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MODIFIED SUBGRADIENT EXTRAGRADIENT ALGORITHM FOR PSEUDOMONOTONE EQUILIBRIUM PROBLEMS

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Abstract. The paper introduces a modified subgradient extragradient method for solving equilibrium problems involving pseudomonotone and Lipschitz-type bifunctions in Hilbert spaces. Theorem of weak convergence is established under suitable conditions. Several experiments are implemented to illustrate the numerical behavior of the new algorithm and compare it with a well known extragradient method.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : H \times H \to \Re$ be a bifunction with $f(x,x) = 0$ for all $x \in C$. The equilibrium problem (EP) for the bifunction f on C is stated as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \ \forall y \in C. \tag{1}$$

Let us denote $EP(f,C)$ the solution set of EP (1). Mathematically, EP (1) can be considered a generalization of many mathematical models including variational inequality problems, optimization problems and fixed point problems [2, 20, 21, 27]. EPs have been considered by many authors in recent years, see, e.g., [1, 8, 13–17, 19, 28, 29, 31]. Some notable methods for EPs have been proposed such as: proximal point methods (PPM) [26], auxiliary problem principle methods [24] and gap function methods [25].

The PPM is often used for monotone EPs and this method is based on a regularized equilibrium problem which is strongly monotone and so the solution of it is unique and can be found more easily than solutions of the original problem. Solution approximations generated by the PPM can converge finitely or asymptotically to some solution of EP.

One of the first methods for solving EPs based on the auxiliary problem principle is the proximal-like method introduced early in [11]. The auxiliary

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problem principle, see, e.g., [24, Proposition 2.1], is an equivalent reformulation of the equilibrium problem thanks to a suitable choice of some bifunction. This reformulation gives us a new problem which is called the auxiliary equilibrium problem (AEP), and of course it is solved more easily than the original one. From the AEP with suitable choice of auxiliary bifunction, we can construct iterative sequences converging to some solution of the given problem EP under certain conditions. Recently, the convergence of the proximal-like method in [11] has been further extended and investigated in [29] under different assumptions that equilibrium bifunctions are pseudomonotone and satisfy the Lipschitz-type condition. More precisely, the method in [29] generates two sequences $\{x_n\}$, $\{y_n\}$ as follows:

$$\text{(EGM)} \qquad \begin{cases} y_n = \arg\min\left\{\lambda f(x_n,y) + \frac{1}{2}||x_n - y||^2 : y \in C\right\}, \\ x_{n+1} = \arg\min\left\{\lambda f(y_n,y) + \frac{1}{2}||x_n - y||^2 : y \in C\right\}, \end{cases}$$

where $\lambda > 0$ is a suitable parameter. The methods in [11, 29] are also called extragradient methods (EGM) due to the results of Korpelevich in [22]. Under some suitable conditions imposed on parameters and bifunctions, solution approximation sequences generated by the extragradient method are proved to be convergent to some solution of EP. In recent years, the extragradient methods have received a lot of attention by several authors, see, e.g., in [5,14,15,17,28,31]. The advantage of the extragradient method [11,29] seems to be more easy to solve numerically than the PPM. Moreover, it can be applied to more general class of pseudomonotone bifunctions.

At this stage, it is emphasized that the EGM requires to solve two optimization problems on the feasible set C and to compute two values of bifunction f at two current approximations x_n and y_n. These can be costly and affect the efficiency of the used method if the bifunction f and the feasible set C have complex structures. Then, in this paper, we wish first to modify method (EGM) in the following form (see, Algorithm 2.4 in Sect. 2),

$$\text{(MEGM)} \qquad \begin{cases} x_{n+1} = \underset{y \in T_n}{\operatorname{argmin}}\{\lambda f(y_n,y) + \frac{1}{2}||x_n - y||^2\}, \\ y_{n+1} = \underset{y \in C}{\operatorname{argmin}}\{\lambda f(y_n,y) + \frac{1}{2}||x_{n+1} - y||^2\}. \end{cases}$$

where T_n is a half space constructed suitably. Since T_n is a half space, the first optimization program in method (MEGM) can be solved effectively by using the available methods of convex quadratic programming [3, Chapter 8]. In the special case where problem (EP) is a variational inequality problem then it is a projection on a half space, and so, it is computed explicitly. Moreover, contrary to algorithm (EGM), per iteration method (MEGM) only requires to compute a value of bifunction f at current approximation y_n. The second purpose of the paper is to study the numerical behavior of algorithm (MEGM) and also to compare it with the well known algorithm (EGM) presented in [11, 29].

This paper is organized as follows: In Section 2, we collect some definitions and preliminary results for further use, and then, present the new algorithm in more details. Section 3 deals with analyzing the convergence of the proposed algorithm. Section 4 reports several numerical results on test problems to illustrate the convergence of the algorithm and compare it with algorithm (EGM).

2. Preliminaries

Let C be a nonempty closed convex subset of H. We begin with some concepts of monotonicity of a bifunction [2,27]. A bifunction $f : H \times H \to \Re$ is said to be:

(i) strongly monotone on C if there exists a constant $\gamma > 0$ such that

$$f(x, y) + f(y, x) \leq -\gamma||x - y||^2, \ \forall x, y \in C;$$

(ii) monotone on C if

$$f(x, y) + f(y, x) \leq 0, \ \forall x, y \in C;$$

(iii) pseudomonotone on C if

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0, \ \forall x, y \in C;$$

(iv) to satisfy Lipschitz-type condition on C if there exist two positive constants c_1, c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1||x - y||^2 - c_2||y - z||^2, \quad \forall x, y, z \in C.$$

From the definitions above, it is obvious that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). For solving problem EP (1), we assume that bifunction $f : H \times H \to \Re$ satisfying the following conditions.

Condition 1

(A1) f is pseudomonotone on C with $f(x, x) = 0$ for all $x \in C$;

(A2) f satisfies Lipschitz-type condition on H with two constants c_1 and c_2;

(A3) $f(\cdot, y)$ is sequentially weakly upper semicontinuous on C for each fixed point $y \in C$, i.e., if $\{x_n\} \subset C$ is a sequence converging weakly to $x \in C$, then $\limsup_{n\to\infty} f(x_n, y) \leq f(x, y)$;

(A4) $f(x, \cdot)$ is convex and subdifferentiable on H for every fixed $x \in C$.

It is easy to show that under Condition 1, the solution set $EP(f, C)$ of EP (1) is closed and convex, see for instance [29]. In this paper, we assume that $EP(f, C)$ is nonempty.

