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ON THE MINIMUM ORDER OF 4-LAZY COPS-WIN GRAPHS

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Abstract. We consider the minimum order of a graph G with a given lazy cop number $c_L(G)$. Sullivan, Townsend and Werzanski [7] showed that the minimum order of a connected graph with lazy cop number 3 is 9 and $K_3\square K_3$ is the unique graph on nine vertices which requires three lazy cops. They conjectured that for a graph G on n vertices with $\Delta(G) \geq n - k^2$, $c_L(G) \leq k$. We proved that the conjecture is true for $k = 4$. Furthermore, we showed that the Petersen graph is the unique connected graph G on 10 vertices with $\Delta(G) \leq 3$ having lazy cop number 3 and the minimum order of a connected graph with lazy cop number 4 is 16.

1. Introduction

The game of Cops and Robbers is a well-known two-player game played on a finite connected undirected graph. It was independently introduced by Quilliot [6], and by Nowakowski and Winkler [4]. The first player occupies some vertices with some number of cops (multiple cops may occupy a single vertex) and the second player occupies a vertex with a single robber. After that they move alternately along the edges of the graph. On the cops' turn, each of the cops may remain stationary or move to an adjacent vertex. On the robber's turn, he may remain stationary or move to an adjacent vertex. A *round* of the game is a cop move together with the subsequent robber move. The cops win if after a finite number of rounds, one of them can move to *catch* the robber, that is, the cop and the robber occupy the same vertex.

The main object of study in the game of Cops and Robbers on a graph G is the *cop number* $c(G)$, the minimum number of cops required to catch the robber, introduced by Aigner and Fromme [1]. For a fixed positive integer k, we say a graph G is $k-cop-win$ if $c(G) = k$. For example, a path is 1-cop-win and the Petersen graph is 3-cop-win [2]. We define M_k to be the minimum order of a connected k-cop-win graph and m_k to be the minimum order of a connected graph G satisfying $c(G) \geq k$. Clearly, we have $m_k \leq M_k$. The exact values

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of these parameters are only known for first three values of k. Baird et al. [2] showed that $m_1 = M_1 = 1$, $m_2 = M_2 = 4$ and $m_3 = M_3 = 10$. Moreover, they proved that the Petersen graph is the unique 3-cop-win graph with order 10. They also used a computer search to calculate the cop number of every connected graph on 10 or fewer vertices. They performed this categorization by checking for cop-win orderings [4] and using an algorithm provided in [3].

We are interested in a variant of cops and robber introduced by Offner and Ojakian [5], where at most one cop moves in any round. It is called the game of *Lazy Cops and Robbers* and the *lazy cop number* is the minimum number of cops required to catch the robber in this setting. We write $c_L(G)$ for the lazy cop number of a graph G. Let P_n and C_n be the n-path and n-cycle, respectively. It is straightforward that $c(P_n) = c_L(P_n) = 1 = c(C_3) = c_L(C_3)$ for $n \geq 1$ and $c(C_n) = c_L(C_n) = 2$ for $n \geq 4$. A graph satisfying $c_L(G) = k$ is k-lazy cop-win. Define M_k^l to be the minimum order of a connected k-lazy cop-win graph and define m_k^l to be the minimum order of a connected graph G with $c_L(G) \geq k$. It is easy to see that $m_1^l = M_1^l = 1$. For $k = 2$, we must have $m_2^l = M_2^l = 4$ since the only connected graphs with three vertices are P_3 and C_3, both of which are 1-lazy cop-win graphs.

For a graph G, the degree of a vertex $u \in V(G)$ is denoted as $\deg_G(u)$. The minimum degree and the maximum degree of G are denoted as $\delta(G)$ and $\Delta(G)$, respectively. Given two graphs G and H, their *Cartesian product* $G \square H$ is a graph with vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \square H$ if and only if either

(i) $u_1 = u_2$ and v_1 is adjacent to v_2 in H, or
(ii) $v_1 = v_2$ and u_1 is adjacent to u_2 in G.

Sullivan, Townsend and Werzanski [7] proved that for the game of lazy cops and robber, $K_3 \square K_3$ is the unique 3-lazy cop-win graph on nine vertices. In addition, all other graphs on 9 or fewer vertices have lazy cop number at most two. Hence $m_3^l = M_3^l = 9$. They also showed that $c_L(K_n \square K_n) = n$ and interestingly, noted that $K_n \square K_n$ can be interpreted as an $n \times n$ grid with edges representing a Rook's move in chess. Furthermore, they conjectured that for a graph G on n vertices with $\Delta(G) \geq n - k^2$, we must have $c_L(G) \leq k$. In this paper, we compute the exact values for m_4^l and M_4^l and prove some related results, including the above conjecture for the case $k = 4$ (see Corollary 4.7).

Theorem 1.1. *If G is a connected graph with 10 vertices and $\Delta(G) \leq 3$, then $c_L(G) \leq 3$. Furthermore, equality holds if and only if G is the Petersen graph.*

Theorem 1.2. *If G is a connected graph with at most 15 vertices, then $c_L(G) \leq 3$.*

The exact values for m_4^l and M_4^l can be deduced easily from Theorem 1.2 and the fact that $K_4 \square K_4$ is a 4-lazy cop-win graph [7].

Corollary 1.3. $m_4^l = M_4^l = 16$.

Given a vertex $v \in V(G)$, its *neighborhood* $N_G(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and $N_G[v]$ is the set $\{v\} \cup N_G(v)$. Furthermore, for any subset $U \subseteq V(G)$, $N_G(U) = \bigcup_{u \in U} N_G(u)$ and $N_G[U] = \bigcup_{u \in U} N_G[u]$. We write neighborhood of the vertex set $U = \{v_1, v_2, v_3, \ldots, v_i\}$ as $N_G[v_1, v_2, v_3, \ldots, v_i]$. If the graph in question is clear, we shall write $N(v)$, $N[v]$, $N(U)$, $N[U]$ and $N[v_1, v_2, v_3, \ldots, v_i]$. A vertex occupied by a cop or robber is also called a *position*.

Let the cops c_i, $i = 1, 2, \ldots$, be at our disposal to play on a graph G. A *winning strategy* of the cops on G refers to a set of instructions for the cops c_i, $i = 1, 2 \ldots$, if followed, guarantees that the cops can win any game played on G, regardless of how the robber r moves throughout the game. If e is a cop or a robber and is at position $u \in V(G)$, we shall write $N_G(e)$ instead of $N_G(u)$. Similarly, $N_G[e] = N_G[u]$.

When we say a cop c moves *one step at a time* to a vertex w, we mean that c will move towards w in all cop's turn regardless of the movement of r in each robber's turn. So c will occupy w in finite steps.

Lemma 1.4 ([7, Theorem 2.5])**.** *Assume $G = (V, E)$ has a vertex $v \in V$ with $deg(v) = 1$; say $uv \in E$ is the unique edge incident to v. Define G' to be the graph with vertex set $V' = V - \{v\}$ and edge set $E' = E - \{uv\}$. Then $c_L(G') = c_L(G)$.*

By virtue of Lemma 1.4, we may ignore graphs that have a vertex of degree 1. By removing vertices of degree 1, we obtain a graph with the same lazy cop number but with smaller number of vertices.

In Section 2, we will show that $c_L(P(n, 2)) = 3$ for $n \geq 5$ (Lemma 2.1). This result is of interest on its own. Then we prove Theorem 1.1 and Theorem 1.2 in Section 3 and Section 4 respectively.

2. $c_L(P(n, 2))$

The generalized Petersen graph $P(n, 2)$ is the graph with vertex set

$$V(P(n, 2)) = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$$

and edge set

$$E(P(n, 2)) = \{u_i v_i, u_i u_{i+1}, v_i v_{i+2} \ : \ i \geq 1\},$$

where the subscripts are taken modulo n. Note that $P(5, 2)$ is the Petersen graph.

