

## ON CERTAIN MULTIPLES OF LITTLEWOOD AND NEWMAN POLYNOMIALS

PAULIUS DRUNGILAS, JONAS JANKAUSKAS, GRINTAS JUNEVIČIUS,  
LUKAS KLEBONAS, AND JONAS ŠIURYS

ABSTRACT. Polynomials with all the coefficients in  $\{0, 1\}$  and constant term 1 are called Newman polynomials, whereas polynomials with all the coefficients in  $\{-1, 1\}$  are called Littlewood polynomials. By exploiting an algorithm developed earlier, we determine the set of Littlewood polynomials of degree at most 12 which divide Newman polynomials. Moreover, we show that every Newman quadrinomial  $X^a + X^b + X^c + 1$ ,  $15 > a > b > c > 0$ , has a Littlewood multiple of smallest possible degree which can be as large as 32765.

### 1. Introduction

A polynomial

$$(1.1) \quad P(X) = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$$

with a nonzero constant term  $P(0)$  is called a *Borwein polynomial*, if  $a_j \in \{-1, 0, 1\}$  for each  $0 \leq j \leq d$ . To avoid trivialities, we consider only polynomials with non-zero leading and constant terms  $a_d \cdot a_0 \neq 0$ . In such case, both  $P(X)$  and its reciprocal polynomial  $P^*(X) := X^d P(1/X)$  are of the same degree  $d$ . For example,  $P(X) = X^5 - X^3 + X - 1$  is a Borwein polynomial.

Similarly, the polynomial  $P(X)$  in (1.1) is called a *Newman polynomial* if all coefficients  $a_j \in \{0, 1\}$  and  $P(0) = 1$ . For instance,  $P(X) = X^4 + X + 1$  is a Newman polynomial. The subset of  $\mathbb{Z}[X]$  of all Newman polynomials is denoted by  $\mathcal{N}$ . Finally, a Borwein polynomial  $P(X)$  in (1.1) is called a *Littlewood polynomial* if all of its coefficients are non-zero, i.e.,  $a_j \in \{-1, 1\}$  for

---

Received October 3, 2017; Revised January 26, 2018; Accepted May 2, 2018.

2010 *Mathematics Subject Classification*. 11R09, 11Y16, 12D05, 11R06.

*Key words and phrases*. Borwein polynomial, Littlewood polynomial, Newman polynomial, Salem number, complex Salem number, polynomials of small height.

This research of the first author was funded by a grant (No. S-MIP-17-66/LSS-110000-1274) from the Research Council of Lithuania. This research was performed in cooperation with the Department Mathematik und Statistik, Montanuniversität Leoben.

The research of J. Jankauskas was supported by FWF grant “Number Systems, Spectra and Rational Fractal Tiles” number M 2259-N35 under Lise Meitner Programme.

each  $0 \leq j \leq d$ . For example,  $P(X) = X^4 + X^3 - X^2 - X - 1$  is a Littlewood polynomial. The set of all Littlewood polynomials is denoted by  $\mathcal{L}$ .

We say that a polynomial  $P(X)$  in (1.1) is a *trinomial* if it has only three non-zero coefficients  $a_j$  for  $0 \leq j \leq d$ . If the number of non-zero coefficients is four,  $P(X)$  is called a *quadrinomial*.

We say that a polynomial  $P(X)$  has a Littlewood multiple if it divides some polynomial in the set  $\mathcal{L}$ . We say that  $P(X)$  has a Newman multiple if  $P(X)$  divides some polynomial in  $\mathcal{N}$ . When we need to restrict our attention only to polynomials of fixed degree, we use the subscript  $d$  in  $\mathcal{N}_d$  and  $\mathcal{L}_d$  to denote the sets of Newman and Littlewood polynomials of degree  $d$ , respectively. We use the subscript “ $\leq d$ ” to indicate the sets of polynomials of degree *at most*  $d$ , that is

$$\mathcal{N}_{\leq d} = \bigcup_{j=0}^d \mathcal{N}_j, \quad \mathcal{L}_{\leq d} = \bigcup_{j=0}^d \mathcal{L}_j.$$