Note that if $f(x, y) = \langle Ax, y - x\rangle$, where $A : H \to H$ is a Lipschitz continuous operator, i.e., there exists a number $L > 0$ such that

$$||Ax - Ay|| \leq L||x - y||, \ \forall x, y \in H,$$

then f satisfies the Lipschitz-type condition with $c_1 = c_2 = L/2$, see [17, Corollary 2].

Next, we review the definition of the metric projection and several its properties. For every $x \in H$, the metric projection $P_C x$ of x onto C is defined by

$$P_C x = \arg\min \{\|y - x\| : y \in C\}.$$

Since C is nonempty, closed and convex, $P_C x$ exists and is unique. From the definition of the metric projection, it is easy to show that $P_C : H \to C$ has the following characterizations [12].

Lemma 2.1. (i) *P_C is 1-inverse strongly monotone on H, i.e., for all $x, y \in H$,*

$$\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2.$$

(ii) *For all $y \in H, x \in C$,*

$$\|x - P_C y\|^2 + \|P_C y - y\|^2 \leq \|x - y\|^2. \tag{2}$$

(iii) *$z = P_C x$ if and only if*

$$\langle x - z, z - y\rangle \geq 0, \quad \forall y \in C. \tag{3}$$

The normal cone N_C to C at a point $x \in C$ is defined by

$$N_C(x) = \{w \in H : \langle w, x - y\rangle \geq 0, \forall y \in C\}.$$

For proving the convergence of the proposed algorithm, we need the following two technical lemmas.

Lemma 2.2 ([9]). *Let C be a nonempty subset of H and let $\{x_n\}$ be a sequence in H. Suppose that, for every $x \in C$, there exists a summable sequence $\{\epsilon_n\}$ in $[0, +\infty)$ such that*

$$||x_{n+1} - x||^2 \leq ||x_n - x||^2 + \epsilon_n, \ \forall n \geq 0,$$

and that every weak cluster point of $\{x_n\}$ is in C. Then $\{x_n\}$ converges weakly to a point $p \in C$.

Lemma 2.3 ([30]). *Let C be a nonempty closed convex subset of H and $g : C \to \Re$ be a convex, subdifferentiable and lower semicontinuous function on C. Then, x^* is a solution to the following convex problem*

$$\min\{g(x) : x \in C\}$$

if and only if

$$0 \in \partial g(x^*) + N_C(x^*),$$

where $\partial g(.)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone of C at x^*.

Now, we are in a position to present a modification of algorithm (EGM) in [29] for equilibrium problems. The algorithm is described as follows:

Algorithm 2.4 (Modified subgradient extragradient algorithm for EPs)**.**

Initialization. Choose $x_0 \in H$, $y_0 \in C$, a control parameter $\lambda > 0$, and compute

$$x_1 = \underset{y\in C}{\mathrm{argmin}}\{\lambda f(y_0, y) + \frac{1}{2}||x_0 - y||^2\},$$

$$y_1 = \underset{y\in C}{\mathrm{argmin}}\{\lambda f(y_0, y) + \frac{1}{2}||x_1 - y||^2\}.$$

Iterative step. For $n \geq 1$,

Step 1. Select $w_n \in \partial_2 f(y_{n-1}, y_n) = \partial f(y_{n-1}, \cdot)(y_n)$ and construct a half-space

$$T_n = \{z \in H : \langle x_n - \lambda w_n - y_n, z - y_n\rangle \leq 0\}.$$

Step 2. Solve two strongly convex optimization programs

$$\begin{cases} x_{n+1} = \underset{y\in T_n}{\mathrm{argmin}}\{\lambda f(y_n, y) + \frac{1}{2}||x_n - y||^2\}, \\ y_{n+1} = \underset{y\in C}{\mathrm{argmin}}\{\lambda f(y_n, y) + \frac{1}{2}||x_{n+1} - y||^2\}. \end{cases}$$

Stopping criterion. If $y_{n+1} = y_n = x_{n+1}$ then stop.

Remark 2.5. In the special case where problem (EP) is a variational inequality problem, i.e., $f(x, y) = \langle Ax, y - x\rangle$, where $A : H \to H$ is an operator, then Algorithm 2.4 becomes the following algorithm: Choose $x_0 \in H$, $y_0 \in C$ and compute $x_1 = P_C(x_0 - \lambda Ay_0)$, $y_1 = P_C(x_1 - \lambda Ay_0)$ and

$$\begin{cases} T_n = \{z \in H : \langle x_n - \lambda Ay_{n-1} - y_n, z - y_n\rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda Ay_n), \\ y_{n+1} = P_C(x_{n+1} - \lambda Ay_n), \end{cases} \tag{4}$$

which is studied in [23]. Then, Algorithm 2.4 can be considered a generalization of the result in [23] from variational inequalities to equilibrium problems. It is also worth mentioning here that algorithm (4) only requires to compute a value of A at y_n and a projection on C for y_{n+1}. The first projection for x_{n+1} is inherently explicit. For this reason, we can say that algorithm (4) is almost equivalent to the classical gradient method. Another algorithm, namely the subgradient extragradient algorithm, which has a same feature to algorithm (4), has been studied in [4–7]. However, contrary to algorithm (4), the subgradient extragradient method needs to find two values of cost value A at current approximations. It is emphasized here that the geometric meaning of the half-space T_n has been described and explained in details in [4–7].

Remark 2.6. In *Initialization* of Algorithm 2.4, if C is expressed in an explicit form, the choice $y_0 \in C$ is an easy task. However, if C is given in an implicit form, for instance, a generalized convex feasible set [32, Definition 4.1], it is difficult to choose immediately a point $y_0 \in C$. In this case, we see that C can be expressed by the fixed point set of a mapping [32, Proposition 4.2b]). Then, we can use a fixed point iterative procedure to find this starting point. In Section 4, we have also implemented an experiment (Experiment 5) in this case.

In order to establish the convergence of Algorithm 2.4 in the next section, we consider the stepsize λ satisfying the following condition.

Condition 2

$$0 < \lambda < \frac{1}{2(c_2 + 2c_1)},$$

where c_1 and c_2 are two Lipschitz-type constants of f.

3. Main results

In this section, we will analyze the convergence of Algorithm 2.4. We start with the following lemma which plays an important role in proving the convergence of the proposed algorithm.