Lemma 2.1. *For $P(n, 2)$ of girth ≥ 5, we have $c_L(P(n, 2)) = 3$.*

Proof. [1] shows that for any graph G with girth at least 5, $c(G) \geq \delta(G)$. Since $P(n, 2)$ is 3-regular and $c(G) \leq c_L(G)$ (see [8]), this indicates that $c_L(P(n, 2)) \geq 3$.

Now, it is left to show that $c_L(P(n, 2)) \leq 3$. Here we describe a winning strategy for three cops. Suppose we have 3 cops at our disposal, say c_1, c_2 and

c_3. The robber will be denoted by r. If at round t, the robber is at position u_j or v_j, we set $W_t(r) = j$. We do the same for the cop c_i. We may consider W_t as a weight of a cop or the robber at round t.

Initially we place c_1 at position u_n, c_2 at position v_1 and c_3 at position v_2 (see Figure 1).

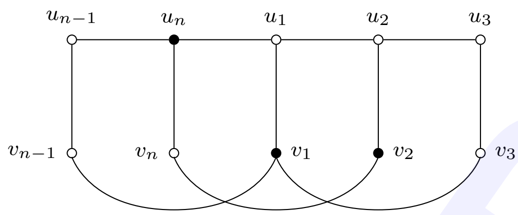


FIGURE 1

Therefore $W_1(c_1) = n$, $W_1(c_2) = 1$ and $W_1(c_3) = 2$. Note that r cannot be placed at positions $\{u_1, v_1, u_2, v_2, v_3, u_{n-1}, u_n, v_{n-1}, v_n\}$. So, initially we must have

$$\max(W_1(c_2), W_1(c_3)) = 2 < W_1(r) < n - 1 = W_1(c_1) - 1,$$

and $W_1(c_2)$ and $W_1(c_3)$ are consecutive integers. The size of the interval that $W_1(r)$ can lie within is $W_1(c_1) - 1 - \max(W_1(c_2), W_1(c_3)) = n - 3$.

We prove this by induction on t. Suppose that at round t, we have

$$\max(W_t(c_2), W_t(c_3)) < W_t(r) < W_t(c_1) - 1,$$

and $W_t(c_2)$ and $W_t(c_3)$ are consecutive integers. The size of the interval that $W_t(r)$ can lie within is $s = W_t(c_1) - 1 - \max(W_t(c_2), W_t(c_3))$. Now we shall give a strategy depending on the value of $W_t(r)$ that will reduce the size of the interval that $W_{t+1}(r)$ can lie within.

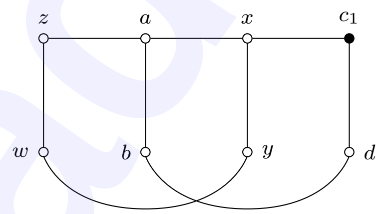


FIGURE 2

Scenario 1. Suppose $W_t(r) = W_t(c_1) - 3$ (see Figure 2). So r is at position z or w. We move the cop c_1 to position x. At robber's turn, if r is at position z, he cannot move to a, otherwise he will be caught in the next round. Similarly, if r is at position w, he cannot move to y. Thus, at round $t+1$, we must have $W_{t+1}(r) < W_{t+1}(c_1) - 1 = W_t(c_1) - 2$. Note that $W_{t+1}(c_2) = W_t(c_2)$, $W_{t+1}(c_3) = W_t(c_3)$ and $W_{t+1}(r) = W_t(r), W_t(r) - 1$ or $W_t(r) - 2$. So, $W_{t+1}(c_2)$ and $W_{t+1}(c_3)$ are still consecutive integers. We now consider two cases.

First, we suppose $W_{t+1}(r) > \max(W_{t+1}(c_2), W_{t+1}(c_3))$, then we have achieved our objective for the size of the interval that $W_{t+1}(r)$ can lie within is $W_{t+1}(c_1) - 1 - \max(W_{t+1}(c_2), W_{t+1}(c_3)) = W_t(c_1) - 2 - \max(W_t(c_2), W_t(c_3)) = s-1$. Recall that the size of the interval that $W_t(r)$ can lie within is s.

Next, we suppose $W_{t+1}(r) \le \max(W_{t+1}(c_2), W_{t+1}(c_3))$. We may assume that $W_{t+1}(c_2) = W_{t+1}(c_3) - 1$ because $W_{t+1}(c_2)$ and $W_{t+1}(c_3)$ are consecutive integers. Since $\max(W_t(c_2), W_t(c_3)) < W_t(r)$, this can only happen if $W_{t+1}(r) = W_t(r) - 1$ or $W_t(r) - 2$. If $W_{t+1}(r) = W_t(r) - 2$, then r must be at position w or f at round t (see Figure 3), and at his turn, he moves to the position a cop is occupying. This is absurd. If $W_{t+1}(r) = W_t(r) - 1$, then r must be at position z at round t, and at his turn, he moves to e. The robber will be caught at round $t+1$ by the cop c_3.

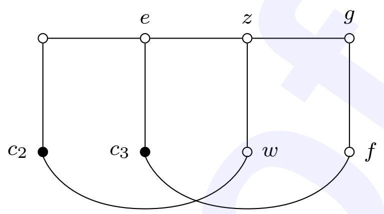


FIGURE 3

Scenario 2. Suppose $W_t(r) \neq W_t(c_1) - 3$. Assume that $W_t(c_2) = W_t(c_3) - 1$ (see Figure 3). We move the cop c_2 to position w. At the robber's turn, he cannot move to z, otherwise he will be caught at round $t+1$ by the cop c_2. Moreover, if the robber was already at z, he cannot remain there, nor can he move to e, so he must move to g (the vertex above f). So we must have $W_t(c_3) + 1 = \max(W_{t+1}(c_2), W_{t+1}(c_3)) < W_{t+1}(r)$. Since $W_t(r) < W_t(c_1) - 1$ and $W_t(r) \neq W_t(c_1) - 3$, either $W_t(r) = W_t(c_1) - 2$ or $W_t(r) < W_t(c_1) - 3$. If $W_t(r) = W_t(c_1) - 2$, then r is at position a or b (see Figure 2). If r is at a, he cannot move to x, otherwise he will be caught at round $t+1$ by the cop c_1. Similarly, if r is at b, he cannot move to d. Thus, $W_{t+1}(r) < W_{t+1}(c_1) - 1 = W_t(c_1) - 1$. If $W_t(r) < W_t(c_1) - 3$, then $W_{t+1}(r) < W_t(c_1) - 1$, for $W_{t+1}(r) \le W_t(r) + 2$. Hence we must have $W_{t+1}(r) < W_{t+1}(c_1) - 1$. We have achieved our objective for the size of the interval that $W_{t+1}(r)$ can lie within is $W_{t+1}(c_1) - 1 - \max(W_{t+1}(c_2), W_{t+1}(c_3)) = W_t(c_1) - 1 - (W_t(c_3) + 1) = s - 1$.

From Scenario 1 and 2, we see that either the robber is caught or the interval is getting smaller and smaller. This process cannot go on indefinitely. So the robber will be caught eventually.

This completes the proof of the lemma. □

3. Proof of Theorem 1.1

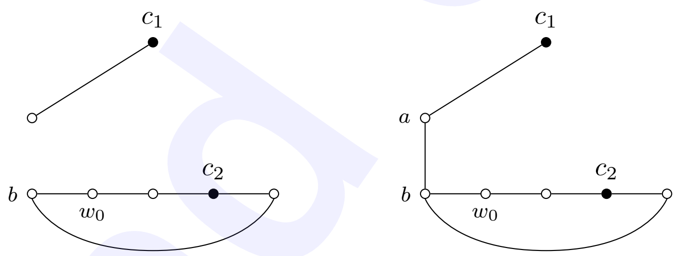
Lemma 3.1. *Let G be a connected graph on 10 vertices with $\Delta(G) = 3$. If $G - N[v]$ is not a 6-cycle for all $v \in V(G)$ with $\deg(v) = 3$, then $c_L(G) \leq 2$.*

Proof. Let c_1 and c_2 be the two cops at our disposal to catch the robber r in G. Recall that for a vertex $u \in V(G)$, we write $\deg_G(u) = k$ to mean the degree of u in the graph G, as a whole, is k, and that $\Delta(G - N[u])$ is the maximum degree of the subgraph $G - N[u]$.