Let  $\mathcal{A} \subset \mathbb{Z}[X]$ . We will employ the notation  $\mathcal{L}(\mathcal{A})$  to denote the set of polynomials  $P(X) \in \mathcal{A}$  which divide some Littlewood polynomial. In the same way, denote by  $\mathcal{N}(\mathcal{A})$  the set of polynomials  $P(X) \in \mathcal{A}$  which divide some Newman polynomial. In particular, the set  $\mathcal{N}_d \setminus \mathcal{L}(\mathcal{N})$  consists of those Newman polynomials of degree  $d$  that do not divide any Littlewood polynomial, whereas the set  $\mathcal{L}_{\leq d} \setminus \mathcal{N}(\mathcal{L})$  consists of those Littlewood polynomials of degree at most  $d$  that do not divide any Newman polynomial.

Let  $\mathcal{D} \subset \mathbb{Z}$  be a finite set. We call  $\mathcal{D}$  a *digit set*. Consider the following question.

**Question 1.** Given a polynomial  $P \in \mathbb{Z}[X]$ , does there exist a nonzero polynomial with coefficients in  $\mathcal{D}$  which is divisible by  $P$ ?

In [3] we implemented an algorithm which answers the above question provided that the polynomial  $P(X)$  has no roots on the unit circle  $|z| = 1$  in the complex plane. As far as we know the first instance of such an algorithm appears in the work of Frougny [5] on the digit representations of numbers in algebraic integer bases produced by finite automata. Lau [8] describes a slightly different version of this algorithm used to determine discreteness property of Bernoulli measures, focused to the case when  $P(X)$  is a minimal polynomial of Pisot number. Subsequently, Hare and his coauthors [2, 6] used the same algorithm to verify the uniform discreteness property of the spectra of certain Pisot numbers. Stankov [13] considered spectra of non-Pisot algebraic integers. Akiyama, Thuswaldner and Zaïmi [1, Th. 3] treated Question 1 in the context of *height reducing problem* and arrived to the algorithm that is essentially similar to the one that was devised by Frougny [5]. Thus, Question 1 has been fully answered for separable monic polynomials  $P(X) \in \mathbb{Z}[X]$  with no roots on the unit circle. The version of this algorithm that was implemented in [3] is also capable to deal with non-separable cases  $P(X) \in \mathbb{Z}[X]$  (i.e., when  $P(X) \in \mathbb{Z}[X]$  have multiple roots). It is not known whether the non-vanishing

on  $|z| = 1$  can be dropped or not; it seems to be essential to the proof that the search terminates. In many practical cases this restriction can be circumvented (see [3], Sec. 4.1 and the note at the end of Sec. 3 therein).

In [3], we implemented this algorithm to answer Question 1 for all Borwein polynomials of degree up to 9 and the digit sets  $\mathcal{D} = \{0, 1\}$  and  $\mathcal{D} = \{-1, 1\}$ . For every Borwein polynomial of degree at most 9 it was determined whether it has a Littlewood multiple and whether it divides some Newman polynomial. Also, for every Newman polynomial  $P(X)$  of degree at most 11, it was checked whether  $P(X) \in \mathcal{L}(\mathcal{N})$ . These computations extended the results previously obtained by Dubickas and Jankauskas [4], Borwein and Hare [2], Hare and Mossinghoff [6].

In the present paper we show that all Newman quadrinomials of degree at most 15, with the small number of possible exceptions that are given in Table 1, have Littlewood multiples. Moreover, we prove that every polynomial  $P \in \mathcal{L}(\mathcal{N}_{\leq 10})$  possesses a Littlewood multiple of smallest possible degree  $\deg_2 \tilde{P}$  (see Section 4 for the details) as well as every quadrinomial  $Q \in \mathcal{N}_{\leq 14}$  does. We also compute the set  $\mathcal{N}(\mathcal{L}_{\leq 12})$  and show that it contains exactly nine polynomials which are not products of cyclotomic polynomials.