Lemma 3.1. *Let $\{x_n\}$, $\{y_n\}$ be two sequences generated by Algorithm 2.4. Then*

(i) $\lambda\left(f(y_n, y) - f(y_n, y_{n+1})\right) \geq \langle x_{n+1} - y_{n+1}, y - y_{n+1}\rangle$ *for all* $y \in C$ *and* $n \geq 0$.

(ii) *For all $x^* \in EP(f, C)$, the following estimate holds*

$$\begin{aligned} ||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 - (1 - 2\lambda c_2)||y_n - x_{n+1}||^2 + 4\lambda c_1 ||y_{n-1} - x_n||^2 \\ - (1 - 4\lambda c_1)||x_n - y_n||^2. \end{aligned} \tag{5}$$

Proof. (i) From the definition of y_{n+1} and Lemma 2.3,

$$0 \in \partial_2 \left(\lambda f(y_n, y) + \frac{1}{2}||x_{n+1} - y||^2\right)(y_{n+1}) + N_C(y_{n+1}).$$

Thus, there exist $w \in \partial_2 f(y_n, y_{n+1}) := \partial f(y_n, \cdot)(y_{n+1})$ and $\bar{w} \in N_C(y_{n+1})$ such that

$$\lambda w + y_{n+1} - x_{n+1} + \bar{w} = 0.$$

Hence, it follows from the definition of N_C that

$$\begin{aligned} \langle x_{n+1} - y_{n+1}, y - y_{n+1}\rangle &= \lambda \langle w, y - y_{n+1}\rangle + \langle \bar{w}, y - y_{n+1}\rangle \\ &\leq \lambda \langle w, y - y_{n+1}\rangle, \ \forall y \in C. \end{aligned} \tag{6}$$

By $w \in \partial_2 f(y_n, y_{n+1})$,

$$\langle w, y - y_{n+1}\rangle \leq f(y_n, y) - f(y_n, y_{n+1}), \ \forall y \in C. \tag{7}$$

From relations (6) and (7), we obtain

$$\lambda\left(f(y_n, y) - f(y_n, y_{n+1})\right) \geq \langle x_{n+1} - y_{n+1}, y - y_{n+1}\rangle, \ \forall y \in C.$$

(ii) It follows from the definition of T_n and $x_{n+1} \in T_n$ that

$$\langle x_n - \lambda w_n - y_n, x_{n+1} - y_n\rangle \leq 0.$$

Thus

$$\langle x_n - y_n, x_{n+1} - y_n\rangle \leq \lambda \langle w_n, x_{n+1} - y_n\rangle . \tag{8}$$

Since $w_n \in \partial_2 f(y_{n-1}, y_n)$,

$$\langle w_n, y - y_n\rangle \leq f(y_{n-1}, y) - f(y_{n-1}, y_n),\ \forall y \in H. \tag{9}$$

Relations (8) and (9) with $y = x_{n+1}$ come to

$$\langle x_n - y_n, x_{n+1} - y_n\rangle \leq \lambda(f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n)). \tag{10}$$

From the definition of x_{n+1}, by arguing as in the proof of (i) we obtain

$$\lambda\left(f(y_n, y) - f(y_n, x_{n+1})\right) \geq \langle x_n - x_{n+1}, y - x_{n+1}\rangle ,\ \forall y \in T_n.$$

This with $y = x^*$ implies that

$$\lambda\left(f(y_n, x^*) - f(y_n, x_{n+1})\right) \geq \langle x_n - x_{n+1}, x^* - x_{n+1}\rangle . \tag{11}$$

Since $x^* \in EP(f, C)$, $f(x^*, y_n) \geq 0$. Thus, from the pseudomonotonicity of f, we obtain $f(y_n, x^*) \leq 0$. It follows from relation (11) that $-\lambda f(y_n, x_{n+1}) \geq \langle x_n - x_{n+1}, x^* - x_{n+1}\rangle$. Thus

$$\langle x_n - x_{n+1}, x_{n+1} - x^*\rangle \geq \lambda f(y_n, x_{n+1}). \tag{12}$$

From the Lipschitz-type condition of f,

$$f(y_n, x_{n+1}) \geq (f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n)) - c_1||y_{n-1} - y_n||^2 - c_2||y_n - x_{n+1}||^2.$$

This together with relations (10) and (12) implies that

$$\begin{aligned} &\langle x_n - x_{n+1}, x_{n+1} - x^*\rangle \\ \geq\ &\langle x_n - y_n, x_{n+1} - y_n\rangle - c_1\lambda||y_{n-1} - y_n||^2 - c_2\lambda||y_n - x_{n+1}||^2. \end{aligned} \tag{13}$$

We have the following facts:

$$\langle x_n - x_{n+1}, x_{n+1} - x^*\rangle = \frac{1}{2}\left\{||x_n - x^*||^2 - ||x_n - x_{n+1}||^2 - ||x_{n+1} - x^*||^2\right\},$$

$$\langle x_n - y_n, x_{n+1} - y_n\rangle = \frac{1}{2}\left\{||x_n - y_n||^2 + ||x_{n+1} - y_n||^2 - ||x_n - x_{n+1}||^2\right\}.$$

Combining the last two equalities with relation (13), we obtain

$$\begin{aligned} ||x_{n+1} - x^*||^2 \leq\ &||x_n - x^*||^2 + 2c_1\lambda||y_{n-1} - y_n||^2 - (1 - 2c_2\lambda)||y_n - x_{n+1}||^2 \\ &- ||x_n - y_n||^2. \end{aligned} \tag{14}$$

From the triangle inequality and the Cauchy-Schwarz inequality,

$$\begin{aligned} ||y_{n-1} - y_n||^2 &\leq (||y_{n-1} - x_n|| + ||x_n - y_n||)^2 \\ &\leq 2\left(||y_{n-1} - x_n||^2 + ||x_n - y_n||^2\right). \end{aligned} \tag{15}$$

Two inequalities (14) and (15) imply that

$$\begin{aligned} ||x_{n+1} - x^*||^2 \leq\ &||x_n - x^*||^2 - (1 - 2c_2\lambda)||y_n - x_{n+1}||^2 + 4c_1\lambda||y_{n-1} - x_n||^2 \\ &- (1 - 4c_1\lambda)||x_n - y_n||^2. \end{aligned}$$

□

The next lemma gives us a stopping criterion of Algorithm 2.4.

