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ON A WARING-GOLDBACH PROBLEM INVOLVING SQUARES, CUBES AND BIQUADRATES

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Abstract. Let P_r denote an almost-prime with at most r prime factors, counted according to multiplicity. In this paper, it is proved that for every sufficiently large even integer N, the equation

$$N = x^2 + p_1^2 + p_2^3 + p_3^3 + p_4^4 + p_5^4$$

is solvable with x being an almost-prime P_4 and the other variables primes. This result constitutes an improvement upon that of Lü [7].

1. Introduction

Let N, k_1, k_2, $\ldots$, k_s be natural numbers such that $2 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_s, N > s$. Waring's problem of mixed powers concerns the representation of N as the form

$$N = x_1^{k_1} + \cdots + x_s^{k_s}. \tag{1.1}$$

Not very much is known about results of this type. For references in this aspect, we refer the reader to section P12 of LeVeque's *Reviews in number theory*, the bibliography in Vaughan [9] and the recent papers by J. Brüdern and by T. D. Wooley.

In principle the Hardy-Littlewood method is applicable to problems of this kind, but one has to overcome various difficulties not experienced in the pure Waring's problem (1.1) with $k_1 = k_2 = \cdots = k_s$. In particular, the choice of the relevant parameters in the definitions of major and minor arcs tends to become complicated if a deeper representation problem (1.1) is under consideration.

In 1969, Vaughan [8] investigated the equation

$$x_1^2 + x_2^2 + x_3^3 + x_4^3 + x_5^4 + x_6^4 = N.$$

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He proved that for any sufficiently large integer N, the following asymptotic formula

$$\sum_{x_1^2+x_2^2+x_3^3+x_4^3+x_5^4+x_6^4=N} 1 = \frac{\Gamma^2(\frac{3}{2})\Gamma^2(\frac{4}{3})\Gamma^2(\frac{5}{4})}{\Gamma(\frac{13}{6})}\widetilde{\mathfrak{S}}(N)N^{\frac{7}{6}} + O(N^{\frac{7}{6}-\frac{1}{96}+\varepsilon})$$

holds, where

$$\widetilde{\mathfrak{S}}(N) = \sum_{q=1}^{\infty} q^{-6} \sum_{\substack{a=1\\(a,q)=1}}^{q} S_2^2(q,a)S_3^2(q,a)S_4^2(q,a)e\left(\frac{-aN}{q}\right),$$

$$S_k(q,a) = \sum_{r=1}^{q} e\left(\frac{ar^k}{q}\right), \quad e(\alpha) = e^{2\pi i\alpha}.$$

Let P_r denote an almost-prime with at most r prime factors, counted according to multiplicity. In 2015, motivated by Brüdern [1,2], Lü [7] proved that for every sufficiently large even integer N, the equation

$$N = x^2 + p_1^2 + p_2^3 + p_3^3 + p_4^4 + p_5^4 \tag{1.2}$$

is solvable with x being an almost-prime P_6 and the $p_j (j = 1,2,3,4,5)$ primes.

In this paper, we shall obtain the following sharper result.

Theorem. *For every sufficiently large even integer N, the number of solutions of the equation*

$$N = x^2 + p_1^2 + p_2^3 + p_3^3 + p_4^4 + p_5^4$$

with x being an almost-prime P_4 and the other variables primes, is

$$\gg \frac{N^{\frac{7}{6}}}{\log^6 N}.$$

In the proof of the Theorem, we shall employ the Hardy-Littlewood method and the linear sieve theory. The improvement of our Theorem upon that of Lü [7] stems from the use of the linear sieve theory with the bilinear error term instead of the linear sieve theory with the linear error term utilized by Lü [7].