This paper is organized as follows. Main results regarding Littlewood multiples of Newman polynomials and those regarding Newman multiples of Littlewood polynomials are given in Sections 2 and 3, respectively. In Section 4 we consider Littlewood multiples of least degree. A short description of the algorithm used in this paper is given in Section 5. Finally, our computations are described in Section 6.

## 2. Littlewood multiples of Newman quadrinomials

In [3, 4] the sets  $\mathcal{L}(\mathcal{N}_d)$ ,  $\mathcal{N}_d \setminus \mathcal{L}(\mathcal{N})$  have been completely determined for  $d \leq 11$ . In particular, it was proved in [4] that every Newman polynomial of degree at most 8 divides some Littlewood polynomial, that is,  $\mathcal{L}(\mathcal{N}_{\leq 8}) = \mathcal{N}_{\leq 8}$ . On the other hand,  $\#\mathcal{N}_9 \setminus \mathcal{L}(\mathcal{N}) = 18$ ,  $\#\mathcal{N}_{10} \setminus \mathcal{L}(\mathcal{N}) = 36$  and  $\#\mathcal{N}_{11} \setminus \mathcal{L}(\mathcal{N}) = 174$ . It was also proved in [4] that every Borwein trinomial

$$X^a \pm X^b \pm 1, \quad 1 \leq b < a, \quad a, b \in \mathbb{Z}$$

(including Newman trinomials  $X^a + X^b + 1$ ) has a Littlewood multiple, as well as certain Borwein quadrinomials  $X^a \pm X^b \pm X^c \pm 1$  do. Dubickas and Jankauskas [4] asked:

**Question 2** ([4]). Does there exist a Newman quadrinomial with no Littlewood multiple? Equivalently, does the set  $\mathcal{N} \setminus \mathcal{L}(\mathcal{N})$  contain a quadrinomial?

In [3], this question was extended to include Borwein quadrinomials; 20 such quadrinomials in  $\mathcal{B}_{\leq 9} \setminus \mathcal{L}(\mathcal{B})$  were found. However, none of them is a Newman quadrinomial (each of them has coefficients  $-1$  and  $1$ ).

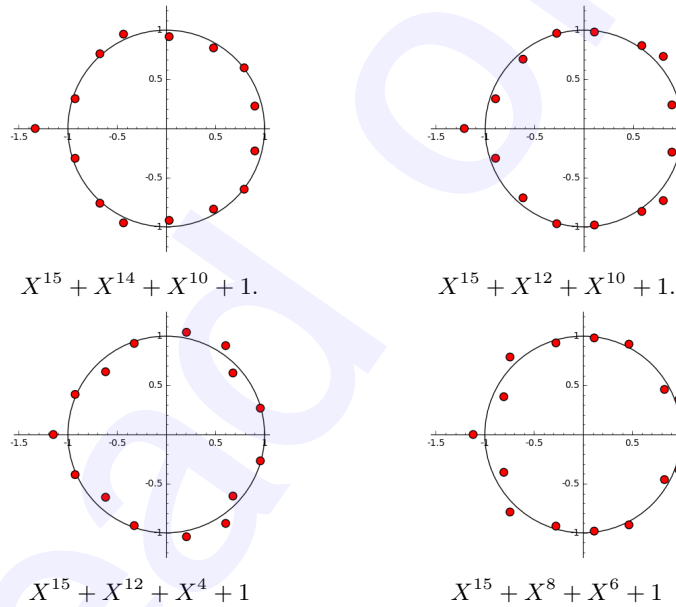
Our computations show that:

TABLE 1. Newman quadrinomials  $Q \in \mathcal{N}_{\leq 15}$  which are not known to have Littlewood multiples (reciprocals omitted).