Lemma 3.2. *If Algorithm 2.4 terminates at some iterative step n, then $x_{n+1} \in EP(f, C)$.*

Proof. If $y_{n+1} = y_n = x_{n+1}$, then, from Lemma 3.1 (i), hypothesis (A1) and $\lambda > 0$, we obtain $f(y_n, y) \geq 0$ for all $y \in C$. Thus, $x_{n+1} = y_n \in EP(f, C)$. $\square$

Thanks to Lemma 3.2, if Algorithm 2.4 stops then a solution of EP (1) can be found. Otherwise, we have the following result.

Lemma 3.3. *Let $\{x_n\}$, $\{y_n\}$ be two (infinite) sequences generated by Algorithm 2.4. Then*

(i) *The sequence $\{x_n\}$ is bounded and*

$$\lim_{n\to\infty} ||x_n - y_n|| = \lim_{n\to\infty} ||x_n - y_{n-1}|| = \lim_{n\to\infty} ||y_n - y_{n-1}|| = 0.$$

(ii) *If p is a weak cluster point of $\{x_n\}$, then $p \in EP(f, C)$.*

Proof. (i) For each fixed $K \geq 1$, we consider inequalities (5) for $n = 1, 2, \ldots, K$. Summing up them, we obtain

$$\begin{aligned} ||x_{K+1} - x^*||^2 \leq\ & ||x_0 - x^*||^2 + 4c_1\lambda_1||y_0 - x_1||^2 \\ & - \sum_{n=1}^{K}(1 - 2c_2\lambda - 4\lambda c_1)||y_{n-1} - x_n||^2 \\ & - \sum_{n=1}^{K}(1 - 4\lambda c_1)||x_n - y_n||^2. \end{aligned} \tag{16}$$

From Condition 2, $1 - 2c_2\lambda - 4\lambda c_1 > 0$ and so $1 - 4\lambda c_1 > 0$. These together with inequality (16) imply that $\left\{||x_{K+1} - x^*||^2\right\}_{K\geq 1}$ is bounded. Thus, $\{x_n\}$ is also bounded. Hence, it follows from inequality (16) that

$$\sum_{n=1}^{\infty}(1-2c_2\lambda-4\lambda c_1)||y_{n-1}-x_n||^2 < \infty \text{ and } \sum_{n=1}^{\infty}(1-4\lambda c_1)||x_n-y_n||^2 < \infty. \tag{17}$$

Therefore, from Condition 2, we obtain

$$\lim_{n\to\infty} ||x_n - y_n|| = \lim_{n\to\infty} ||x_n - y_{n-1}|| = 0.$$

Thus, it follows from the triangle inequality that

$$\lim_{n\to\infty} ||y_n - y_{n-1}|| = 0.$$

(ii) Suppose that $\{x_m\}$ is a subsequence of $\{x_n\}$ such that $x_m \rightharpoonup p$. Since $||x_m - y_m|| \to 0$, $y_m \rightharpoonup p$. Since C is closed and convex in H, C is weakly closed. It follows from $\{y_m\} \subset C$ that $p \in C$. From Lemma 3.1(i) with $n = m$ and the Lipschitz-type condition of f, we have

$$\begin{aligned} \lambda f(y_m, y) &\geq \lambda f(y_m, y_{m+1}) + \langle x_{m+1} - y_{m+1}, y - y_{m+1}\rangle \\ &\geq \lambda \left\{f(y_{m-1}, y_{m+1}) - f(y_{m-1}, y_m)\right\} - \lambda c_1||y_{m-1} - y_m||^2 \\ &\quad - \lambda c_2||y_m - y_{m+1}||^2 + \langle x_{m+1} - y_{m+1}, y - y_{m+1}\rangle. \end{aligned} \tag{18}$$

It also follows from Lemma 3.1(i) with $n = m - 1$ that

$$\lambda\left(f(y_{m-1}, y) - f(y_{m-1}, y_m)\right) \geq \langle x_m - y_m, y - y_m \rangle, \ \forall y \in C.$$

This with $y = y_{m+1}$ leads to

$$\lambda\left(f(y_{m-1}, y_{m+1}) - f(y_{m-1}, y_m)\right) \geq \langle x_m - y_m, y_{m+1} - y_m \rangle. \tag{19}$$

Combining relations (18) and (19), we have

$$\begin{aligned} \lambda f(y_m, y) \geq\ & \langle x_m - y_m, y_{m+1} - y_m \rangle - \lambda c_1 ||y_{m-1} - y_m||^2 - \lambda c_2 ||y_m - y_{m+1}||^2 \\ & + \langle x_{m+1} - y_{m+1}, y - y_{m+1} \rangle, \ \forall y \in C. \end{aligned} \tag{20}$$

Passing to the limit in inequality (20) as $m \to \infty$ and using hypothesis (A3), Lemma 3.3(i) and Condition 2, we obtain $f(p, y) \geq 0$ for all $y \in C$ or $p \in EP(f, C)$. □

Finally, we have the following main result.

Theorem 3.4. *Let C be a nonempty closed convex subset of H and f be a function satisfying Condition 1. Suppose that parameter λ satisfies Condition 2. In addition, the solution set $EP(f, C)$ is nonempty. Then, two sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 2.4 converge weakly to some point $p \in EP(f, C)$. Moreover, $p = \lim_{n\to\infty} P_{EP(f,C)}(x_n)$.*

Proof. It follows from Lemma 3.1(ii) and Condition 2 that

$$||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 + 4\lambda c_1 ||y_{n-1} - x_n||^2,$$

or

$$||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 + \epsilon_n, \ \forall n \geq 0, \ \forall x^* \in EP(f, C), \tag{21}$$

where $\epsilon_n = 4\lambda c_1 ||y_{n-1} - x_n||^2$. It follows from (17) that $\sum_{n=1}^{\infty} \epsilon_n < \infty$. Moreover, from Lemma 3.3(ii), we see that every weak cluster point of $\{x_n\}$ is in $EP(f, C)$. These together with (21) and Lemma 2.2 imply that the whole sequence $\{x_n\}$ converges weakly to some point $p \in EP(f, C)$. From Lemma 3.3(i), we conclude that $\{y_n\}$ also converges weakly to p. Let $a_n = ||x_n - P_{EP(f,C)} x_n||^2$. From the definition of the metric projection and relation (21), we have

$$||x_{n+1} - P_{EP(f,C)} x_{n+1}||^2 \leq ||x_{n+1} - P_{EP(f,C)} x_n||^2 \leq ||x_n - P_{EP(f,C)} x_n||^2 + \epsilon_n,$$

