some Lie algebras as follows.

Proposition 2.8. *Up to isomorphism, the post-Lie algebras in Theorem 2.6 give rise to the following* 11 *Lie algebras on the space with* $\mathbb{C}$*-basis* $\{L_i, H_i \mid i \in \mathbb{Z}\}$, *and with the bracket* $\{,\}$ *defined by Proposition 1.2* (*in every case* $\{H_m, H_n\} = 0$):

$$(\mathcal{LW}_1^{\mathcal{P}_1}): \ \{L_m, L_n\}_1^{\mathcal{P}_1} = (m-n)L_{m+n} \text{ for all } m,n \in \mathbb{Z};$$
$$\{L_m, H_n\}_1^{\mathcal{P}_1} = (m-n)H_{m+n} \text{ for all } m,n \in \mathbb{Z};$$
$$(\mathcal{LW}_2^{\mathcal{P}_1}): \ \{L_m, L_n\}_2^{\mathcal{P}_1} = (m-n)L_{m+n} \text{ for all } m,n \in \mathbb{Z};$$
$$\{L_m, H_n\}_2^{\mathcal{P}_1} = 0 \text{ for all } m,n \in \mathbb{Z};$$

$$(\mathcal{LW}_{1,\mu}^{\mathcal{P}_3^a}): \ \{L_m, L_n\}_{1,\mu}^{\mathcal{P}_3^a} = \begin{cases} (n-m)L_{m+n}, & m,n>0, \\ (m-n)L_{m+n}, & m,n<0, \\ -naL_n, & m=0, n>0, \\ -n(a+1)L_n & m=0, n<0, \\ 0, & \textit{otherwise}; \end{cases}$$

$$\{L_m, H_n\}_{1,\mu}^{\mathcal{P}_3^a} = \begin{cases} (m-n)H_{m+n}, & m<0, \\ (-n(a+1)+\mu)H_n, & m=0, \\ 0, & m>0; \end{cases}$$

$$(\mathcal{LW}_{2,\mu}^{\mathcal{P}_3^a}): \ \{L_m, L_n\}_{2,\mu}^{\mathcal{P}_3^a} = \{L_m, L_n\}_{1,\mu}^{\mathcal{P}_3^a},$$

$$\{L_m, H_n\}_{2,\mu}^{\mathcal{P}_3^a} = \begin{cases} (n-m)H_{m+n}, & m>0, \\ (-na+\mu)H_n, & m=0, \\ 0, & m<0; \end{cases}$$

$$(\mathcal{LW}_{3,\mu}^{\mathcal{P}_3^{a,b}}): \ \{L_m, L_n\}_{3,\mu}^{\mathcal{P}_3^{a,b}} = \{L_m, L_n\}_{1,\mu}^{\mathcal{P}_3^a},$$

$$\{L_m, H_n\}_{3,\mu}^{\mathcal{P}_3^{a,b}} = \begin{cases} (n-m)H_{m+n}, & m,n>0, \\ (m-n)H_{m+n}, & m,n<0, \\ (-na+\mu)H_n, & m=0, n>0, \\ (-n(a+1)+\mu)H_n & m=0, n<0, \\ mbH_m & m>0, n=0, \\ m(b+1)H_m & m<0, n=0, \\ 0, & \textit{otherwise}; \end{cases}$$

$$(\mathcal{LW}_{4,\mu}^{\mathcal{P}_3^a}):\ \{L_m,L_n\}_{4,\mu}^{\mathcal{P}_3^a}=\{L_m,L_n\}_{1,\mu}^{\mathcal{P}_3^a},$$

$$\{L_m,H_n\}_{4,\mu}^{\mathcal{P}_3^a}=\begin{cases}(n-m)H_{m+n}, & m>0,n\geqslant 2,\\ (m-n)H_{m+n}, & m<0,n\leqslant 1,\\ (-na+\mu)H_n, & m=0,n\geqslant 2,\\ (-n(a+1)+\mu)H_n & m=0,n\leqslant 1,\\ 0, & otherwise;\end{cases}$$

$$(\mathcal{LW}_{5,\mu}^{\mathcal{P}_3^a}):\ \{L_m,L_n\}_{5,\mu}^{\mathcal{P}_3^a}=\{L_m,L_n\}_{1,\mu}^{\mathcal{P}_3^a},$$

$$\{L_m,H_n\}_{5,\mu}^{\mathcal{P}_3^a}=\begin{cases}(n-m)H_{m+n}, & m>0,n\geqslant -1,\\ (m-n)H_{m+n}, & m<0,n\leqslant -2,\\ (-na+\mu)H_n, & m=0,n\geqslant -1,\\ (-n(a+1)+\mu)H_n & m=0,n\leqslant -2,\\ 0, & otherwise;\end{cases}$$