Case 1. Suppose there is a vertex $u_0 \in V(G)$ with $\deg_G(u_0) = 3$ such that $\Delta(G - N[u_0]) \leq 2$.

Since $\Delta(G - N[u_0]) \leq 2$, every component in $G - N[u_0]$ is a path or a cycle. Initially, we place the two cops at position u_0. Then r can only be placed at a component H in $G - N[u_0]$. As long as there is a cop occupying u_0, r will have to remain in H.

- If H is a path, then we keep c_1 at u_0 and move c_2 to a vertex in H. Since $c_L(H) = 1$, r will be caught by c_2 eventually.
- Suppose H is a cycle. By the hypothesis of the lemma, H cannot be a 6-cycle. We shall assume H is a 5-cycle. The case H is a 4-cycle or a 3-cycle can be proved similarly.
  - Assume there is a vertex $w_0 \in V(H)$ with $\deg_G(w_0) = 2$. Then w_0 is not adjacent to any vertices in $N[u_0]$. There are two possibilities (see Figure 4). We keep c_1 at u_0 and move c_2 into position as in Figure 4.



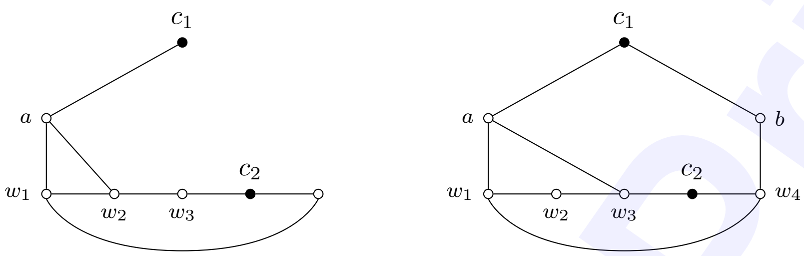
(a) (b)

FIGURE 4

Since $\deg_G(b) \leq 3$ and $\deg_H(b) = 2$, b is not adjacent to any vertices in $N(u_0)$ (Figure 4(a)) or b is adjacent to $a \in N(u_0)$ (Figure 4(b)). In either case, r can only stay at positions b or w_0.

In Figure 4(a), we keep c_2 at his position and move c_1 to position w_0 one step at a time. In Figure 4(b), we keep c_2 at his position and move c_1 to position b via a. In either case, r will be caught.

– Assume $\deg_G(w) = 3$ for all $w \in V(H)$.



(a) (b)

FIGURE 5

Since $\deg_H(w) = 2$, $N(w) \cap N(u_0) = 1$ for all $w \in V(H)$. This means there is a vertex $a \in N(u_0)$ with $|N(a) \cap V(H)| = 2$. We keep c_1 at u_0 and move c_2 into position as in Figure 5. Note that r can only stay at positions w_1 or w_2. In Figure 5(a), we keep c_2 at his position and move c_1 to a. The robber will be caught. In Figure 5(b), we move c_2 to w_3. Then r can be at positions w_1 or w_4 only. Now move c_1 to b. At robber's turn, he can only remain at w_1. In the next round, we move c_1 from b to w_4. The robber will be caught.

Case 2. Suppose $\Delta(G - N[u]) = 3$ for all $u \in V(G)$ with $\deg_G(u) = 3$.

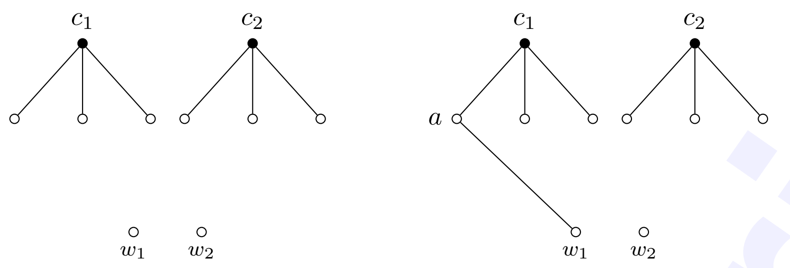
Pick a vertex $u_0 \in V(G)$ with $\deg_G(u_0) = 3$ and pick a $v_0 \in V(G - N[u_0])$ with $\deg_{G-N[u_0]}(v_0) = 3$. Initially we place c_1 at u_0 and c_2 at v_0. Note that $G - N[u_0] - N[v_0]$ is a disjoint union of 2 vertices or a 2-path. Let $V(G - N[u_0] - N[v_0]) = \{w_1, w_2\}$.

Suppose $G - N[u_0] - N[v_0]$ is a disjoint union of 2 vertices. We may assume r is at position w_1. Since $\deg(w_1) \leq 3$, there is a c_i such that $|N(c_i) \cap N(w_1)| \leq 1$ for some $i = 1, 2$. We may assume $|N(c_1) \cap N(w_1)| \leq 1$ (see Figure 6).

In Figure 6(a), we keep c_2 at his position and move c_1 to w_1 one step at a time. Note that r can only remain at w_1 for c_2 is occupying v_0. So the robber will be caught. In Figure 6(b), we keep c_2 at his position and move c_1 to a. The robber will also be caught.

Suppose $G - N[u_0] - N[v_0]$ is a 2-path.

(i) $|N(w_2) \cap N(c_1)| = 0$ and $|N(w_1) \cap N(c_1)| \leq 1$.



(a) $|N(c_1) \cap N(w_1)| = 0$ (b) $|N(c_1) \cap N(w_1)| = 1$

FIGURE 6

This situation is quite similar like the one in Figure 6 except that w_1 and w_2 are adjacent. So we use the same cop- winning strategy, that is, keep c_2 at his position and move c_1 towards w_1. The robber will be caught.

(ii) $|N(w_2) \cap N(c_1)| = 0$ and $|N(w_1) \cap N(c_1)| = 2$.
Since $\deg_G(w_1) = 3$, w_1 is not adjacent to any vertices in $N(c_2)$, i.e., $|N(w_1) \cap N(c_2)| = 0$. If $|N(w_2) \cap N(c_2)| \leq 1$, then the cops will have a winning strategy similar to (i). So we may assume $|N(w_2) \cap N(c_2)| = 2$ (see Figure 7). If r is at w_1, then we move c_2 to b and in the next round from b to w_2. The robber will be caught. If r is at w_2, then we move c_1 to a and in the next round from a to w_1. The robber will also be caught.

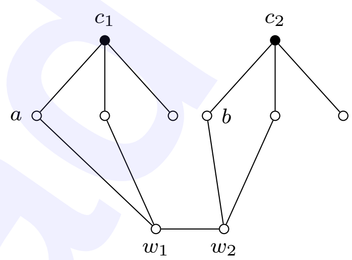


FIGURE 7

From (i) and (ii), we may assume that $|N(w_i) \cap N(c_1)| = 1 = |N(w_i) \cap N(c_2)|$ for $i = 1, 2$ (see Figure 8). There are two possibilities. In Figure 8(a), we have the case where w_1 and w_2 have a common neighbor in (without loss of

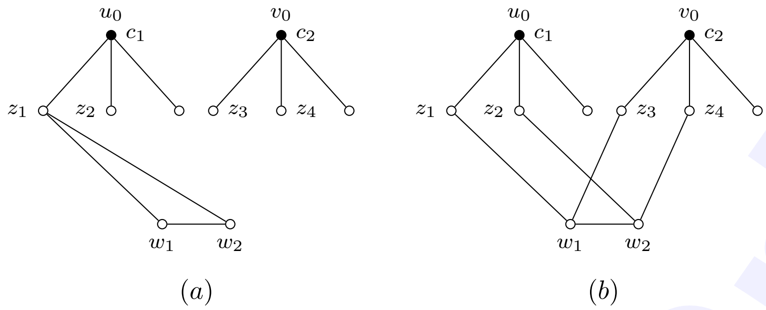


FIGURE 8

generality) $N(c_1)$. In that case, we move c_1 to z_1 and the robber will be caught. In Figure 8(b), we have the case where w_1 and w_2 have no common neighbors at all. Now, from the graph in Figure 8(b), we remove $N[w_1]$ from G (see Figure 9).