or $a_{n+1} \leq a_n + \epsilon_n$. Thus, it follows from $\sum_{n=1}^{\infty} \epsilon_n < \infty$ that there exists the limit of the sequence $\{a_n\}$ [9, Lemma 3.1]. Put $u_n = P_{EP(f,C)} x_n$. From Lemma 2.1(ii) and relation (21), for all $m > n$, we have

$$\begin{aligned} ||u_n - u_m||^2 &= ||P_{EP(f,C)} x_n - P_{EP(f,C)} x_m||^2 \\ &\leq ||x_m - P_{EP(f,C)} x_n||^2 - ||x_m - P_{EP(f,C)} x_m||^2 \\ &\leq ||x_{m-1} - P_{EP(f,C)} x_n||^2 + \epsilon_{m-1} - ||x_m - P_{EP(f,C)} x_m||^2 \\ &\leq \cdots \end{aligned}$$

$$\leq ||x_n - P_{EP(f,C)}x_n||^2 + \sum_{k=n}^{m-1} \epsilon_k - ||x_m - P_{EP(f,C)}x_m||^2$$

$$= a_n - a_m + \sum_{k=n}^{m-1} \epsilon_k. \tag{22}$$

Passing to the limit in inequality (22) as $m, n \to \infty$, we obtain $\lim_{m,n\to\infty} ||u_n - u_m||^2 = 0$. Thus, $\{u_n\}$ is a Cauchy sequence, or there exists the limit

$$\lim_{n\to\infty} u_n = u \in EP(f, C).$$

It follows from $u_n = P_{EP(f,C)}x_n$ and Lemma 2.1(iii) that $\langle p - u_n, x_n - u_n\rangle \leq 0$. Passing to the limit in the last inequality as $n \to \infty$, we obtain $||p - u||^2 = \langle p - u, p - u\rangle \leq 0$. Thus, $u = p$ or $p = \lim_{n\to\infty} P_{EP(f,C)}x_n$. □

4. Numerical experiments

In this section, we perform several numerical examples to illustrate the convergence of the proposed algorithm. The first two examples are to study the numerical behavior of Algorithm 2.4 on two test problems for different stepsizes of λ. The last example presents a comparison of the proposed algorithm with an extended extragradient method (EGM) to EPs in [29].

Example 1. We first consider the bifunction f in $\Re^m$ $(m = 50)$ defined by

$$f(x, y) = \langle Px + Qy + q, y - x\rangle ,$$

where $q \in \Re^m$ and $P,\ Q$ are two matrices of order m such that Q is symmetric and positive semidefinite and $Q - P$ is negative semidefinite. The feasible set C is the intersection of two balls defined by $B_1 = \left\{x \in \Re^m : ||x||^2 \leq 4\right\}$ and $B_2 = \left\{x \in \Re^m : ||x - (2, 0, 0, \dots, 0)||^2 \leq 1\right\}$. In all the experiments, f satisfies Condition 1 with $c_1 = c_2$. Thus, Condition 2 becomes $0 < \lambda < \frac{1}{6c_1}$. Five stepsizes of λ are chosen to study the behavior of Algorithm 2.4 as $\lambda \in \left\{\frac{1}{6.001c_1}, \frac{1}{10c_1}, \frac{1}{100c_1}, \frac{1}{500c_1}, \frac{1}{1000c_1}\right\}$ (Experiments 1, 2 and 3). The starting points here are $x_0 = (1, 1, \dots, 1)^T$ and $y_0 = (2, 0, \dots, 0)^T \in C$. The first optimization program in Step 2 of Algorithm 2.4 is a quadratic convex problem over a half-space (polyhedral convex set) while the second one is not. We use respectively two functions *quadprog* and *fmincon* in Matlab 7.0 Optimization Toolbox to solve these auxiliary optimization subproblems. All the programs are written in Matlab version 7.0. and performed on a PC Desktop Intel(R) Core(TM) i5-3210M CPU @ 2.50GHz 2.50 GHz, RAM 2.00 GB.

Experiment 1. Suppose that $q = 0$ and $P = Q$ is a diagonal matrix with diagonal entries being $1, 2, \dots, m$. In this case, the bifunction f satisfies Condition 1 for all $c_1,\ c_2 > 0$ and we here chose $c_1 = c_2 = 5$. It is easy to imply that the exact solution of EP is $x^* = (1, 0, \dots, 0)^T$. We use the sequence $D_n = ||x_n - x^*||^2,\ n = 1, 2, \dots$ to check the convergence of $\{x_n\}$ generated by

Algorithm 2.4. The convergence of $\{D_n\}$ to 0 implies that the sequence $\{x_n\}$ converges to the solution x^* of EP.

Figure 1 shows the results of D_n generated by Algorithm 2.4 with the chosen stepsizes of λ for the first 2000 iterations. In this figure, the y-axis represents for the value of D_n while the x-axis represents for the number of iterations n. We see that the smaller the stepsize of λ is, the slower the convergence of Algorithm 2.4 is. More precisely, the convergence of the sequence $\{D_n\}$ with $\lambda = \frac{1}{6.001c_1}$ is the best while that one with $\lambda = \frac{1}{1000c_1}$ is the slowest. The execution times in second (resp. with the given stepsizes of λ above) are 93.72, 88.10, 88.67, 85.33 and 85.81s.

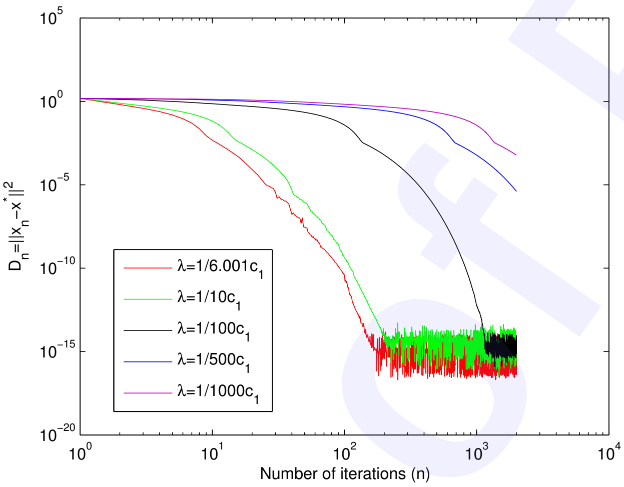

FIGURE 1. Behavior of $D_n = ||x_n - x^*||^2$ for Algorithm 2.4 in Experiment 1 for the first 2000 iterations (Execution times (CPU in second) are 93.72, 88.10, 88.67, 85.33 and 85.81s, resp.)