$$(\mathcal{LW}_{1}^{\mathcal{P}_5}):\ \{L_m,L_n\}_{1}^{\mathcal{P}_5}=\begin{cases}(n-m)L_{m+n}, & m,n\geqslant 2,\\ (m-n)L_{m+n}, & m,n\leqslant 1,\\ 0, & otherwise;\end{cases}$$

$$\{L_m,H_n\}_{1}^{\mathcal{P}_5}=\begin{cases}0, & m\geqslant 2,\\ (m-n)H_{m+n}, & m\leqslant 1;\end{cases}$$

$$(\mathcal{LW}_{2}^{\mathcal{P}_5}):\ \{L_m,L_n\}_{2}^{\mathcal{P}_5}=\{L_m,L_n\}_{1}^{\mathcal{P}_5},$$

$$\{L_m,H_n\}_{2}^{\mathcal{P}_5}=\begin{cases}(n-m)H_{m+n}, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}$$

$$(\mathcal{LW}_{3}^{\mathcal{P}_5}):\ \{L_m,L_n\}_{3}^{\mathcal{P}_5}=\{L_m,L_n\}_{1}^{\mathcal{P}_5},$$

$$\{L_m,H_n\}_{3}^{\mathcal{P}_5}=\begin{cases}(n-m)H_{m+n}, & m,n\geqslant 2,\\ (m-n)H_{m+n}, & m,n\leqslant 1,\\ 0, & otherwise;\end{cases}$$

$$(\mathcal{LW}_{4}^{\mathcal{P}_5}):\ \{L_m,L_n\}_{4}^{\mathcal{P}_5}=\{L_m,L_n\}_{1}^{\mathcal{P}_5},$$

$$\{L_m,H_n\}_{4}^{\mathcal{P}_5}=\begin{cases}(n-m)H_{m+n}, & m\geqslant 2,n>0,\\ (m-n)H_{m+n}, & m\leqslant 1,n\leqslant 0,\\ 0, & otherwise\end{cases}$$

where $a,b,\mu\in\mathbb{C}$.

Proof. Theorem 2.6 tells us that, up to isomorphism, there are 17 types of graded post-Lie algebra structure on $W(2,2)$ satisfying (5)-(8), which induced 17 types of Lie algebras by Proposition 1.2, and here are denoted by $\mathcal{LW}_k^{\mathcal{P}_1}$, $\mathcal{LW}_k^{\mathcal{P}_2}$, $\mathcal{LW}_{i,\mu}^{\mathcal{P}_3^a}$, $\mathcal{LW}_j^{\mathcal{P}_5}$ and $\mathcal{LW}_j^{\mathcal{P}_6}$ where $k\in\{1,2\}$, $i\in\{1,2,3,4,5\}$ and $j\in\{1,2,3,4\}$. On the other hand, the Lie algebras $\mathcal{LW}_k^{\mathcal{P}_1}$, $\mathcal{LW}_j^{\mathcal{P}_5}$ are isomorphic to the Lie algebras $\mathcal{LW}_k^{\mathcal{P}_2}$, $\mathcal{LW}_j^{\mathcal{P}_6}$ respectively through the linear transformation $L_m \to -L_{-m}, H_m \to -H_{-m}$. The conclusions are easily deducible. □

3. Application to Rota-Baxter operators

Lemma 3.1 (see [1]). *Let L be a complex Lie algebra and $R : L \to L$ a Rota-Baxter operator of weight 1. Define a new operation $x \circ y = [R(x), y]$ on L. Then $(L, [,], \circ)$ is a post-Lie algebra.*

In this section, by using Lemma 3.1 and Theorem 2.6, we mainly consider the homogeneous Rota-Baxter operator R of weight 1 on the W-algebra $W(2,2)$ given by

$$R(L_m) = f(m)L_m, \quad R(H_m) = g(m)H_m \tag{36}$$

for all $m \in \mathbb{Z}$, where f, g are complex-valued functions on $\mathbb{Z}$. We will prove the following.

Theorem 3.2. *A homogeneous Rota-Baxter operator R of weight 1 satisfying (36) on the W-algebra $W(2,2)$ must be one of the following types (where $a, b \in \mathbb{C}$) for all $m, n \in \mathbb{Z}$,*

$(\mathcal{R}_1^{\mathcal{P}_1})$: $R(L_m) = 0, R(H_m) = 0$;
$(\mathcal{R}_2^{\mathcal{P}_1})$: $R(L_m) = 0, R(H_m) = -H_m$;
$(\mathcal{R}_1^{\mathcal{P}_2})$: $R(L_m) = -L_m, R(H_m) = 0$;
$(\mathcal{R}_2^{\mathcal{P}_2})$: $R(L_m) = -L_m, R(H_m) = -H_m$;

$$(\mathcal{R}_1^{\mathcal{P}_3^a}): \ R(L_m) = \begin{cases} -L_m, & m > 0, \\ aL_0, & m = 0, \\ 0, & m < 0; \end{cases} \qquad R(H_n) = 0;$$