2. Notation and some preliminary lemmas

Throughout this paper, $\varepsilon \in (0, 10^{-10})$. By N we denote a sufficiently large even integer in terms of ε. The letter p, with or without subscript, is reserved for a prime number. The constants in O-term and $\ll$-symbol depend at most on ε. By $A \sim B$ we mean that $B < A \leqslant 2B$. We denote by (m,n) the greatest common divisor of m and n. By $\tau(n)$ we denote the divisor function. As usual, $\varphi(n)$ stands for Euler's function. We use $e(\alpha)$ to denote $e^{2\pi i\alpha}$ and $e_q(\alpha) = e(\alpha/q)$. By $a(m)$, $b(n)$ we denote arithmetic functions satisfying $|a(m)| \leqslant 1$ and

$|b(n)| \leqslant 1$. We denote by $\sum\limits_{r(q)}$ and $\sum\limits_{r(q)^*}$ sums with r running over a complete system and a reduced system of residues modulo q respectively. We set

$$A = 10^{10},\ \ Q_0 = \log^{20A} N,\ \ Q_1 = N^{\frac{1}{3}+10\varepsilon},\ \ Q_2 = N^{\frac{1}{2}},$$

$$D = N^{\frac{1}{8}-10\varepsilon},\ \ z = D^{\frac{1}{3}},\ \ U_k = 0.5N^{\frac{1}{k}},$$

$$\mathcal{M}_r = \{m \,|\, m \sim U_2, m = p_1p_2\cdots p_r, z \leqslant p_1 \leqslant p_2 \leqslant \cdots \leqslant p_r\}\ (5 \leqslant r \leqslant 12),$$

$$\mathcal{N}_r = \{n \,|\, n = p_1\cdots p_{r-1}, z \leqslant p_1 \leqslant p_2 \leqslant \cdots \leqslant p_{r-1},\ p_1\cdots p_{r-2}p_{r-1}^2 \leqslant 2U_2\}\quad (5 \leqslant r \leqslant 12),$$

$$f_k(\alpha) = \sum_{p\sim U_k} (\log p) e(\alpha p^k),\ \ g_r(\alpha) = \sum_{\substack{n\in\mathcal{N}_r,\\ np\sim U_2}} e(\alpha(np)^2)\frac{\log p}{\log\frac{U_2}{n}},$$

$$S_k^*(q,a) = \sum_{r(q)^*} e_q(ar^k),\ \ S_k(q,a) = \sum_{r(q)} e_q(ar^k),$$

$$B_d(q,N) = \sum_{a(q)^*} S_2(q,ad^2)S_2^*(q,a){S_3^*}^2(q,a){S_4^*}^2(q,a)e_q(-aN),$$

$$A_d(q,N) = \frac{B_d(q,N)}{q\varphi^5(q)},\ \ \mathfrak{S}_d(N) = \sum_{q=1}^{\infty} A_d(q,N),\ \ \mathfrak{S}(N) = \mathfrak{S}_1(N).$$

For $\alpha = \dfrac{a}{q} + \beta$, let

$$u_k(\beta) = \int_{U_k}^{2U_k} e(\beta u^k)\,\mathrm{d}u,\quad U_k(\alpha) = \frac{S_k^*(q,a)}{\varphi(q)}u_k(\beta),$$

$$W(\alpha) = \sum_{m\leqslant D^{\frac{2}{3}}, n\leqslant D^{\frac{1}{3}}} \frac{a(m)b(n)}{mnq} S_2(q,am^2n^2)u_2(\beta),$$

$$\mathfrak{I}(N) = \int_{-\infty}^{\infty} u_2^2(\beta)u_3^2(\beta)u_4^2(\beta)e(-\beta N)\,\mathrm{d}\beta.$$

Lemma 2.1. *Let*

$$h(\alpha) = \sum_{m\leqslant D^{\frac{2}{3}}} a(m) \sum_{n\leqslant D^{\frac{1}{3}}} b(n) \sum_{l\sim \frac{U_2}{mn}} e(\alpha(mnl)^2). \tag{2.1}$$

Then for $\alpha \in \mathfrak{m}_2$, we have

$$h(\alpha) \ll N^{\frac{1}{3}-3\varepsilon}.$$

Proof. It follows from (4.6) in Brüdern and Kawada [3] that

$$h(\alpha) \ll \frac{N^{\frac{1}{2}+\varepsilon}}{q^{\frac{1}{2}}(1+N|\beta|)^{\frac{1}{2}}} + N^{\frac{1}{3}-3\varepsilon} \ll N^{\frac{1}{3}-3\varepsilon}.$$