Experiment 2. All entries of q are generated randomly and uniformly in $[-m, m]$. Two matrices P and Q are also generated randomly[1] such that they satisfy all conditions of the problem. In this case, two Lipschitz-type constants are $c_1 = c_2 = ||Q - P||/2$ [29]. Since the solution of the problem is not known, we use the sequence $F_n = ||y_{n+1} - y_n||^2 + ||y_{n+1} - x_{n+1}||^2,\ n = 0, 1, \dots$ to

[1]Two matrices P, Q are generated randomly as follows: we randomly choose $\lambda_{1k} \in [0, m]$, $\lambda_{2k} \in [-m, 0]$, $k = 1, \dots, m$. Set $\widehat{Q}$, $\widehat{T}$ as two diagonal matrices with eigenvalues $\{\lambda_{1k}\}_{k=1}^m$ and $\{\lambda_{2k}\}_{k=1}^m$, respectively. Then, we make a positive semidefinite matrix Q and a negative semidefinite matrix T by using $\widehat{Q}$ and $\widehat{T}$ with two random orthogonal matrices, respectively. Finally, we set $P = Q - T$.

check the convergence of $\{x_n\}$ to the solution of the problem. Note that, from Lemma 3.2, if $y_{n+1} = y_n = x_{n+1}$ then x_{n+1} is a solution of the problem. The convergence of $\{F_n\}$ to 0 can be considered as the sequence $\{x_n\}$ converging to the solution of the problem. Figure 2 describes the behavior of $\{F_n\}$ generated by Algorithm 2.4 for given stepsizes of λ in the first 2000 iterations. This figure shows that the convergence of F_n with $\lambda = \frac{1}{6.001c_1}$ is better than the convergence of F_n with others. As the previous experiment, the smaller the stepsize is, the slower the convergence of F_n is. The execution times are respectively 102.50, 103.49, 96.97, 92.55 and 92.08s.

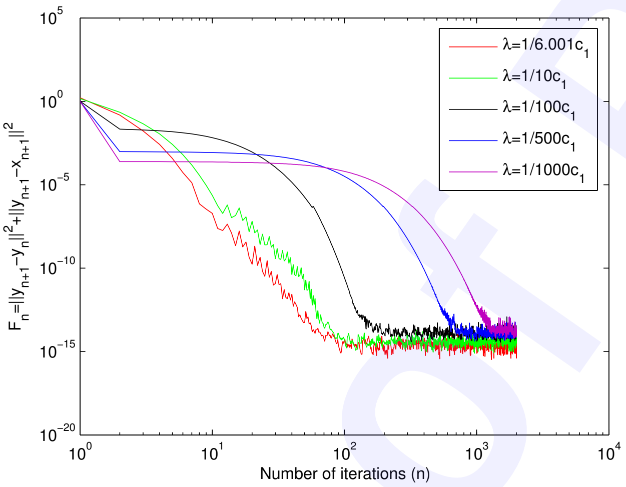

FIGURE 2. Behavior of $F_n = ||y_{n+1} - y_n||^2 + ||y_{n+1} - x_{n+1}||^2$ for Algorithm 2.4 in Experiment 2 for the first 2000 iterations (Execution times (CPU in second) are 102.50, 103.49, 96.97, 92.55 and 92.08s, resp.)

Example 2. We consider the nonsmooth bifunction f in $\Re^m$ ($m = 50$) defined by

$$f(x, y) = \langle Px + Qy + q, y - x\rangle + h(y) - h(x),$$

where $P,\ Q,\ q$ are defined as in Example 1 and $h(x) = \sum_{j=1}^{m} h_j(x_j)$ with

$$h_j(x_j) = \max\left\{\bar{h}_j(x_j), \widehat{h}_j(x_j)\right\},$$
$$\bar{h}_j(x_j) = \bar{a}_j x_j^2 + \bar{b}_j x_j + \bar{c}_j,$$
$$\widehat{h}_j(x_j) = \widehat{a}_j x_j^2 + \widehat{b}_j x_j + \widehat{c}_j,$$

and $\bar{a}_j,\ \bar{b}_j,\ \bar{c}_j,\ \widehat{a}_j,\ \widehat{b}_j,\ \widehat{c}_j$ are real numbers such that $\bar{a}_j > 0,\ \widehat{a}_j > 0$ for all $j = 1, \ldots, m$. The feasible set C is a box defined by $C = \{x \in \Re^m : x_{\min} \le x \le x_{\max}\}$, where $x_{\min} = (0, 0, \ldots, 0)^T$ and $x_{\max} = (5, 5, \ldots, 5)^T$. The bifunction f is generalized from the Nash-Cournot market equilibrium model investigated in [10, 29] . The function $h(x)$ is nonsmooth and convex. The bifunction f also satisfies Condition 1 with $c_1 = c_2 = ||Q - P||/2$. The function h is subdifferentiable and its subdifferential at x is given by $\partial h(x) = (\partial h_1(x_1), \ldots, \partial h_m(x_m))^T$, where

$$\partial h_j(x_j) = \begin{cases} 2\bar{a}_j x_j + \bar{b}_j & \text{if} \quad \bar{h}_j(x_j) > \widehat{h}_j(x_j), \\ \left[2\bar{a}_j x_j + \bar{b}_j, 2\widehat{a}_j x_j + \widehat{b}_j\right] & \text{if} \quad \bar{h}_j(x_j) = \widehat{h}_j(x_j), \\ 2\widehat{a}_j x_j + \widehat{b}_j, & \text{if} \quad \bar{h}_j(x_j) < \widehat{h}_j(x_j). \end{cases}$$

Experiment 3. In an experiment, $P,\ Q,\ q$ are generated randomly as in Example 1. The numbers $\bar{a}_j,\ \widehat{a}_j$ are generated uniformly and randomly in $[1, m]$ and $\bar{b}_j,\ \bar{c}_j,\ \widehat{b}_j,\ \widehat{c}_j$ are in $[-m, m]$. The starting points are here chosen as $x_0 = (1, 1, \ldots, 1)^T$ and $y_0 = (0, 0, \ldots, 0)^T$. The optimization problems are still solved effectively by the function *fmincon* in Matlab 7.0 Optimization Toolbox. Figure 3 describes the behavior of $\{F_n\}$ generated by Algorithm 2.4 for given stepsizes of λ in the first 5000 iterations. There is a slight difference to the previous experiments. More precisely, although the sequences $\{F_n\}$ with $\lambda = \frac{1}{6.001c_1}$ and $\lambda = \frac{1}{10c_1}$ are more fastly convergent than others in the early iterations, but they only obtain to the approximation error 10^{-10} while $\{F_n\}$ with other stepsizes can obtain to a smaller approximation one as 10^{-15}. The execution time for $\lambda = \frac{1}{6.001c_1}$ is significantly less than that one for other stepsizes of λ.