Quadrinomial $Q(X)$	$\deg_2 \tilde{Q} - 1$
$X^{15} + X^{14} + X^{10} + 1$	10921
$X^{15} + X^{12} + X^{10} + 1$	32765
$X^{15} + X^{12} + X^4 + 1$	31681
$X^{15} + X^8 + X^6 + 1$	32765

**Theorem 3.** *Every quadrinomial  $Q \in \mathcal{N}_{\leq 15}$ , except possibly for those that are given in Table 1, divides some Littlewood polynomial. Moreover, every quadrinomial  $Q \in \mathcal{N}_{\leq 14}$  possesses a Littlewood multiple of smallest possible degree  $\deg_2 \tilde{Q} - 1$  (this quantity will be defined in Section 4).*

The second column in Table 1 indicates the smallest possible degree of a Littlewood multiple (if it exists) of  $Q \in \mathcal{N}_{15}$ . Figure 1 depicts the roots of polynomials from Table 1.

FIGURE 1. Complex roots of quadrinomials  $Q(X)$  from Table 1.

### 3. Newman multiples of Littlewood polynomials

We already mentioned, that the sets  $\mathcal{L}(\mathcal{N}_d)$  have been completely determined for  $d \leq 11$ . Also, it was proved in [3] (see Proposition 18) that if  $P(X)$  is a product of cyclotomic polynomials and  $P(1) \neq 0$ , then  $P(X)$  divides some Newman polynomial. Therefore Littlewood polynomials which are products of cyclotomic polynomials and do not vanish at  $X = 1$  have Newman multiples too.

Motivated by this result, we considered *the complementary problem* of computing the sets  $\mathcal{N}(\mathcal{L}_d)$  for small degrees  $d$ . By running the algorithm, described in Section 5, on every Littlewood polynomial  $L(X) \in \mathcal{L}_{\leq 12}$ , we calculated explicitly all non-trivial elements of the set  $\mathcal{N}(\mathcal{L}_{\leq 12})$ . They are listed in Table 2.

**Theorem 4.** *There are exactly 9 monic Littlewood polynomials  $P(X)$  of degree  $\leq 12$  that possess Newman multiples and are not themselves the products of cyclotomic polynomials.*

This result contrasts very sharply with the fact that *most* Newman polynomials of small degree have Littlewood multiples. Roots of these 9 polynomials are depicted in Figure 2.

Recall that a real algebraic integer  $\alpha > 1$  is called a *Salem number* (see, e.g., [10–12]), if all other conjugates of  $\alpha$  lie in the unit circle  $|z| \leq 1$ , with at least one conjugate on the unit circle  $|z| = 1$ . Similarly, an algebraic integer  $\alpha$  is called a *negative Salem number* if  $-\alpha$  is a Salem number. Finally, a *complex Salem number* (see, e.g., [15]) is a nonreal algebraic integer  $\alpha$  of modulus  $|\alpha| > 1$  whose other conjugates, except for  $\bar{\alpha}$ , are of moduli  $\leq 1$ , with at least one conjugate of modulus = 1. Note that the noncyclotomic parts (polynomials, obtained omitting their cyclotomic factors) of polynomials  $P_1(X)$ ,  $P_2(X)$  and  $P_7(X)$  from Table 2 are minimal polynomials of negative Salem numbers, whereas the noncyclotomic parts of polynomials  $P_3(X)$ ,  $P_4(X)$ ,  $P_5(X)$  and  $P_6(X)$  are minimal polynomials of complex Salem numbers. As a result we have the following corollary.