□

For $(a,q)=1, 1 \leqslant a \leqslant q$, put

$$\mathfrak{M}_0(q,a) = \left(\frac{a}{q} - \frac{Q_0^5}{N}, \frac{a}{q} + \frac{Q_0^5}{N}\right], \ \mathfrak{M}_0 = \bigcup_{1 \leqslant q \leqslant Q_0^5} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \mathfrak{M}_0(q,a),$$

$$\mathfrak{M}(q,a) = \left(\frac{a}{q} - \frac{1}{qQ_2}, \frac{a}{q} + \frac{1}{qQ_2}\right], \ \mathfrak{M} = \bigcup_{1 \leqslant q \leqslant Q_0^5} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \mathfrak{M}(q,a),$$

$$\mathfrak{J}_0 = \left(-\frac{1}{Q_2}, 1 - \frac{1}{Q_2}\right], \ \mathfrak{m}_0 = \mathfrak{M} \setminus \mathfrak{M}_0,$$

$$\mathfrak{m}_1 = \bigcup_{Q_0^5 < q \leqslant Q_1} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \mathfrak{M}(q,a), \ \mathfrak{m}_2 = \mathfrak{J}_0 \setminus \left(\mathfrak{M} \bigcup \mathfrak{m}_1\right).$$

Then we have the Farey dissection

$$\mathfrak{J}_0 = \mathfrak{M}_0 \bigcup \mathfrak{m}_0 \bigcup \mathfrak{m}_1 \bigcup \mathfrak{m}_2. \tag{2.2}$$

Lemma 2.2. *For $\alpha = \frac{a}{q} + \beta \in \mathfrak{M}_0$, we have*

$$g_r(\alpha) = \frac{c_r U_2(\alpha)}{\log U_2} + O\left(U_2 \exp(-\log^{\frac{1}{3}} N)\right), \ 5 \leqslant r \leqslant 12, \tag{2.3}$$

where

$$\begin{aligned} c_r = &\ \big(1+O(\varepsilon)\big) \\ &\times \int_{r-1}^{11} \frac{\mathrm{d}t_1}{t_1} \int_{r-2}^{t_1-1} \frac{\mathrm{d}t_2}{t_2} \cdots \int_{3}^{t_{r-4}-1} \frac{\mathrm{d}t_{r-3}}{t_{r-3}} \int_{2}^{t_{r-3}-1} \frac{\log(t_{r-2}-1)\,\mathrm{d}t_{r-2}}{t_{r-2}}. \end{aligned} \tag{2.4}$$

Proof. It follows from the arguments used in the proof of Lemma 4 in Cai [4]. □

3. Mean value estimations

In this section, we give two propositions for the proof of the Theorem.

Proposition 3.1. *Define*

$$J_d(N) = \sum_{\substack{(dl)^2+p_1^2+p_2^3+p_3^3+p_4^4+p_5^4=N\\ dl\sim U_2,\ p_1\sim U_2\\ p_2\sim U_3,\ p_3\sim U_3\\ p_4\sim U_4,\ p_5\sim U_4}} \prod_{j=1}^{5} \log p_j.$$

Then we have

$$\sum_{m\leqslant D^{\frac{2}{3}}} a(m) \sum_{n\leqslant D^{\frac{1}{3}}} b(n) \left(J_{mn}(N) - \frac{\mathfrak{S}_{mn}(N)}{mn}\mathfrak{J}(N) \right) \ll \frac{N^{\frac{7}{6}}}{\log^A N}.$$

Proof. The proof of Proposition 3.1 follows from the arguments used in the proof of Lemma 3.1 in Lü [7] and Lemma 2.1. □

By Lemma 2.2 and arguments similar to that used in the proof of Proposition 3.1, we have:

Proposition 3.2. *For* $5 \leqslant r \leqslant 12$*, let*

$$J_d^{(r)}(N) = \sum_{\substack{(dl)^2+(np)^2+p_1^3+p_2^3+p_3^4+p_4^4=N \\ dl\sim U_2,\, np\sim U_2,\, n\in\mathcal{N}_r \\ p_1\sim U_3,\ p_2\sim U_3 \\ p_3\sim U_4,\, p_4\sim U_4}} \left(\frac{\log p}{\log \frac{U_2}{n}} \prod_{j=1}^{4} \log p_j \right).$$

Then we have

$$\sum_{m\leqslant D^{\frac{2}{3}}} a(m) \sum_{n\leqslant D^{\frac{1}{3}}} b(n) \left(J_{mn}^{(r)}(N) - c_r \frac{\mathfrak{S}_{mn}(N)}{mn \log U_2}\mathfrak{J}(N) \right) \ll \frac{N^{\frac{7}{6}}}{\log^A N},$$

where c_r *is defined by* (2.4).