Example 3. Consider the bifunction f defined as in **Example 1** with $P,\ Q,\ q$ being generated randomly. In this example, we perform two experiments to compare the proposed algorithm with an extended extragradient method (EGM) to EPs in [29]. Note that in method (EGM) if $y_n = x_n$ then $x_n \in EP(f, C)$. We have used the following stopping criterions:

$$\text{EGM}: \quad ||x_n - y_n|| \le \text{TOL}, \tag{23}$$

$$\text{Alg. 2.4}: \quad ||y_n - y_{n+1}|| + ||x_{n+1} - y_n|| \le \text{TOL}. \tag{24}$$

For each $u \in C$, we consider the proximal mapping of $f(u, \cdot)$ with $\lambda > 0$,

$$\text{prox}_{\lambda f(u,\cdot)}(v) := \arg\min \left\{ \lambda f(u, y) + \frac{1}{2}||v - y||^2 : y \in C \right\},\ v \in H.$$

It is emphasized that if $v = \text{prox}_{\lambda f(v,\cdot)}(v)$ then $v \in EP(f, C)$. The stopping criterion (23) means that $||x_n - \text{prox}_{\lambda f(x_n,\cdot)}(x_n)|| \le \text{TOL}$. We do not use this stopping criterion for Algorithm 2.4 because we do not want to compute $\text{prox}_{\lambda f(x_n,\cdot)}(x_n)$ extra. From the triangle inequality, we have

$$||y_n - \text{prox}_{\lambda f(y_n,\cdot)}(y_n)|| \le ||y_n - y_{n+1}|| + ||y_{n+1} - \text{prox}_{\lambda f(y_n,\cdot)}(y_n)||. \tag{25}$$

From the definitions of y_{n+1} (in Algorithm 2.4) and the (firm) nonexpansiveness of proximal mapping, we obtain $||y_{n+1} - \text{prox}_{\lambda f(y_n,\cdot)}(y_n)|| \le ||x_{n+1} - y_n||$. This together with relations (24) and (25) implies that $||y_n - \text{prox}_{\lambda f(y_n,\cdot)}(y_n)|| \le$

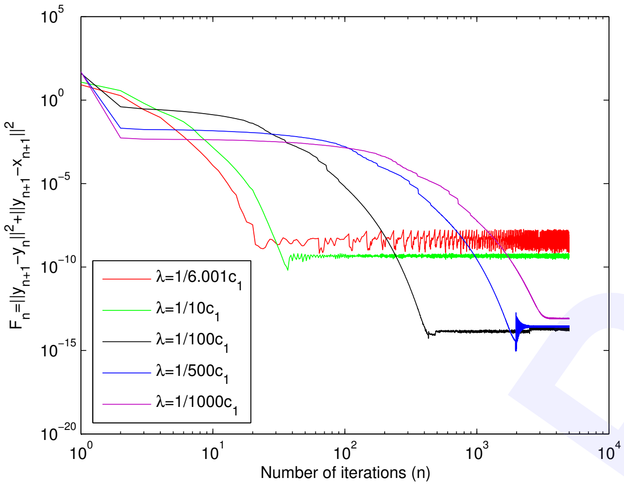

FIGURE 3. Behavior of $F_n = ||y_{n+1} - y_n||^2 + ||y_{n+1} - x_{n+1}||^2$ for Algorithm 2.4 in Example 2 in the first 5000 iterations (Execution times (CPU in second) are 382.62, 624.97, 859.92, 835.66, and 762.01s, resp.)

$||y_n - y_{n+1}|| + ||x_{n+1} - y_n|| \leq \text{TOL}$. The last inequality means that the approximation solution generated by Algorithm 2.4 is not worse than the solution generated by the EGM if we measure the quality of the solution by the proximal error bound.

Experiment 4. Consider three feasible sets as

$$C_1 = \Re^m_+,$$
$$C_2 = \left\{x \in \Re^m_+ : x_1 + x_2 + \cdots + x_m = m\right\},$$
$$C_3 = \{x \in \Re^m : Ax \leq b\},$$

where $A \in \Re^{k\times m}$ ($k = 100$) is a matrix with its entries being generated randomly in $[-m, m]$ and $b \in \Re^k_+$ is a positive vector with its entries in $[1, m]$. TOL is chosen as TOL $= 10^{-6}$, the stepsize is $\lambda = \frac{1}{10c_1}$, the starting point y_0 is $y_0 = (m, 0, 0, \ldots, 0)^T$ with C_1, C_2 and $y_0 = (0, 0, \ldots, 0)^T$ with C_3. The comparison includes the number of iterations (iter.) and the execution time in second (time). Table 1 shows the results for $x_0 = (1, 1, \ldots, 1)^T$ and for different spaces $\Re^m$ while Table 2 is in $\Re^{100}$ for several different starting points x_0. The numerical results show that the proposed algorithm has an advantage of execution time over the EGM, especially in larger dimensional spaces.

Experiment 5. In this experiment, we consider feasible set as a *generalized convex feasible set* [32, Definition 4.1]. Let K, $C_1, \ldots, C_l$ be nonempty closed

TABLE 1. Results in *Experiment 4* for different spaces $\Re^m$.

m	$C = C_1$				$C = C_2$				$C = C_3$			
	Alg. 2.4		EGM		Alg. 2.4		EGM		Alg. 2.4		EGM	
	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time
10	49	1.63	50	1.57	49	1.09	50	1.17	38	0.90	39	1.02
50	82	4.69	82	6.06	120	5.61	121	6.94	68	3.85	66	6.08
100	78	6.05	76	8.37	129	9.07	129	12.26	101	14.21	102	24.13
150	104	13.79	103	19.82	143	27.97	144	47.73	133	38.53	133	68.14
300	100	38.62	101	62.47	156	240.39	156	472.87	145	257.62	146	503.38

TABLE 2. Results in *Experiment 4* for different starting points x_0.