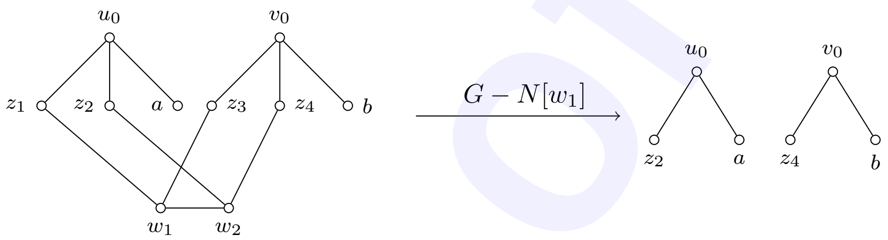


FIGURE 9

Let $J_1 = G - N[w_1]$. From what we assume in Case 2, there is a vertex of degree 3 in J_1. Note that u_0, v_0, z_2 and z_4 are at most of degree 2 in J_1. We may assume a is of degree 3 in J_1.

- Suppose a is adjacent to vertices z_2 and b (see Figure 10(a)). We move c_1 to z_1. Then r can only move to w_2 or z_2. Next, move c_2 to z_4. Then r can only move to z_2 or a. Next, move c_1 back to u_0. Then r can only move to b. Now move c_2 back to v_0. Since the robber's potential moves are $N(b) \subseteq \{v_0, a, z_1, z_3, z_4\}$, the robber cannot move back to w_1 or w_2. Hence the robber will be caught.
- Suppose a is adjacent to z_4 (see Figure 10(b)). Note that a cannot be adjacent to z_1 or z_3 since $deg_{J_1}(a) = 3$. It may be adjacent to z_2 or b. We move c_2 to z_3. Then r can only move to w_2 or z_4. Next, move c_1 to z_2. Then r can only move to z_4 or a. Next, move c_2 back to v_0. Then

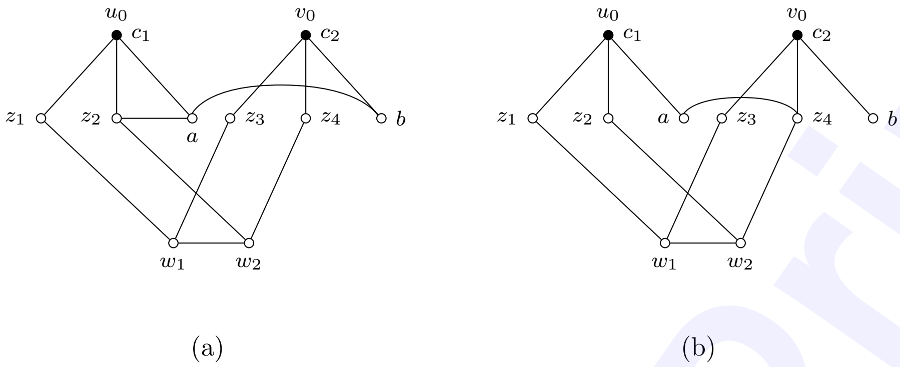


FIGURE 10

r can only move to a. Now move c_1 back to u_0. Since a is adjacent to b or z_i, the robber cannot move back to w_1 or w_2. Hence the robber will be caught.

This completes the proof of the lemma. □

Now, we are ready to prove Theorem 1.1.

Theorem 1.1. *If G is a connected graph with 10 vertices and $\Delta(G) \leq 3$, then $c_L(G) \leq 3$. Furthermore, equality holds if and only if G is the Petersen graph.*

Proof. By Lemma 2.1, $c_L(P(5,2)) = 3$. So it is sufficient to show that if G is not the Petersen graph $P(5,2)$, then $c_L(G) \leq 2$. If $\Delta(G) \leq 2$, then G is a path or a cycle, and thus, $c_L(G) \leq 2$. So we may assume that $\Delta(G) = 3$ and G is not the Petersen graph. By Lemma 3.1, we may further assume that there is a vertex $u_0 \in V(G)$ with $\deg(u_0) = 3$ and $J = G - N[u_0]$ is a 6-cycle. Note that each vertex in $V(J)$ is adjacent to at most one vertex in $N(u_0)$. Initially we may place two cops c_1 and c_2 at u_0. Note that the robber r can only remain in J as long as a cop is occupying u_0.

Case 1. Suppose there are two vertices $a, b \in V(J)$ such that a and b are not adjacent to any vertices in $N(u_0)$. We consider three cases where (i) a is adjacent to b in J, (ii) a and b are separated by a vertex in J or (iii) a and b are separated by two vertices in J.

(i) Suppose a is adjacent to b in J. We keep c_1 at u_0 and move c_2 into position as in Figure 11.

Note that r can only stay at a, b or v. In Figure 11(a), v is not adjacent to any vertices in $N(u_0)$. So we move c_1 towards v, one step at a time. We keep c_2 at his position. At each robber's turn, he can only remain at a, b or v. Thus he will be caught by c_1. In Figure 11(b),

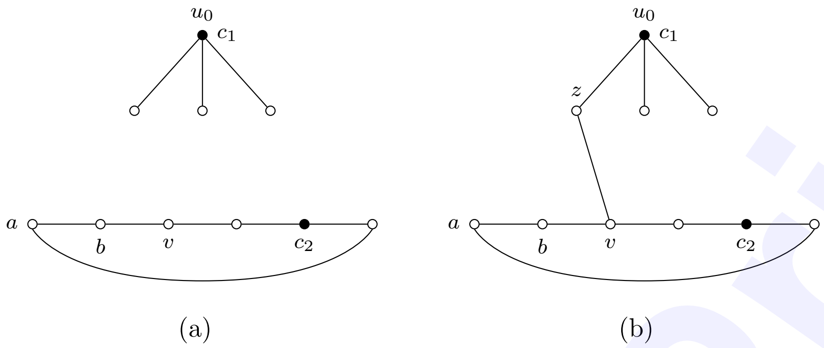


FIGURE 11

v is adjacent to the vertex $z \in N(u_0)$. So we move c_1 to z and then from z to v. The robber will also be caught.

(ii) The case where a and b are separated by a vertex is almost identical to case (i). The cop's winning strategy can be argued analogously, starting with keeping c_1 at u_0 and moving c_2 into position as in Figure 12.

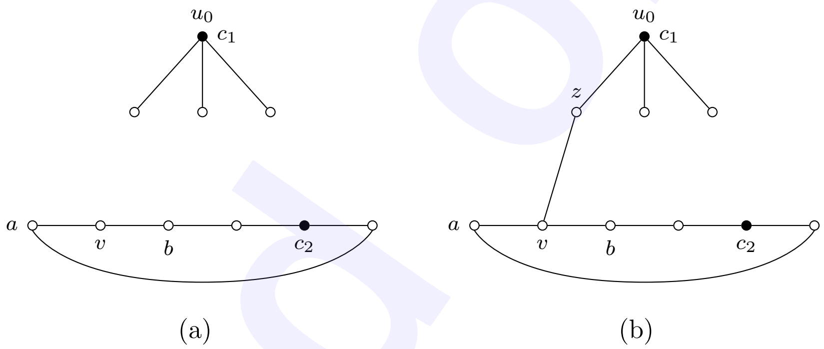


FIGURE 12

(iii) Suppose a and b are separated by two vertices. Each $w_i \in V(J)\backslash\{a, b\}$ is adjacent to a vertex in $N(u_0)$, or else we would be in the situation of Case (i) or (ii). Thus there is a vertex $z \in N(u_0)$ that is adjacent to two vertices w_1 and w_2 in J. There are three possibilities (see Figure 13). We keep c_1 at u_0 and move c_2 into position as in Figure 13.