**Corollary 5.** *Every polynomial  $P \in \mathcal{N}(\mathcal{L}_{\leq 12})$  that is not a product of cyclotomic polynomials, has a complex root on  $|z| = 1$  which is not a root of unity.*

### 4. Littlewood multiples of minimal degree

Let  $\mathbb{F}_2$  be a finite field with two elements. Consider a polynomial  $p(X) \in \mathbb{F}_2[X]$  with a non-zero constant term  $p(0) \in \mathbb{F}_2$ . Then  $p(X)$  has a unique representation as a product

$$p(X) = (X + 1)^m \prod_{j=1}^r \phi_j(X)^{m_j},$$

where  $m \geq 0$  and  $\phi_j(X) \in \mathbb{F}_2[X]$  are irreducible polynomials of degree greater than or equal to 2 and multiplicity  $m_j \geq 1$ ,  $j = 1, 2, \dots, r$ . The product is

TABLE 2. Monic non-trivial elements of  $\mathcal{N}(\mathcal{L}_{\leq 12})$ 

$k$	$P_k(X)$
1	$X^6 + X^5 - X^4 - X^3 - X^2 + X + 1$
2	$X^7 + X^6 - X^5 + X^4 + X^3 - X^2 + X + 1$
3	$X^8 + X^7 - X^6 - X^5 + X^4 - X^3 - X^2 + X + 1$
4	$X^9 + X^8 + X^7 - X^6 - X^5 - X^4 - X^3 + X^2 + X + 1$
5	$X^{10} - X^9 + X^8 + X^7 - X^6 + X^5 - X^4 + X^3 + X^2 - X + 1$
6	$X^{10} + X^9 + X^8 - X^7 - X^6 - X^5 - X^4 - X^3 + X^2 + X + 1$
7	$X^{12} + X^{11} - X^{10} - X^9 - X^8 + X^7 + X^6 + X^5 - X^4 - X^3 - X^2 + X + 1$
8	$X^{12} - X^{11} + X^{10} + X^9 - X^8 + X^7 + X^6 + X^5 - X^4 + X^3 + X^2 - X + 1$
9	$X^{12} + X^{11} + X^{10} - X^9 - X^8 - X^7 + X^6 - X^5 - X^4 - X^3 + X^2 + X + 1$

empty if  $r = 0$ . Every polynomial  $\phi_j(X)$  divides a unique cyclotomic polynomial  $\Phi_{e_j}(X)$  of odd index  $e_j$  (see, e.g., [4, Section 4]). Let  $s$  be the least positive integer satisfying  $2^s \geq \max\{m + 1, m_1, \dots, m_r\}$ . Following [4], we define the number

$$\deg_2 p = 2^s \cdot \text{lcm}(e_1, \dots, e_r).$$

Given a polynomial  $P(X) \in \mathbb{Z}[X]$ , denote by  $\tilde{P}(X) \in \mathbb{F}_2[X]$  its reduction modulo 2. In [4], it was shown that whenever  $P(X) \in \mathbb{Z}[X]$  divides a Littlewood polynomial  $L(X)$ , then  $\deg L = k \cdot \deg_2 \tilde{P} - 1$  for some  $k \in \{1, 2, \dots\}$ . Therefore,  $\deg L \geq \deg_2 \tilde{P} - 1$ . For example,  $\deg_2 \tilde{P} = 32766$  for the polynomial  $P(X) = X^{15} + X^8 + X^6 + 1$  (see Table 1). Hence, if this particular quadrinomial has a Littlewood multiple  $L(X)$ , then  $\deg(L) \geq 32765$ . Another example is the quadrinomial  $Q(X) = X^{15} + X^{10} + X^8 + 1$ : we have computed a Littlewood multiple of  $Q(X)$  of degree  $\deg_2 \tilde{Q} - 1 = 32765$ . In general there might be more than one monic Littlewood multiple of minimal degree: Indeed, we found four distinct monic Littlewood multiples of  $P(X) = X^{12} + X^{11} + X^{10} + 1$  of degree  $\deg_2 \tilde{P} - 1 = 1189$ .

Our computations yield the following result.