x_0	$C = C_1$				$C = C_2$				$C = C_3$			
	Alg. 2.4		EGM		Alg. 2.4		EGM		Alg. 2.4		EGM	
	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time
(0)	94	6.95	95	9.85	150	11.29	150	15.88	119	15.42	120	25.50
(1)	84	6.28	84	9.03	150	10.67	150	15.66	114	13.61	116	24.41
rand.	124	9.65	124	14.39	152	11.62	152	16.05	143	18.90	144	30.70
rand.	129	9.85	130	14.41	172	13.52	173	18.89	154	19.96	154	32.66
rand.	128	10.07	129	14.71	164	12.49	165	17.45	155	21.49	156	34.91

convex subsets of $\Re^m$ such that $K \cap \left(\cap_{j=1}^{l} C_j\right) = \emptyset$ and at least one $K,\ C_j,\ j = 1, \ldots, l$ ($l = 100, 200$) is bounded. For each $x \in \Re^m$, we set

$$\Phi(x) = \frac{1}{2} \sum_{j=1}^{l} w_j d^2(x, C_j),$$

where $\{w_j\}_{j=1}^{l} \subset (0,1)$, $\sum_{j=1}^{l} w_j = 1$ and $d(x, C_j) = \min\{||x - y|| : y \in C_j\}$. Consider the feasible set C which is called *a generalized convex feasible set* [32, Definition 4.1] as follows:

$$C = \left\{ x \in K : \Phi(x) = \min_{y \in K} \Phi(y) \right\}. \tag{26}$$

Note that C is a nonempty closed convex subset of K, see [32, Proposition 4.2 and Remark 4.3]. For an experiment, we chose $w_j = \frac{1}{l}$ and

$$K = \{x \in \Re^m : ||x - a|| \leq 4.5\},$$
$$C_1 = \{x \in \Re^m : ||x|| \leq 1\},$$
$$C_j = \left\{x \in \Re^m : \langle c_j^T, x \rangle \leq b_j \right\},\ j = 2, \ldots l,$$

where $a = (10, 0, \ldots, 0)^T \in \Re^m$, and real numbers b_j and vectors c_j are generated randomly with their entries in $[1, m]$. It is easy to see that the first optimization subproblem in our algorithm is a convex quadratic problem because T_n is a half-space. A question is how to solve other optimization subproblems on C when the feasible set C is formulated in implicit form (26). We would like to implement an experiment in this case. Now, we set

$$T = P_K \left((1-\beta)I + \beta \sum_{j=1}^{l} w_j P_{C_j} \right),\ 0 < \beta \leq 2,$$

then T is nonexpansive (also, quasi-nonexpansive) and $C = Fix(T)$ is the fixed point set of T, see [32, Proposition 4.2b and Remark 4.3c]. Thus, optimization problems over C become optimization ones over the fixed point set of T. We have chosen $\beta = 1$ and solved these optimization problems by the subgradient method in [18, Algorithm 4.1] to obtain their solution approximations for tolerance TOL with a posterior stopping criterion. For a starting point y_0 in $C = Fix(T)$, we have used the Mann fixed point iterative procedure with tolerance TOL. Table 3 shows the numerical results with the starting point $x_0 = (1, 1, \ldots, 1)^T$, the stepsize $\lambda = \frac{1}{8c_1}$ and stopping criterions (23) and (24) for respectively Algorithm 2.4 and the EGM. It is not easy to describe the solution of EP in this case. We have used the sequence $\{||x_n - T(x_n)||\}$ to check whether $\{x_n\}$ converges to a point in $Fix(T) = C$ or not[2]. Figure 4 describes the behaviors of $\{||x_n - T(x_n)||\}$ for Algorithm 2.4 and the EGM with $m = 5,\ l = 200$ and TOL $= 10^{-8}$.

As Experiment 4, in this experiment Algorithm 2.4 also has an advantage of execution time over algorithm EGM. A reasonable explanation here is the fact that over each iteration algorithm EGM must require to proceed two values of bifunction f at x_n and y_n while Algorithm 2.4 only needs to compute a value of f at y_n. Besides, the solving of the first optimization program in Step 2 of Algorithm 2.4, which is with only one linear constraint from T_n, is simpler than the one in algorithm EGM coming from the constraints of feasible set C.

TABLE 3. Results in *Experiment 5 with* $\lambda = \frac{1}{8c_1}$.

		$l = 100$				$l = 200$			
		Alg. 2.4		EGM		Alg. 2.4		EGM	
m	TOL	iter.	time	iter.	time	iter.	time	iter.	time
5	10^{-5}	6	0.74	5	0.88	6	1.11	5	1.52
	10^{-8}	9	1.01	8	1.32	11	1.84	9	2.31
10	10^{-5}	7	0.95	6	1.33	7	2.18	6	3.01
	10^{-8}	10	1.34	9	1.92	10	2.94	9	4.35
30	10^{-5}	7	2.35	8	3.83	8	4.35	8	6.99
	10^{-8}	11	3.19	11	5.15	12	6.28	12	9.95
50	10^{-5}	8	3.10	8	4.57	8	6.61	9	10.97
	10^{-8}	11	4.01	12	6.58	12	9.39	13	14.99

[2] $||x - T(x)|| = 0$ if and only if $x \in Fix(T) = C$

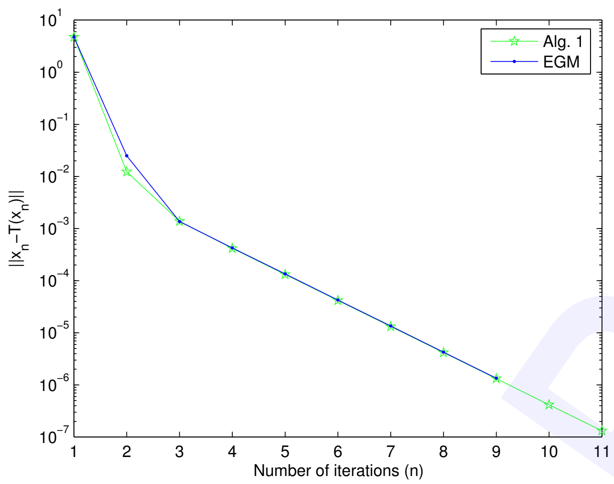

FIGURE 4. Behaviors of $\{||x_n - T(x_n)||\}$ for Algorithm 2.4 and the EGM.

5. Conclusions

The paper has proposed a modified subgradient extragradient method for approximating solutions of equilibrium problems in Hilbert spaces. The weakly convergent theorem is established under standard assumptions imposed on equilibrium bifunctions. This paper has studied the numerical behavior of the proposed algorithm over several test problems and also compared it with an existing extragradient method. The performed results have illustrated several advantages of the new algorithm over algorithm (EGM).

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