Note that r can only stay in a, x or y. In Figure 13(a), w_1 and w_2 are adjacent. We move c_1 to z and then from z to x. The robber will

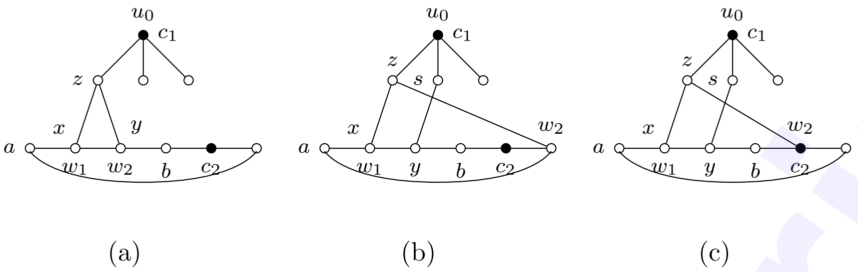


FIGURE 13

be caught. In Figure 13(b), w_1 and w_2 are separated by one vertex in J. We move c_2 to w_2, and so r can only stay in $\{x, y, b\}$. Then we move c_1 to s and then from s to y. The robber will be caught. In Figure 13(c), w_1 and w_2 are separated by two vertices in J. We move c_1 to s, and then from s to y. Now r can only stay in a. We move c_1 from y to x and r will be caught.

Henceforth, we may assume there is at most one $v \in J$ such that $N(v) \cap N(u_0) = \emptyset$. So, with 5 or 6 vertices in J, each having exactly 1 neighbor in $N(u_0)$, there must be at least one $z \in N(u_0)$ which has 2 neighbors in J.

Case 2. Suppose there is a $z \in N(u_0)$ with $N(z) \cap V(J) = \{a, b\}$ such that (i) a is adjacent to b in J or (ii) a and b are separated by a vertex in J.

(i) Suppose a is adjacent to b in J. We keep c_1 at u_0 and move c_2 into position as in Figure 14.

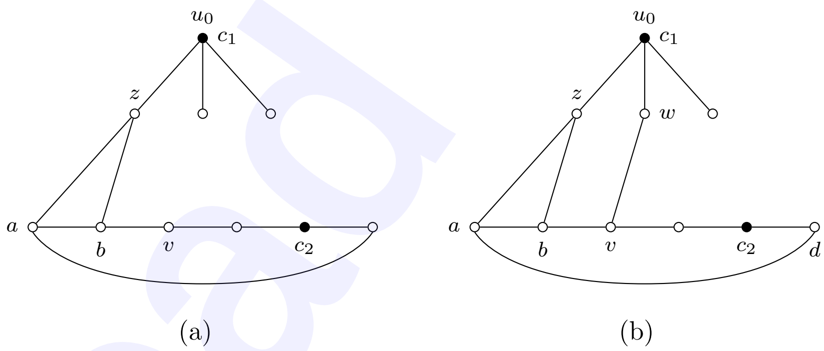
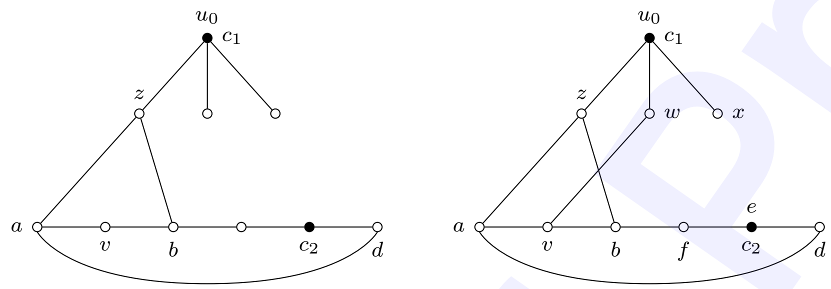


FIGURE 14

Note that r can only stay at a, b or v. In Figure 14(a), v is not adjacent to any vertices in $N(u_0)$. So we move c_1 to z and then from z

to b. The robber will be caught. In Figure 14(b), v is adjacent to the vertex $w \in N(u_0)$. So we move c_1 to w. Note that r can only stay at a, b or z. Next, move c_2 to d and then from d to a. The robber will be caught.

(ii) Suppose a and b are separated by a vertex. We keep c_1 at u_0 and move c_2 into position as in Figure 15.



(a) (b)

FIGURE 15

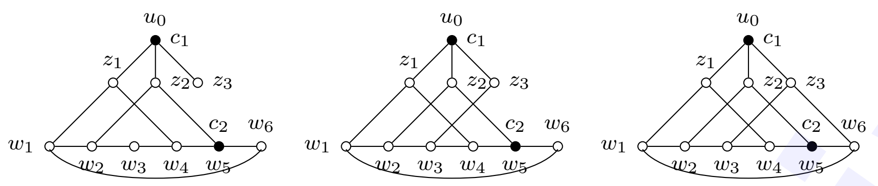
In Figure 15(a), v is not adjacent to any vertices in $N(u_0)$. So, we move c_1 to z. Note that r can only remain at v. Now move c_2 to d and then from d to a. The robber will be caught.

In Figure 15(b), v is adjacent to the vertex $w \in N(u_0)$.

- Suppose w is not adjacent to any vertices in $V(J)$ except v. By Case 1, we may assume that x is adjacent to exactly 2 vertices in $\{d, e, f\}$. Therefore, $\deg_G(w) = 2$. We move c_1 to z. Note that r can only remain at v or w. Next, move c_2 to f and then from f to b and from b to v. The robber will be caught.
- Suppose w is adjacent to a vertex y in $V(J)$. Note that $y \in \{d, e, f\}$. We move c_1 to z. Note that r can only remain at v or w. Next, move c_2 to y and then from y to w. The robber will be caught.

By Case 1 and 2, we may assume that if there is a $z \in N(u_0)$ with $N(z) \cap V(J) = \{a, b\}$, then a and b are separated by exactly 2 vertices in J (a and b are of distance 3 in J). We deduce that G is isomorphic to one of the three graphs shown in Figure 16. Note that Figure 16(c) is the Petersen graph.

In Figure 16(a), w_3 is not adjacent to z_3. We move c_1 to z_1. Then r can only stay at w_2 or w_3. Next, move c_2 to z_2 and then from z_2 to w_2. The robber will be caught. In Figure 16(b), w_3 is adjacent to z_3. Suppose z_3 is not adjacent to w_6. We move c_1 to z_1. Then r can only stay at w_2, w_3 or z_3. Next, move c_2 to



(a) (b) (c)

FIGURE 16

z_2. Then r can only stay at w_3 or z_3. Next, move c_2 from z_2 to w_2 and then from w_2 to w_3. The robber will be caught.

This completes the proof of the theorem. $\square$

4. Proof of Theorem 1.2

Here, we provide some known results and prove the following lemmas which will be useful in proving Theorem 1.2.

Theorem 4.1 ([7, Theorem 3.1])**.** *If G is a connected graph on at most 8 vertices, then $c_L(G) \leq 2$.*

Theorem 4.2 ([7, Theorem 2.4])**.** *The graph $G = K_3 \square K_3$ is the unique graph on 9 vertices with $c_L(G) = 3$. All other graphs H on 9 vertices have $c_L(H) \leq 2$.*

Lemma 4.3. *If G is a connected graph with $\Delta(G) \leq 2$, then $c_L(G) \leq 2$.*

Proof. Since $\Delta(G) \leq 2$, G is a path or a cycle. Hence $c_L(G) \leq 2$. $\square$

Lemma 4.4. *If G is a connected graph on n vertices with $\Delta(G) \leq 3$, then $c_L(G) \leq \max\left(3, \lfloor \frac{n}{4} \rfloor\right)$.*

Proof. Let $\lfloor \frac{n}{4} \rfloor = t$. We shall show that the lemma holds by using induction on t. If $t = 1$, then $n \leq 7$ and the lemma follows from Theorem 4.1. Assume that the lemma holds for all $1 \leq t < m$. We shall show that the lemma also holds for $t = m$; that is, we shall show it holds for $n = 4m + q$, where $0 \leq q \leq 3$.