**Theorem 6.** *Every polynomial  $P(X) \in \mathcal{L}(\mathcal{N}_{\leq 10})$ , except possibly for those given in Table 3, has a Littlewood multiple of smallest possible degree  $\deg_2 \tilde{P} - 1$ .*

The first polynomial in Table 3 is a product of cyclotomic polynomials:

$$P_1(X) = (X + 1)^3(X^2 - X + 1)(X^4 - X^3 + X^2 - X + 1).$$

Second and third polynomials in Table 3 factor in  $\mathbb{Z}[X]$  as follows:

$$P_2(X) = (X^2 + 1)(X^8 + X^7 - X^5 + X^3 + 1),$$

$$P_3(X) = (X + 1)(X^9 + X^6 + X^4 - X^3 + X^2 - X + 1).$$

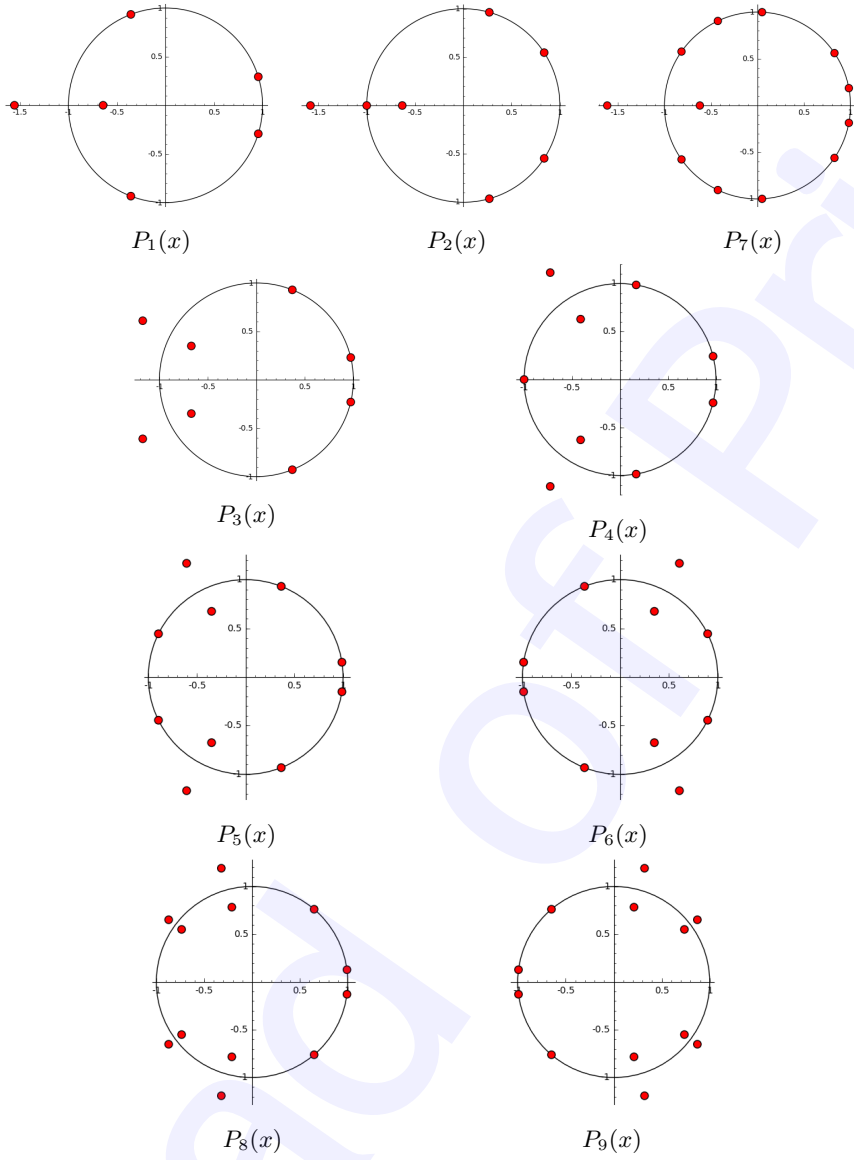


FIGURE 2. Complex roots of polynomials  $P_k(X) \in \mathcal{N}(\mathcal{L}_{\leq 12})$  from Table 2

The roots of  $P_2(X)$  and  $P_3(X)$  are depicted in Figure 3.