$$(\mathcal{R}_2^{\mathcal{P}_3^a}): \ R(L_m) = \begin{cases} -L_m, & m > 0, \\ aL_0, & m = 0, \\ 0, & m < 0; \end{cases} \qquad R(H_n) = -H_n;$$

$$(\mathcal{R}_3^{\mathcal{P}_3^{a,b}}): \ R(L_m) = \begin{cases} -L_m, & m > 0, \\ aL_0, & m = 0, \\ 0, & m < 0; \end{cases} \qquad R(H_n) = \begin{cases} -H_n, & n > 0, \\ bH_0, & n = 0, \\ 0, & n < 0; \end{cases}$$

$$(\mathcal{R}_4^{\mathcal{P}_3^a}): \ R(L_m) = \begin{cases} -L_m, & m > 0, \\ aL_0, & m = 0, \\ 0, & m < 0; \end{cases} \qquad R(H_n) = \begin{cases} -H_n, & n \geqslant 2, \\ 0, & n \leqslant 1; \end{cases}$$

$$(\mathcal{R}_5^{\mathcal{P}_3^a}): \ R(L_m) = \begin{cases} -L_m, & m > 0, \\ aL_0, & m = 0, \\ 0, & m < 0; \end{cases} \qquad R(H_n) = \begin{cases} -H_n, & n \geqslant -1, \\ 0, & n \leqslant -2; \end{cases}$$

$$(\mathcal{R}_1^{\mathcal{P}_4^a}): \ R(L_m) = \begin{cases} -L_m, & m < 0, \\ aL_0, & m = 0, \\ 0, & m > 0; \end{cases} \qquad R(H_n) = 0;$$

$$(\mathcal{R}_2^{\mathcal{P}_4^a}):\ R(L_m)=\begin{cases}-L_m, & m<0,\\ aL_0, & m=0,\\ 0, & m>0;\end{cases}\qquad R(H_n)=-H_n;$$

$$(\mathcal{R}_3^{\mathcal{P}_4^a}):\ R(L_m)=\begin{cases}-L_m, & m<0,\\ aL_0, & m=0,\\ 0, & m>0;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n<0,\\ bH_0, & n=0,\\ 0, & m>0;\end{cases}$$

$$(\mathcal{R}_4^{\mathcal{P}_4^a}):\ R(L_m)=\begin{cases}-L_m, & m<0,\\ aL_0, & m=0,\\ 0, & m>0;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\leqslant -2,\\ 0, & n\geqslant -1;\end{cases}$$

$$(\mathcal{R}_5^{\mathcal{P}_4^a}):\ R(L_m)=\begin{cases}-L_m, & m<0,\\ aL_0, & m=0,\\ 0, & m>0;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\leqslant 1,\\ 0, & n\geqslant 2;\end{cases}$$

$$(\mathcal{R}_1^{\mathcal{P}_5}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}\qquad R(H_n)=0;$$

$$(\mathcal{R}_2^{\mathcal{P}_5}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}\qquad R(H_n)=-H_n;$$

$$(\mathcal{R}_3^{\mathcal{P}_5}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\geqslant 2,\\ 0, & n\leqslant 1;\end{cases}$$

$$(\mathcal{R}_4^{\mathcal{P}_5}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant 2,\\ 0, & m\leqslant 1;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n>0,\\ 0, & n\leqslant 0;\end{cases}$$

$$(\mathcal{R}_1^{\mathcal{P}_6}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant 1,\\ 0, & m\geqslant 2;\end{cases}\qquad R(H_n)=0;$$

$$(\mathcal{R}_2^{\mathcal{P}_6}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant 1,\\ 0, & m\geqslant 2;\end{cases}\qquad R(H_n)=-H_n;$$

$$(\mathcal{R}_3^{\mathcal{P}_6}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant 1,\\ 0, & m\geqslant 2;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\leqslant 1,\\ 0, & n\geqslant 2;\end{cases}$$

$$(\mathcal{R}_4^{\mathcal{P}_6}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant 1,\\ 0, & m\geqslant 2;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\leqslant 0,\\ 0, & n>0;\end{cases}$$

$$(\mathcal{R}_1^{\mathcal{P}_7}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant -2,\\ 0, & m\geqslant -1;\end{cases}\qquad R(H_n)=0;$$

$$(\mathcal{R}_2^{\mathcal{P}_7}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant -2,\\ 0, & m\geqslant -1;\end{cases}\qquad R(H_n)=-H_n;$$

$$(\mathcal{R}_3^{\mathcal{P}_7}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant -2,\\ 0, & m\geqslant -1;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\geqslant -1,\\ 0, & n\leqslant -2;\end{cases}$$

$$(\mathcal{R}_4^{\mathcal{P}_7}):\ R(L_m)=\begin{cases}-L_m, & m\leqslant -2,\\ 0, & m\geqslant -1;\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n<0,\\ 0, & n\geqslant 0;\end{cases}$$