Let $u \in V(G)$ be of degree 3. If $\Delta(G - N[u]) \leq 2$, then by Lemma 4.3, $c_L(G - N[u]) \leq 2$. Thus, $c_L(G) \leq 3$. So we may assume $\Delta(G - N[u]) = 3$. The number of vertices in $G - N[u]$ is $n' = 4(m-1) + q$. If $m - 1 \geq 3$, then by induction, $c_L(G - N[u]) \leq m - 1$, and hence $c_L(G) \leq m$, the lemma holds. So we may assume $m \leq 3$, i.e., G is a graph with at most 15 vertices. We shall show that 3 cops are enough to catch the robber.

Let $S \subseteq V(G)$ be the set of all vertices of degree 3. A subset $M \subseteq S$ is said to be *independent* if $N[s] \cap N[s'] = \emptyset$ for all $s, s' \in M$. $M \subseteq S$ is a maximal independent set if $|M|$ is of the largest size. Note that $|M| \leq 3$, as $|V(G)| \leq 15$.

Case 1. Suppose $|M| = 3$.

Let $u_1, u_2, u_3 \in M$. Initially, we place c_i at u_i for $i = 1, 2, 3$ (see Figure 17).

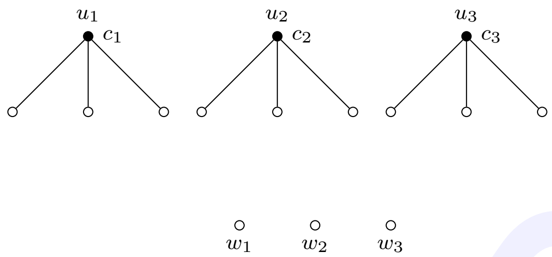


FIGURE 17

Let r be in a component J in $G - N[u_1, u_2, u_3]$. Let $\{w_j\} \in V(J)$ for some $j = 1, 2, 3$. Since $\deg_G(w_j) \le 3$, for any possible graph of J, $|N(J) \cap N(u_i)| \le 1$ for some $i = 1, 2, 3$. We may assume $|N(J) \cap N(u_1)| \le 1$. Now we move c_1 to r in J one step at a time or via the vertex of $N(J) \cap N(u_1)$ if exists. In the latter scenario, as c_1 moves into $V(J)$ via that common neighbor, the robber cannot sneak around and then escape via that neighbor. This is because $|V(J)| \le 3$. The robber r will have to remain in J as long as c_2 and c_3 are occupying u_2 and u_3, respectively. The robber will be caught.

Case 2. Suppose $|M| = 2$.

Let $u_1, u_2 \in M$. Initially we place c_1 at u_1 and c_2, c_3 at u_2. Let r be in a component J in $G - N[u_1] - N[u_2]$.

(i) Suppose J is a path or a 3-cycle. Then we keep c_1 and c_2 at their positions and use c_3 to catch the robber in J. The robber will be caught because $c_L(J) = 1$.

(ii) Suppose J is a t-cycle, $t = 4, 5, 6$ with vertex set $\{w_1, w_2, \ldots, w_t\}$ and edge set $\{w_j w_{j+1}\}$ where the subscripts are taken modulo t. We move c_3 to a vertex w_{t-1} in J as in Figure 18.

Note that $|N(w_1, w_2, w_3) \cap N(u_i)| \le 1$ for some $i = 1, 2$. We may assume $|N(w_1, w_2, w_3) \cap N(u_1)| \le 1$. If $|N(w_1, w_2, w_3) \cap N(u_1)| = 0$, we move c_1 towards w_2 one step at a time. If $|N(w_1, w_2, w_3) \cap N(u_1)| = 1$, we move c_1 towards w_2 through that common vertex. We keep c_2 and c_3 at their positions all the while. The robber can only remain at w_1, w_2 or w_3. So he will be caught.

(iii) Suppose J is a 7-cycle with vertex set $\{w_1, w_2, \ldots, w_7\}$ and edge set $\{w_j w_{j+1}\}$ where the subscripts are taken modulo 7.

Suppose there exists a vertex in the 7-cycle that is not adjacent to $N(u_1, u_2)$. We may assume w_4 is not adjacent to $N(u_1, u_2)$. Then we move c_3 to w_6 in J as in Figure 19. Note that $|N(w_1, w_2, w_3) \cap N(u_i)| \le 1$ for some $i = 1, 2$. We may assume $|N(w_1, w_2, w_3) \cap N(u_1)| \le 1$. Then

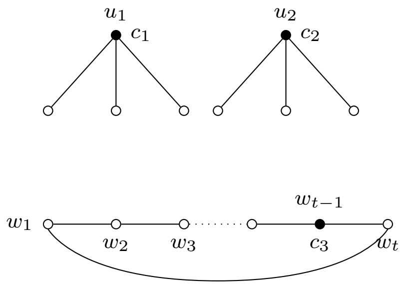


FIGURE 18

we move c_1 similarly as in Case 2(ii). Then, we move c_1 to w_3 if the robber is at w_4. The robber can only remain at w_1, w_2, w_3 or w_4 as c_2 and c_3 remain throughout the game. So he will be caught.

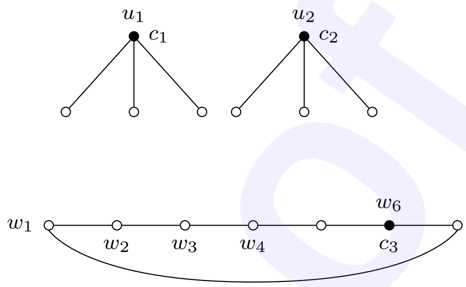


FIGURE 19

Now suppose each vertex in J is adjacent to $N(u_1, u_2)$. By the Pigeonhole Principle, $N(u_1)$ or $N(u_2)$ has at least 4 neighbors in J; suppose, without loss of generality, that $N(u_1)$ does. Since J is a 7-cycle, then the Pigeonhole Principle further tells us that there are four consecutive vertices on the 7-cycle such that at least three of them have neighbors in $N(u_1)$. Let $\{w_1, w_2, w_3, w_4\}$ be those four consecutive vertices. This means $|N(w_1, w_2, w_3, w_4) \cap N(u_2)| \leq 1$; let x be that common neighbor, if it exists. We now move c_3 to w_6. The robber must be in $\{w_1, w_2, w_3, w_4\}$. Then we move c_2 towards r one step at a time, via that vertex x (if it exists), while c_1 and c_3 remain still. The robber must be caught.

Case 3. Suppose $|M| = 1$. Let $u \in M$. Then $\Delta(G - N[u]) \leq 2$. By Lemma 4.3, $c_L(G - N[u]) \leq 2$. Hence $c_L(G) \leq 3$.

This completes the proof. □

The following lemma is a direct modification of Lemma 3.2 in [7], and the proof is essentially the same.

Lemma 4.5. *If G is a connected graph on n vertices with $\Delta(G) \geq n-9$, then $c_L(G) \leq 3$.*

Proof. Place a cop at a vertex u with $\deg(u) = \Delta(G)$ and keep it stationary at all time. Then by Theorem 4.1, two cops are sufficient to catch the robber in any component of $G - N[u]$. □

Lemma 4.6. *Let G be a connected graph with 15 vertices and there is at least one vertex of degree 4. If $G - N[u]$ is the Petersen graph for all $u \in V(G)$ with $\deg_G(u) = 4$, then $\Delta(G) \geq 5$.*

Proof. It is sufficient to show that there is a vertex in $V(G)$ with degree 5. Let $u_1 \in V(G)$ with $\deg_G(u_1) = 4$. Since G is connected, there is a vertex u_2 in $N(u_1)$ adjacent to a vertex v_1 in $V(G - N[u_1])$. We may assume the graph is as in Figure 20.