The corresponding result on Littlewood multiples of Newman quadrinomials was stated in Theorem 3 of Section 2.

TABLE 3. Polynomials  $P(X) \in \mathcal{L}(\mathcal{N}_{\leq 10})$  which are not known to have Littlewood multiples of degree  $\deg_2 \tilde{P} - 1$  (reciprocals omitted).

$n$	$P_n(X)$	$\deg_2 \tilde{P}_n$
1	$X^9 + X^8 + X^6 + X^5 + X^4 + X^3 + X + 1$	60
2	$X^{10} + X^9 + X^8 + X^3 + X^2 + 1$	1020
3	$X^{10} + X^9 + X^7 + X^6 + X^5 + 1$	1022

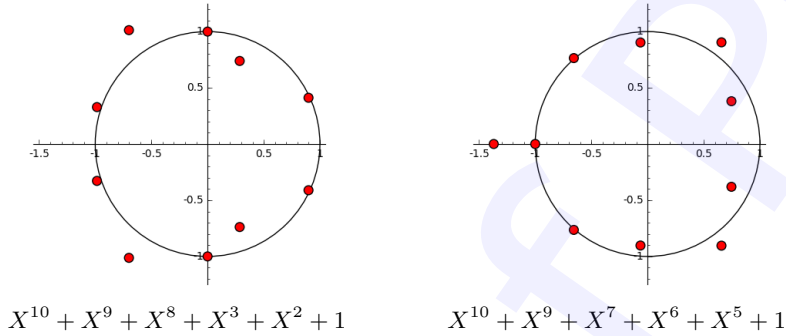


FIGURE 3. Complex roots of  $P_2(X)$  and  $P_3(X)$  from Table 3.

### 5. A short description of the algorithm

We need the following definition.

**Definition 7.** Let  $P(X)$  be a nonconstant polynomial with integer coefficients with no roots on the complex unit circle  $|z| = 1$ . Suppose that the factorization of  $P$  in  $\mathbb{C}[X]$  is

$$P(X) = a \cdot (X - \alpha_1)^{e_1} (X - \alpha_2)^{e_2} \cdots (X - \alpha_s)^{e_s},$$

where  $\alpha_1, \alpha_2, \dots, \alpha_s$  are distinct complex numbers and  $e_j \geq 1$  for  $j = 1, 2, \dots, s$ . Let  $B$  be arbitrary positive number. Define  $\mathcal{R}(P, B)$  to be the set of all polynomials  $R \in \mathbb{Z}[X]$ ,  $\deg R < \deg P$ , which, for each  $j \in \{1, 2, \dots, s\}$ , satisfy the inequalities

$$(5.1) \quad |R(\alpha_j)| \leq \frac{B}{\|\alpha_j - 1\|}, \quad |R'(\alpha_j)| \leq \frac{1!B}{\|\alpha_j - 1\|^2}, \dots, \quad |R^{(e_j-1)}(\alpha_j)| \leq \frac{(e_j-1)!B}{\|\alpha_j - 1\|^{e_j}}.$$

Here  $R^{(k)}$  denotes the  $k$ th derivative of the polynomial  $R$ , and  $R^{(0)} := R$ .