$$(\mathcal{R}_1^{\mathcal{P}_8}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant -1,\\ 0, & m\leqslant -2;\end{cases}\qquad R(H_n)=0;$$

$$(\mathcal{R}_2^{\mathcal{P}_8}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant -1,\\ 0, & m\leqslant -2,\end{cases}\qquad R(H_n)=-H_n;$$

$$(\mathcal{R}_3^{\mathcal{P}_8}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant -1,\\ 0, & m\leqslant -2,\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\geqslant -1,\\ 0, & n\leqslant -2,\end{cases}$$

$$(\mathcal{R}_4^{\mathcal{P}_8}):\ R(L_m)=\begin{cases}-L_m, & m\geqslant -1,\\ 0, & m\leqslant -2,\end{cases}\qquad R(H_n)=\begin{cases}-H_n, & n\geqslant 0,\\ 0, & n<0.\end{cases}$$

Proof. In view of Lemma 3.1, if we define a new operation $x\circ y=[R(x),y]$ on $W(2,2)$, then $(W(2,2),[,],\circ)$ is a post-Lie algebra. By (36), we have

$$L_m\circ L_n=[R(L_m),L_n]=(m-n)f(m)L_{m+n},$$

$$L_m\circ H_n=[R(L_m),H_n]=(m-n)f(m)H_{m+n},$$

$$H_m\circ L_n=[R(H_m),L_n]=(m-n)g(m)H_{m+n},$$

and $H_m\circ H_n=[R(H_m),H_n]=0$ for all $m,n\in\mathbb{Z}$. This means that $(W(2,2),[,],\circ)$ is a graded post-Lie algebra structure satisfying (5)-(8) with $\phi(m,n)=(m-n)f(m)$, $\varphi(m,n)=(m-n)f(m)$ and $\theta(m,n)=(m-n)g(m)$. By Theorem 2.6, we see that f, g must be of the 30 cases listed in Table 2, which can yield the 30 forms of R one by one. It is easy to verify that every form of R listed in the above is a Rota-Baxter operator of weight 1 satisfying (36). The proof is completed. □

4. Application to Yang-Baxter equation

First we give some notations. Let $\mathrm{ad}:\mathfrak{g}\to gl(\mathfrak{g})$ be the adjoint representation of a Lie algebra $\mathfrak{g}$ defined by $\mathrm{ad}(x)(y)=[x,y]$ for any $x,y\in\mathfrak{g}$. Let $\mathrm{ad}^*:\mathfrak{g}\to gl(\mathfrak{g}^*)$ be the dual representation of the adjoint representation of $\mathfrak{g}$. On the vector space $\mathfrak{g}\oplus\mathfrak{g}^*$, there is a natural Lie algebra structure (denoted by $\mathfrak{g}\ltimes_{\mathrm{ad}^*}\mathfrak{g}^*$) given by

$$[x_1+f_1,x_2+f_2]=[x_1,x_2]+\mathrm{ad}^*(x_1)f_2-\mathrm{ad}^*(x_2)f_1,\ \forall x_1,x_2\in\mathfrak{g},\ f_1,f_2\in\mathfrak{g}^*.$$

A linear map is said to be of finite rank if its image has finite dimension. A linear operator R on $\mathfrak{g}$ of finite rank can be identified as an element in $\mathfrak{g}\otimes\mathfrak{g}^*\subset(\mathfrak{g}\ltimes_{\mathrm{ad}^*}\mathfrak{g}^*)\otimes(\mathfrak{g}\ltimes_{\mathrm{ad}^*}\mathfrak{g}^*)$ as follows. Let $\{e_i\}_{i\in I}$ be a basis of $\mathrm{Im}R$, then for $x\in g$, $R(x)$ can be written as a linear combination of the basis. Namely, for each $i\in I$ there exists a unique linear functional $R_i\in g^*$ such that

$$R(x)=\sum_{i\in I}R_i(x)e_i,\quad \forall x\in\mathfrak{g}.$$

From R is of finite rank we know that I is finite. Then we have

$$R=\sum_{i\in I} e_i\otimes R_i \in \mathfrak{g}\otimes\mathfrak{g}^* \subset (\mathfrak{g}\ltimes_{\mathrm{ad}^*}\mathfrak{g}^*)\otimes(\mathfrak{g}\ltimes_{\mathrm{ad}^*}\mathfrak{g}^*).$$

Lemma 4.1 ([13])**.** *Let $\mathfrak{g}$ be a Lie algebra and $R:\mathfrak{g}\to\mathfrak{g}$ a balanced linear map. Then R is a Rota-Baxter operator of weight 1 on $\mathfrak{g}$ if and only if both $(R-R^{21})+\mathrm{Id}$ and $(R-R^{21})-\mathrm{Id}^{21}$ are solutions of the formal CYBE on $\mathfrak{g}\ltimes_{\mathrm{ad}^*}\mathfrak{g}^*$.*