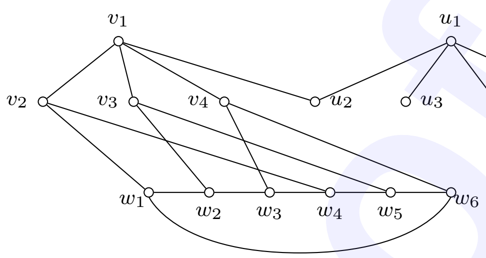


FIGURE 20

Now consider $G - N[v_1]$ (see Figure 21). Since the resulting graph must be the Petersen graph, we may assume u_3 is adjacent to w_1 and w_4, u_4 is adjacent to w_2 and w_5, and u_5 is adjacent to w_3 and w_6 (see Figure 22).

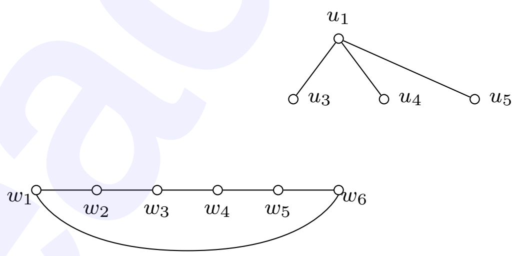


FIGURE 21

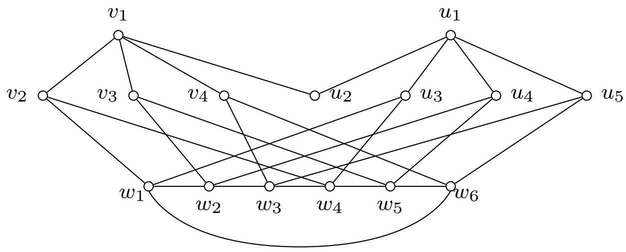


FIGURE 22

Now consider $G - N[w_1]$ (see Figure 23). Since the resulting graph is the Petersen graph, w_4 must be adjacent to u_2. Hence, $\deg_G(w_4) = 5$. □

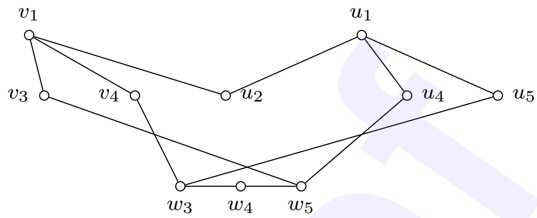


FIGURE 23

We are now ready to prove Theorem 1.2.

Theorem 1.2. *If G is a connected graph with at most 15 vertices, then $c_L(G) \leq 3$.*

Proof. We first consider the case when $|V(G)| \leq 14$. By Lemmas 4.3, 4.4 and 4.5, we shall only need to deal with the case when $\Delta(G) = 4$. Let u be a vertex in G with degree 4. Observe that $G - N[u]$ has at most 9 vertices and that $G - N[u]$ is not the graph $K_3 \square K_3$. So by Theorems 4.1 and 4.2, $c_L(G - N[u]) \leq 2$, implying $c_L(G) \leq 3$.

We now assume that $|V(G)| = 15$. If $\Delta(G) \leq 3$, then by Lemmas 4.3 and 4.4, $c_L(G) \leq 3$. If $\Delta(G) \geq 6$, then by Lemma 4.5, $c_L(G) \leq 3$. Suppose $\Delta(G) = 5$. Let $u \in V(G)$ with $\deg_G(u) = 5$. Initially, place all the three cops c_1, c_2 and c_3 at u. Then the robber r must be at one of the components in $G - N[u]$, say H. Note that r has to remain in H as long as there is a cop occupying u. If H has at most 8 vertices, then by Theorem 4.1, $c_L(H) \leq 2$. If H has 9 vertices and $H \neq K_3 \square K_3$, then by Theorem 4.2, $c_L(H) \leq 2$. In either case, we keep c_1 at u and use c_2 and c_3 to catch the robber in H. Suppose $H = K_3 \square K_3$. There is a vertex $w \in V(H)$ with $\deg_G(w) = 5$ because H is 4-regular and G is connected. Since $K_3\square K_3$ is vertex transitive, we may assume w is the vertex at the top left of $K_3\square K_3$. Now keep c_1 at u and move c_2 to the center vertex of $K_3\square K_3$. After that, move c_3 to the vertex at the bottom right vertex of $K_3\square K_3$. Note that r can only stay at w. Now move c_1 to w through the only vertex in $N(u)\cap N(w)$. The robber will be caught.

So we are only left with the case when $\Delta(G)=4$. Let $S\subseteq V(G)$ be the set of all vertices of degree 4. A subset $M\subseteq S$ is said to be independent if $N[w]\cap N[w']=\emptyset$ for all $w,w'\in M$ with $w\neq w'$. $M\subseteq S$ is a maximal independent set if $|M|$ is of the largest size. Note that $|M|\leq 3$. We shall show that three cops c_1,c_2,c_3 are sufficient to catch the robber for each of the possible size of M.

Case 1. Suppose $|M|=3$.

Let $w_1,w_2,w_3\in M$. Place c_i at w_i for $i=1,2,3$. Since $V(G)=N[w_1]\cup N[w_2]\cup N[w_3]$, the robber will be caught.

Case 2. Suppose $|M|=2$.

Let $w_1,w_2\in M$. Place c_1 and c_3 at w_1 and c_2 at w_2. The robber r must be at one of the components in $G-N[w_1]-N[w_2]$, say J. Since $|M|=2$, $\Delta(J)\leq 3$. Note that r has to remain in J as long as w_1 and w_2 are occupied by cops.

(i) $\Delta(J)=3$.

Let $a\in V(J)$ with $\deg_J(a)=3$. We keep c_1 and c_2 at w_1 and w_2, respectively, and move c_3 to a. Since $|N[w_1]\cup N[w_2]\cup N[a]|=14$, r must be at the remaining vertex, say b. Since $\deg_G(b)\leq 4$, there is a c_i $(1\leq i\leq 3)$ such that $|N(b)\cap N(c_i)|\leq 1$. Now move c_i to b one step at a time or via the vertex in $N(b)\cap N(c_i)$ (if exists). The robber will be caught.

(ii) $\Delta(J)\leq 2$. Then J is a path or a s-cycle where $s\leq 5$.

- If J is a path or a 3-cycle, then we keep c_1 and c_2 at w_1 and w_2, respectively, and use c_3 to catch the robber in J.
- If J is a 4-cycle, then we keep c_1 and c_2 at w_1 and w_2, respectively, and move c_3 to a vertex in J. Note that r must be at the remaining vertex in J, say b (see Figure 24).

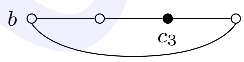


FIGURE 24

Since $\deg_G(b)\leq 4$, there is a c_i $(1\leq i\leq 2)$ such that $|N(b)\cap N(c_i)|\leq 1$. Now move c_i to b one step at a time (if $|N(b)\cap N(c_i)|=0$) or via the vertex in $N(b)\cap N(c_i)$ (if $|N(b)\cap N(c_i)|=1$). The robber will be caught.

- If J is a 5-cycle, then we keep c_1 and c_2 at w_1 and w_2, respectively, and move c_3 to a vertex in J (see Figure 25).

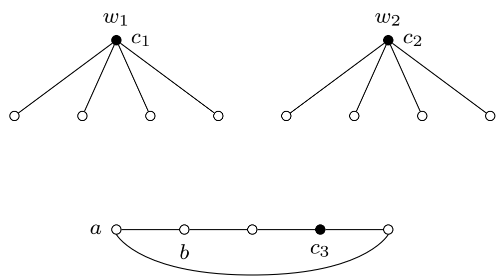


FIGURE 25

If there is no edge connecting a vertex in $\{a, b\}$ with a vertex in $N(c_1)$, then we move c_1 to a one step at a time. Note that r can stay at a or b only, as long as c_2 and c_3 are at their positions. So the robber will be caught. Hence we may assume there is an edge connecting a vertex in $\{a, b\}$ with a vertex in $N(c_1)$. Without loss of generality, we may assume a is adjacent to a vertex z in $N(c_1)$ (see Figure 26).