4. Proof of the Theorem

In this section, $f(s)$ and $F(s)$ denote the classical functions in the linear sieve theory, and $\gamma = 0.577\cdots$ denotes Euler's constant. Then by (8.2.8) and (8.2.9) in Halberstam and Richert [5], we have

$$\begin{aligned} f(s) &= \frac{2e^{\gamma}\log(s-1)}{s}, && 2 \leqslant s \leqslant 4, \\ F(s) &= \frac{2e^{\gamma}}{s}, && 1 \leqslant s \leqslant 3. \end{aligned}$$

In the proof of the Theorem, we adopt the following notation:

$$\begin{aligned} &\omega(d) = \frac{\mathfrak{S}_d(N)}{\mathfrak{S}(N)}, \ \ \mathfrak{P} = \prod_{2<p<z} p, \\ &\mathfrak{N}(z) = \prod_{2<p<z} \left(1 - \frac{\omega(p)}{p}\right), \\ &\log \mathbf{U} = (\log U_2)(\log U_3)^2(\log U_4)^2, \\ &\log 2\mathbf{U} = (\log 2U_2)(\log 2U_3)^2(\log 2U_4)^2. \end{aligned}$$

It follows from Lemma 4.3 in Lü [7] that the function $\omega(d)$ is multiplicative, and

$$0 \leqslant \omega(p) < p, \quad \omega(p) = 1 + O(p^{-1})$$

for each prime p. Then by Mertens's prime number theorem, it is easy to see that

$$\mathfrak{N}(z) \asymp \frac{1}{\log N}. \tag{4.1}$$

Let $R(N)$ denote the number of solutions of the equation (1.2) with x being a P_4 and the other variables primes. Upon noting the fact that the conditions $l \sim U_2,\ (l, \mathfrak{P}) = 1$ imply that l has at most 12 prime factors, counted according to multiplicity, we have

$$\begin{aligned} R(N) &\geqslant \sum_{\substack{l^2+p_1^2+p_2^3+p_3^3+p_4^4+p_5^4=N \\ l\sim U_2, (l,\mathfrak{P})=1,\, p_1\sim U_2 \\ p_2\sim U_3,\, p_3\sim U_3 \\ p_4\sim U_4,\, p_5\sim U_4}} 1 - \sum_{r=5}^{12} \sum_{\substack{h^2+p_1^2+p_2^3+p_3^3+p_4^4+p_5^4=N \\ h\in\mathcal{M}_r, p_1\sim U_2 \\ p_2\sim U_3,\, p_3\sim U_3 \\ p_4\sim U_4,\, p_5\sim U_4}} 1 \\ &\geqslant \sum_{\substack{l^2+p_1^2+p_2^3+p_3^3+p_4^4+p_5^4=N \\ l\sim U_2, (l,\mathfrak{P})=1,\, p_1\sim U_2 \\ p_2\sim U_3,\, p_3\sim U_3 \\ p_4\sim U_4,\, p_5\sim U_4}} 1 - \sum_{r=5}^{12} \sum_{\substack{(np)^2+p_1^2+p_2^3+p_3^3+p_4^4+p_5^4=N \\ n\in\mathcal{N}_r, np\sim U_2, p_1\sim U_2 \\ p_2\sim U_3, p_3\sim U_3 \\ p_4\sim U_4,\, p_5\sim U_4}} 1 \\ &= \mathcal{R}(N) - \sum_{r=5}^{12} \mathcal{R}_r(N), \ \text{ say}, \end{aligned} \tag{4.2}$$

where the fact $\mathcal{M}_r \subseteq \{np \,|\, n \in \mathcal{N}_r, np \sim U_2\}$ is employed.