Lemma 4.2 ([13])**.** *R is a Rota-Baxter operator of weight 1 on a Lie algebra $\mathfrak{g}$ if and only if so is $-R-\mathrm{Id}$ on $\mathfrak{g}$ and*

$$((-R-\mathrm{Id})-(-R-\mathrm{Id})^{21})+\mathrm{Id}=-((R-R^{21})-\mathrm{Id}^{21}).$$

In this paper, we only list the solutions of the CYBE obtained from $(R-R^{21})+\mathrm{Id}$. Note that $\mathrm{Id}=\sum_{m\in\mathbb{Z}}L_m\otimes L_m^*+\sum_{n\in\mathbb{Z}}H_n\otimes H_n^*$ for $W(2,2)$.

By [13], a formal tensor $r=\sum_{i,j\in I}a_{ij}e_i\otimes e_j\in\mathfrak{g}\widehat{\otimes}\mathfrak{g}$, is called a solution of the formal CYBE if it is row-finite and column-finite and satisfies

$$[[r]](e_i,e_j,e_k):=\sum_{s,t\in I}(C_{st}^i a_{sj}a_{tk}+a_{is}C_{st}^j a_{tk}+a_{is}a_{jt}C_{st}^k)=0$$

for all $i,j,k\in I$, where C_{rs}^i are the structural coefficients of $\mathfrak{g}$. A linear operator R on $\mathfrak{g}$ can be identified as an element in $\mathfrak{g}\widehat{\otimes}\mathfrak{g}^*\subset(\mathfrak{g}\ltimes_{\mathrm{ad}^*}\mathfrak{g}^*)\widehat{\otimes}(\mathfrak{g}\ltimes_{\mathrm{ad}^*}\mathfrak{g}^*)$ as follows. Let $\{e_i\}_{i\in I}$ be a basis of $\mathfrak{g}$ and $\{e_i^*\}_{i\in I}$ be its dual defined by

$$e_i^*(e_j)=\delta_{ij},\quad \forall i,j\in I.$$

By Zorn's lemma, $\{e_i^*\}_{i\in I}$ can be extended to a basis of $\mathfrak{g}^*$, say $\{e_i^*\}_{i\in I}\cup\{f_i\}_{i\in J}$. Then we have

$$R=\sum_{i\in I}R(e_i)\otimes e_i^*+\sum_{j\in J}0\otimes f_j\in\mathfrak{g}\widehat{\otimes}\mathfrak{g}^*\subset(\mathfrak{g}\ltimes_{\mathrm{ad}^*}\mathfrak{g}^*)\widehat{\otimes}(\mathfrak{g}\ltimes_{\mathrm{ad}^*}\mathfrak{g}^*).$$

By a similar argument as in [13], we have the following theorem.

Theorem 4.3. *Lemma 4.2 gives the following solutions of the formal CYBE on $W(2,2)\ltimes_{\mathrm{ad}^*}W(2,2)^*$ from the Rota-Baxter operators of weight 1 on $W(2,2)$ given in Theorem 3.2, for some where $a,b\in\mathbb{C}$:*

$(\mathcal{Y}_1^{\mathcal{P}_1}):\ r_1^{\mathcal{P}_1}=\sum_{m\in\mathbb{Z}}L_m\otimes L_m^*+\sum_{n\in\mathbb{Z}}H_n\otimes H_n^*;$
$(\mathcal{Y}_2^{\mathcal{P}_1}):\ r_2^{\mathcal{P}_1}=\sum_{m\in\mathbb{Z}}L_m\otimes L_m^*+\sum_{n\in\mathbb{Z}}H_n^*\otimes H_n;$
$(\mathcal{Y}_1^{\mathcal{P}_2}):\ r_1^{\mathcal{P}_2}=\sum_{m\in\mathbb{Z}}L_m^*\otimes L_m+\sum_{n\in\mathbb{Z}}H_n\otimes H_n^*;$
$(\mathcal{Y}_2^{\mathcal{P}_2}):\ r_2^{\mathcal{P}_2}=\sum_{m\in\mathbb{Z}}L_m^*\otimes L_m+\sum_{n\in\mathbb{Z}}H_n^*\otimes H_n;$
$(\mathcal{Y}_1^{\mathcal{P}_3^a}):\ r_1^{\mathcal{P}_3^a}=\sum_{m<0}L_m\otimes L_m^*+(a+1)L_0\otimes L_0^*+\sum_{m>0}L_m^*\otimes L_m$
$\qquad -aL_0^*\otimes L_0+\sum_{n\in\mathbb{Z}}H_n\otimes H_n^*;$