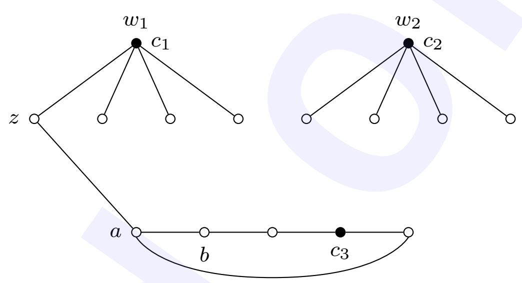


FIGURE 26

Suppose $N(a) \cap N(c_2) = \emptyset$.

i. If $|N(b) \cap N(c_2)| \leq 1$, then keep c_1 and c_3 at their positions and move c_2 to b one step at a time (if $|N(b) \cap N(c_2)| = 0$) or via the vertex in $N(b) \cap N(c_2)$ (if $|N(b) \cap N(c_2)| = 1$). The robber will be caught.

ii. If $|N(b) \cap N(c_2)| = 2$, then $N(b) \cap N(c_1) = \emptyset$. If $|N(a) \cap N(c_1)| = 1$, move c_1 to a via z. The robber will be caught. For if $|N(a) \cap N(c_1)| = 2$ (see Figure 27), we move c_2 to b via x if the robber is at a and move c_1 to a via z if the robber is at b. In either case, the robber will be caught.

So we may assume $|N(a) \cap N(c_2)| = 1$ (see Figure 28).

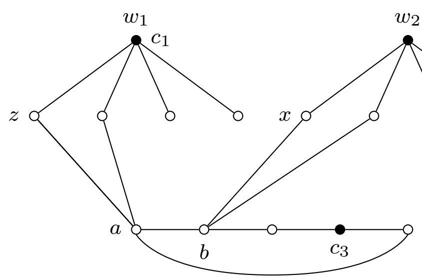


FIGURE 27

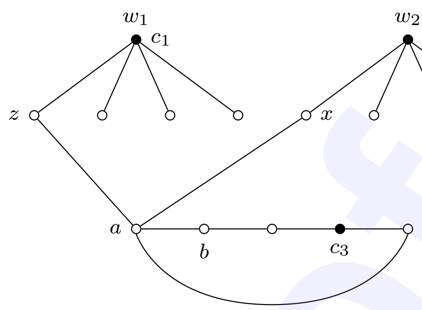


FIGURE 28

If $|N(b) \cap N(c_2)| = 0$, then keep c_1 and c_3 at their positions and move c_2 to a via x. If $|N(b) \cap N(c_1)| = 0$, then keep c_2 and c_3 at their positions and move c_1 to a via z. In either case, the robber will be caught. So we may assume $|N(b) \cap N(c_i)| = 1$ for $i = 1, 2$. If b is adjacent to z, then keep c_2 and c_3 at their positions and move c_1 to z. The robber will be caught. So we may assume b is not adjacent to z. Similarly, we may assume b is not adjacent to x (see Figure 29).

Note that r can be at a or b. We shall assume r is at a. The case r is at b is similar. Move c_2 to x. Then r will have to move to b. Next, move c_1 to e. Then r will have to move to f. Now, move c_2 back to w_2. If f is not adjacent to a vertex in $N(w_1)\backslash\{e\}$, then r will be caught in the next cop's turn. If f is adjacent to a vertex in $N(w_1)\backslash\{e\}$, then r will have to move from f to a vertex in $N(w_1)\backslash\{e\}$. Now, move c_1 back to w_1. At robber's turn, if r is not at z, he will be caught in the next cop's turn. So r must be at z and f is adjacent to z.

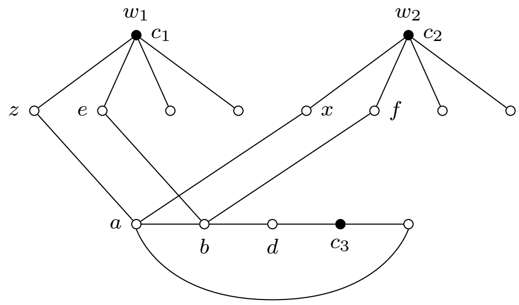


FIGURE 29

Now, reset the movements and assume r is at a. Move c_1 to z. Then r will have to move to b. Next, move c_2 to f. Then r will have to move to e. Now, move c_1 back to w_1. If e is not adjacent to a vertex in $N(w_2)\backslash\{f\}$, then r will be caught in the next cop's turn. If e is adjacent to a vertex in $N(w_2)\backslash\{f\}$, then r will have to move from e to a vertex in $N(w_2)\backslash\{f\}$. Now, move c_2 back to w_2. At robber's turn, if r is not at x, he will caught in the next cop's turn. So r must be at x and e is adjacent to x (see Figure 30).

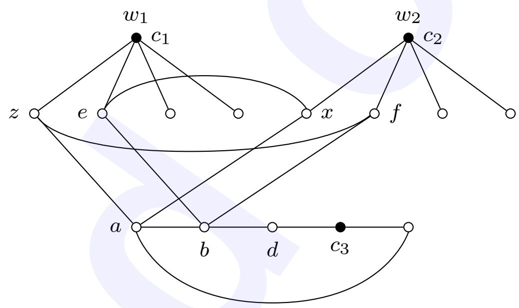


FIGURE 30

Reset the movements and assume r is at a. Now, move c_2 to x. Then r will have to move to b. Next, move c_1 to z. Then r will have to remain at b. Move c_3 to d. The robber will be caught.

Case 3. Suppose $|M| = 1$.

Then $\Delta(G - N[u]) \leq 3$ for all $u \in V(G)$ with $\deg_G(u) = 4$.

Suppose there is a vertex $w \in V(G)$ with $\deg_G(w) = 4$ such that $G - N[w]$ is not connected. Place all the cops at w. The robber r must be at one of the components in $G - N[w]$, say J. If $\Delta(J) \leq 2$, then by Lemma 4.3, $c_L(J) \leq 2$. So we keep one cop at w and use the other two cops to catch the robber in J. Similarly, by Theorems 4.1 and 4.2, we may assume $J = K_3 \Box K_3$ or $|V(J)| = 10$. The former cannot happen because $|M| = 1$. The latter also cannot happen because $G - N[w]$ is not connected.

So we may assume that $G - N[u]$ is connected for all $u \in V(G)$ with $\deg_G(u) = 4$. If there is a vertex $v \in V(G)$ with $\deg_G(v) = 4$ such that $G - N[v]$ is not the Petersen graph, then by Theorem 1.1, $c_L(G - N[v]) \leq 2$. Hence we keep one cop at v and use the other two cops to catch the robber in $G - N[v]$.

Now we may assume that $G - N[u]$ is the Petersen graph for all $u \in V(G)$ with $\deg_G(u) = 4$. By Lemma 4.6, $\Delta(G) \geq 5$, a contradiction.

Hence $c_L(G) \leq 3$ and this completes the proof of the theorem. □

Corollary 4.7. *If G is a connected graph with n vertices and $\Delta(G) \geq n - 16$, then $c_L(G) \leq 4$.*

Proof. Let $u \in V(G)$ with $\deg(u) = \Delta(G)$. Place all the four cops at u. The robber r must be at a component in $G - N[u]$, say H. Note that $|V(H)| \leq 15$. By Theorem 1.2, $c_L(H) \leq 3$. So we keep one cop at u and use the other three cops to catch the robber in H. □

5. Remarks

In this paper, we proved in Theorem 1.2 that if a connected graph has lazy cop number 4, then it must have at least 16 vertices. We also know from [7] that the graph $K_4 \Box K_4$ has lazy cop number 4. However we are unable to show that there is no other connected graph of order 16 with lazy cop number 4.

Question 5.1. *Is it true that $K_4 \Box K_4$ the unique graph of order 16 with lazy cop number 4?*

In fact, the general case of the above question is conjectured in [7], and is still an open problem.

Conjecture 5.2 ([7], Conjecture 5.1)**.** *The graph $K_n \Box K_n$ is the unique smallest graph with $c_L = n$.*

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