In the following subsections we shall give a non-trivial lower bound for $R(N)$ by the linear sieve theory with the bilinear error term.

4.1. The lower bound for $\mathcal{R}(N)$

Let

$$\mathcal{N}(l) = \sum_{\substack{l^2+p_1^2+p_2^3+p_3^3+p_4^4+p_5^4=N \\ p_1\sim U_2,\, p_2\sim U_3 \\ p_3\sim U_3,\, p_4\sim U_4 \\ p_5\sim U_4}} \prod_{j=1}^{5} \log p_j$$

and

$$\mathcal{E}(d) = \sum_{\substack{l\sim U_2 \\ l\equiv 0(\bmod d)}} \mathcal{N}(l) - \frac{\omega(d)}{d}\mathfrak{S}(N)\mathfrak{J}(N).$$

Then by Theorem 1 in Iwaniec [6] and Proposition 3.1, we get

$$\begin{aligned} \mathcal{R}(N) &\geqslant \frac{1}{\log 2\mathbf{U}} \sum_{\substack{l\sim U_2 \\ (l,\mathfrak{P})=1}} \mathcal{N}(l) \\ &\geqslant \left(1 + O\left(\log^{-\frac{1}{3}} D\right)\right) \frac{f(3)\mathfrak{S}(N)\mathfrak{J}(N)\mathfrak{N}(z)}{\log \mathbf{U}} + O\left(\frac{N^{\frac{7}{6}}}{\log^A N}\right). \end{aligned} \tag{4.3}$$

4.2. The upper bound for $\mathcal{R}_r(N)$ $(5 \leqslant r \leqslant 12)$

For $5 \leqslant r \leqslant 12$, let

$$\mathcal{N}_r(l) = \sum_{\substack{(np)^2+l^2+p_1^3+p_2^3+p_3^4+p_4^4=N \\ n\in\mathcal{N}_r, np\sim U_2, p_1\sim U_3 \\ p_2\sim U_3, p_3\sim U_4 \\ p_4\sim U_4}} \left(\frac{\log p}{\log \frac{U_2}{n}} \prod_{j=1}^{4} \log p_j \right)$$

and

$$\mathcal{E}_r(d) = \sum_{\substack{l\sim U_2 \\ l\equiv 0(\bmod d)}} \mathcal{N}_r(l) - \frac{c_r \omega(d)}{d \log U_2} \mathfrak{S}(N)\mathfrak{J}(N),$$

where c_r is defined by (2.4). Then by Theorem 1 in Iwaniec [6] and Proposition 3.2, for $5 \leqslant r \leqslant 12$, we have

$$\begin{aligned} (4.4)\quad \mathcal{R}_r(N) &\leqslant \frac{\log U_2}{\log \mathbf{U}} \sum_{\substack{l\sim U_2 \\ (l,\mathfrak{P})=1}} \mathcal{N}_r(l) \\ &\leqslant \left(1 + O\left(\log^{-\frac{1}{3}} D\right)\right) \frac{F(3) c_r \mathfrak{S}(N)\mathfrak{J}(N)\mathfrak{N}(z)}{\log \mathbf{U}} + O\left(\frac{N^{\frac{7}{6}}}{\log^A N}\right). \end{aligned}$$

4.3. Proof of the Theorem

By numerical integration, we have

$$(4.5)\qquad c_5 < 0.2215, \quad c_r < 0.0280 \text{ for } 6 \leqslant r \leqslant 12$$

and

$$(4.6)\qquad \sum_{r=5}^{12} c_r < 0.4175.$$

We conclude from (4.1)-(4.4) and (4.6) that

$$\begin{aligned} (4.7)\qquad R(N) &\geqslant (0.6931 - 0.4175) \frac{2e^\gamma}{3} \frac{\mathfrak{S}(N)\mathfrak{J}(N)\mathfrak{N}(z)}{\log \mathbf{U}} + O\left(\frac{N^{\frac{7}{6}}}{\log^A N}\right) \\ &\gg \frac{N^{\frac{7}{6}}}{\log^6 N}, \end{aligned}$$

where (3.17) and Lemma 4.2 in Lü [7] are employed. Now, by (4.7), the proof of the Theorem is completed.

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