$$
\begin{aligned}
(\mathcal{Y}_2^{\mathcal{P}_3^a}):\ r_2^{\mathcal{P}_3^a} = &\sum_{m<0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m>0} L_m^* \otimes L_m \\
&- aL_0^* \otimes L_0 + \sum_{n\in\mathbb{Z}} H_n^* \otimes H_n;
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{Y}_3^{\mathcal{P}_3^{a,b}}):\ r_3^{\mathcal{P}_3^{a,b}} = &\sum_{m<0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* \\
&+ \sum_{m>0} L_m^* \otimes L_m - aL_0^* \otimes L_0 \\
&+ \sum_{n<0} H_n \otimes H_n^* + (b+1)H_0 \otimes L_0^* \\
&+ \sum_{n>0} H_n^* \otimes H_n - bH_0^* \otimes H_0;
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{Y}_4^{\mathcal{P}_3^a}):\ r_4^{\mathcal{P}_3^a} = &\sum_{m<0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m>0} L_m^* \otimes L_m \\
&- aL_0^* \otimes L_0 + \sum_{n\leq 1} H_n \otimes H_n^* + \sum_{n\geqslant 2} H_n^* \otimes H_n;
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{Y}_5^{\mathcal{P}_3^a}):\ r_5^{\mathcal{P}_3^a} = &\sum_{m<0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m>0} L_m^* \otimes L_m \\
&- aL_0^* \otimes L_0 + \sum_{n\leqslant -2} H_n \otimes H_n^* + \sum_{n\geq -1} H_n^* \otimes H_n;
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{Y}_1^{\mathcal{P}_4^a}):\ r_1^{\mathcal{P}_4^a} = &\sum_{m>0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m<0} L_m^* \otimes L_m \\
&- aL_0^* \otimes L_0 + \sum_{n\in\mathbb{Z}} H_n \otimes H_n^*;
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{Y}_2^{\mathcal{P}_4^a}):\ r_2^{\mathcal{P}_4^a} = &\sum_{m>0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m<0} L_m^* \otimes L_m \\
&- aL_0^* \otimes L_0 + \sum_{n\in\mathbb{Z}} H_n^* \otimes H_n;
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{Y}_3^{\mathcal{P}_4^{a,b}}):\ r_3^{\mathcal{P}_4^{a,b}} = &\sum_{m>0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m<0} L_m^* \otimes L_m \\
&- aL_0^* \otimes L_0 + \sum_{n>0} H_n \otimes H_n^* + (b+1)H_0 \otimes H_0^* \\
&+ \sum_{n<0} H_n^* \otimes H_n - bH_0^* \otimes H_0;
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{Y}_4^{\mathcal{P}_4^a}):\ r_4^{\mathcal{P}_4^a} = &\sum_{m>0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m<0} L_m^* \otimes L_m \\
&- aL_0^* \otimes L_0 + \sum_{n\geqslant -1} H_n \otimes H_n^* + \sum_{n\leq -2} H_n^* \otimes H_n;
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{Y}_5^{\mathcal{P}_4^a}):\ r_5^{\mathcal{P}_4^a} =&\ \sum_{m>0} L_m \otimes L_m^* + (a+1)L_0 \otimes L_0^* + \sum_{m<0} L_m^* \otimes L_m \\
&- aL_0^* \otimes L_0 + \sum_{n\geq 2} H_n \otimes H_n^* + \sum_{n\leqslant 1} H_n^* \otimes H_n; \\
(\mathcal{Y}_1^{\mathcal{P}_5}):\ r_1^{\mathcal{P}_5} =&\ \sum_{m\leqslant 1} L_m \otimes L_m^* + \sum_{m\geq 2} L_m^* \otimes L_m + \sum_{n\in\mathbb{Z}} H_n \otimes H_n^*; \\
(\mathcal{Y}_2^{\mathcal{P}_5}):\ r_2^{\mathcal{P}_5} =&\ \sum_{m\leqslant 1} L_m \otimes L_m^* + \sum_{m\geq 2} L_m^* \otimes L_m + \sum_{n\in\mathbb{Z}} H_n^* \otimes H_n; \\
(\mathcal{Y}_3^{\mathcal{P}_5}):\ r_3^{\mathcal{P}_5} =&\ \sum_{m\leqslant 1} L_m \otimes L_m^* + \sum_{m\geq 2} L_m^* \otimes L_m + \sum_{n\leqslant 1} H_n \otimes H_n^* \\
&+ \sum_{n\geq 2} H_n^* \otimes H_n; \\
(\mathcal{Y}_4^{\mathcal{P}_5}):\ r_4^{\mathcal{P}_5} =&\ \sum_{m\leqslant 1} L_m \otimes L_m^* + \sum_{m\geq 2} L_m^* \otimes L_m + \sum_{n\leq 0} H_n \otimes H_n^* \\
&+ \sum_{n>0} H_n^* \otimes H_n; \\
(\mathcal{Y}_1^{\mathcal{P}_6}):\ r_1^{\mathcal{P}_6} =&\ \sum_{m\geqslant 2} L_m \otimes L_m^* + \sum_{m\leq 1} L_m^* \otimes L_m + \sum_{n\in\mathbb{Z}} H_n \otimes H_n^*; \\
(\mathcal{Y}_2^{\mathcal{P}_6}):\ r_2^{\mathcal{P}_6} =&\ \sum_{m\geqslant 2} L_m \otimes L_m^* + \sum_{m\leq 1} L_m^* \otimes L_m + \sum_{n\in\mathbb{Z}} H_n^* \otimes H_n; \\
(\mathcal{Y}_3^{\mathcal{P}_6}):\ r_3^{\mathcal{P}_6} =&\ \sum_{m\geqslant 2} L_m \otimes L_m^* + \sum_{m\leq 1} L_m^* \otimes L_m + \sum_{n\geqslant 2} H_n \otimes H_n^* \\
&+ \sum_{n\leq 1} H_n^* \otimes H_n; \\
(\mathcal{Y}_4^{\mathcal{P}_6}):\ r_4^{\mathcal{P}_6} =&\ \sum_{m\geqslant 2} L_m \otimes L_m^* + \sum_{m\leq 1} L_m^* \otimes L_m + \sum_{n>0} H_n \otimes H_n^* \\
&+ \sum_{n\leq 0} H_n^* \otimes H_n; \\
(\mathcal{Y}_1^{\mathcal{P}_7}):\ r_1^{\mathcal{P}_7} =&\ \sum_{m\geqslant -1} L_m \otimes L_m^* + \sum_{m\leq -2} L_m^* \otimes L_m + \sum_{n\in\mathbb{Z}} H_n \otimes H_n^*; \\
(\mathcal{Y}_2^{\mathcal{P}_7}):\ r_2^{\mathcal{P}_7} =&\ \sum_{m\geqslant -1} L_m \otimes L_m^* + \sum_{m\leq -2} L_m^* \otimes L_m + \sum_{n\in\mathbb{Z}} H_n^* \otimes H_n; \\
(\mathcal{Y}_3^{\mathcal{P}_7}):\ r_3^{\mathcal{P}_7} =&\ \sum_{m\geqslant -1} L_m \otimes L_m^* + \sum_{m\leq -2} L_m^* \otimes L_m + \sum_{n\leq -2} H_n \otimes H_n^* \\
&+ \sum_{n\geqslant -1} H_n^* \otimes H_n;
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{Y}_4^{\mathcal{P}_7}):\ r_4^{\mathcal{P}_7} = & \sum_{m\geqslant -1} L_m \otimes L_m^* + \sum_{m\leq -2} L_m^* \otimes L_m + \sum_{n\geq 0} H_n \otimes H_n^* \\
& + \sum_{n<0} H_n^* \otimes H_n; \\
(\mathcal{Y}_1^{\mathcal{P}_8}):\ r_1^{\mathcal{P}_8} = & \sum_{m\leqslant -2} L_m \otimes L_m^* + \sum_{m\geq -1} L_m^* \otimes L_m + \sum_{n\in\mathbb{Z}} H_n \otimes H_n^*; \\
(\mathcal{Y}_2^{\mathcal{P}_8}):\ r_2^{\mathcal{P}_8} = & \sum_{m\leqslant -2} L_m \otimes L_m^* + \sum_{m\geq -1} L_m^* \otimes L_m + \sum_{n\in\mathbb{Z}} H_n^* \otimes H_n; \\
(\mathcal{Y}_3^{\mathcal{P}_8}):\ r_3^{\mathcal{P}_8} = & \sum_{m\leqslant -2} L_m \otimes L_m^* + \sum_{m\geq -1} L_m^* \otimes L_m + \sum_{n\leqslant -2} H_n \otimes H_n^* \\
& + \sum_{n\geq -1} H_n^* \otimes H_n; \\
(\mathcal{Y}_4^{\mathcal{P}_8}):\ r_4^{\mathcal{P}_8} = & \sum_{m\leqslant -2} L_m \otimes L_m^* + \sum_{m\geq -1} L_m^* \otimes L_m + \sum_{n<0} H_n \otimes H_n^* \\
& + \sum_{n\geq 0} H_n^* \otimes H_n.
\end{aligned}
$$

Acknowledgments. The authors would like to express their deep gratitude to referees for their very careful reading, important comments and valuable suggestions.

References

[1] C. Bai, L. Guo, and X. Ni, *Nonabelian generalized Lax pairs, the classical Yang-Baxter equation and PostLie algebras*, Comm. Math. Phys. **297** (2010), no. 2, 553–596.

[2] G. Baxter, *An analytic problem whose solution follows from a simple algebraic identity*, Pacific J. Math. **10** (1960), 731–742.

[3] R. J. Baxter, *Partition function of the eight-vertex lattice model*, Ann. Physics **70** (1972), 193–228.

[4] D. Burde and K. Dekimpe, *Post-Lie algebra structures on pairs of Lie algebras*, J. Algebra **464** (2016), 226–245.

[5] D. Burde, K. Dekimpe, and K. Vercammen, *Affine actions on Lie groups and post-Lie algebra structures*, Linear Algebra Appl. **437** (2012), no. 5, 1250–1263.

[6] D. Burde and W. A. Moens, *Commutative post-Lie algebra structures on Lie algebras*, J. Algebra **467** (2016), 183–201.

[7] H. Chen and J. Li, *Left-symmetric algebra structures on the W-algebra $W(2,2)$*, Linear Algebra Appl. **437** (2012), no. 7, 1821–1834.

[8] C. Chu and L. Guo, *Localization of Rota-Baxter algebras*, J. Pure Appl. Algebra **218** (2014), no. 2, 237–251.

[9] K. Ebrahimi-Fard, A Lundervold, I Mencattini, and H. Z Munthe-Kaas, *Post-Lie algebras and isospectral flows*, SIGMA Symmetry Integrability Geom. Methods Appl. **11** (2015), Paper 093, 16 pp.

[10] K. Ebrahimi-Fard, A. Lundervold, and H. Z. Munthe-Kaas, *On the Lie enveloping algebra of a post-Lie algebra*, J. Lie Theory **25** (2015), no. 4, 1139–1165.

[11] K. Ebrahimi-Fard, I. Mencattini, and H. Munthe-Kaas, *Post-Lie algebras and factorization theorems*, J. Geom. Phys. **119** (2017), 19–33.

[12] S. L. Gao, C. P. Jiang, and Y. F. Pei, *Derivations, central extensions and automorphisms of a Lie algebra*, Acta Math. Sinica (Chin. Ser.) **52** (2009), no. 2, 281–288.

[13] X. Gao, M. Liu, C Bai, and N. Jing, *Rota-Baxter operators on Witt and Virasoro algebras*, J. Geom. Phys. **108** (2016), 1–20.

[14] L. Guo, *An Introduction to Rota-Baxter Algebra*, Surveys of Modern Mathematics, **4**, International Press, Somerville, MA, 2012.

[15] W. Jiang and W. Zhang, *Verma modules over the $W(2,2)$ algebras*, J. Geom. Phys. **98** (2015), 118–127.

[16] R. Kashaev, *The Yang-Baxter relation and gauge invariance*, J. Phys. A **49** (2016), no. 16, 164001, 16 pp.

[17] H. Z. Munthe-Kaas and A. Lundervold, *On post-Lie algebras, Lie-Butcher series and moving frames*, Found. Comput. Math. **13** (2013), no. 4, 583–613.

[18] Y. Pan, Q. Liu, C. Bai, and L. Guo, *PostLie algebra structures on the Lie algebra* $\mathrm{SL}(2,\mathbb{C})$, Electron. J. Linear Algebra **23** (2012), 180–197.

[19] G. Radobolja, *Subsingular vectors in Verma modules, and tensor product of weight modules over the twisted Heisenberg-Virasoro algebra and $W(2,2)$ algebra*, J. Math. Phys. **54** (2013), no. 7, 071701, 24 pp.

[20] G.-C. Rota, *Baxter operators, an introduction*, in Gian-Carlo Rota on combinatorics, 504–512, Contemp. Mathematicians, Birkhäuser Boston, Boston, MA, 1995.

[21] X. Tang, *Post-Lie algebra structures on the Witt algebra*, arXiv:1701.00200, 2017.

[22] ———, *Biderivations, linear commuting maps and commutative post-Lie algebra structures on W-algebras*, Comm. Algebra **45** (2017), no. 12, 5252–5261.

[23] X. Tang and Y. Zhang, *Post-Lie algebra structures on solvable Lie algebra $t(2,\mathbb{C})$*, Linear Algebra Appl. **462** (2014), 59–87.

[24] X. Tang, Y. Zhang, and Q. Sun, *Rota-Baxter operators on 4-dimensional complex simple associative algebras*, Appl. Math. Comput. **229** (2014), 173–186.

[25] B. Vallette, *Homology of generalized partition posets*, J. Pure Appl. Algebra **208** (2007), no. 2, 699–725.

[26] Y. Wang, Q. Geng, and Z. Chen, *The superalgebra of $W(2,2)$ and its modules of the intermediate series*, Comm. Algebra **45** (2017), no. 2, 749–763.

[27] C. N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett. **19** (1967), 1312–1315.

[28] W. Zhang and C. Dong, *W-algebra $W(2,2)$ and the vertex operator algebra $L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)$*, Comm. Math. Phys. **285** (2009), no. 3, 991–1004.

Xiaomin Tang
School of Mathematical Science
Heilongjiang University
Harbin 150080, P. R. China
Email address: x.m.tang@163.com

Yongyue Zhong
School of Mathematical Science
Heilongjiang University
Harbin 150080, P. R. China
Email address: 845630692@qq.com