Let  $\mathcal{D} \subset \mathbb{Z}$  be a finite set and let  $\mathcal{G} = \mathcal{G}(P, \mathcal{D})$  be a directed graph whose vertices represent all the distinct polynomials  $R \in \mathcal{R}(P, B) \cup \mathcal{D}$ , where  $B = \max\{|b| : b \in \mathcal{D}\}$ . Two vertices corresponding to polynomials  $R_i$  and  $R_j$  are



connected by a directed edge from  $R_i$  to  $R_j$ , if  $R_j \equiv X \cdot R_i + b \pmod{P}$  in  $\mathbb{Z}[X]/(P)$  for some digit  $b \in \mathcal{D}$ .

It was proved in [1, 3, 5] that, given a polynomial  $P(X) \in \mathbb{Z}[X]$ , which satisfies the conditions of Definition 7, the set  $\mathcal{R}(P, B)$  is finite for every  $B > 0$ . Therefore, the graph  $\mathcal{G} = \mathcal{G}(P, \mathcal{D})$  is finite for such polynomials  $P$ .

We restate Theorem 16 from [3] without proof.

**Theorem 8.** *Let  $P \in \mathbb{Z}[X]$  be a monic polynomial with no roots on the complex unit circle  $|z| = 1$ . Then  $P$  divides an integer polynomial*

$$Q(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 \in \mathbb{C}[X]$$

*with all the coefficients  $a_j \in \mathcal{D}$  and the leading coefficient  $a_n \in \mathcal{D}$ , if and only if the graph  $\mathcal{G} = \mathcal{G}(P, \mathcal{D})$  contains a path which starts at the remainder polynomial  $R(X) = a_n$  and ends at  $R(X) = 0$ . The length of the path is  $n$ , where  $n$  is the degree of  $Q$ .*

The polynomial  $Q$  with the coefficients in the set  $\mathcal{D}$  can be found by running any path finding algorithm on  $\mathcal{G} = \mathcal{G}(P, \mathcal{D})$ .

If  $P(X) \in \mathbb{Z}[X]$  has unimodular roots that are not roots of unity, we exclude them when checking the inequalities (5.1) of Definition 7 and continue building the graph  $\mathcal{G}(P, \mathcal{D})$ . Even in cases where  $\mathcal{G}(P, \mathcal{D})$  is infinite, the component that is accessible from  $R_0 = a_n$  might be finite, or this component might be simple enough so an existing path from  $R_0$  to  $R = 0$  can be found within the reasonable time. Thus, in practice, it is often the case that Question 1 can be answered for such polynomials.

## 6. Computations

We implemented the algorithm described in Section 5 in C using Arb [7] library for arbitrary-precision floating-point ball arithmetic and ran it on the SGI Altix 4700 server at Vilnius University. OpenMP [9] was used for the multiprocessing.

For every  $P \in \mathcal{L}_{\leq 12}$ , we calculated whether it divides some Newman polynomial. In the same way, for every Newman quadrinomial  $Q \in \mathcal{N}_{\leq 15}$ , except for those given Table 1, we found a Littlewood multiple of  $Q$ . A slightly modified code was used to search for the smallest degree Littlewood multiples of Newman polynomials  $P \in \mathcal{L}(\mathcal{N}_{10})$  and Newman quadrinomials  $Q \in \mathcal{N}_{\leq 15}$  (see Section 4). We will briefly explain how these calculations were organized.

By Proposition 18, Section 4.1 of [3], one can omit cyclotomic factors of  $P(X) \in \mathbb{Z}[X]$  when searching for its Littlewood multiples: if  $\Phi_n(X)$  is the  $n$ -th cyclotomic polynomial, then  $\Phi_n P \in \mathcal{L}(\mathbb{Z}[X])$  if and only if  $P \in \mathcal{L}(\mathbb{Z}[X])$ . In particular, every polynomial that is a product of cyclotomic polynomials has a Littlewood multiple. For every quadrinomial  $Q \in \mathcal{N}_{\leq 15}$  we omitted all of its cyclotomic factors and ran the algorithm to check whether  $Q$  has a Littlewood multiple. Similarly, we ran the algorithm for every  $P \in \mathcal{L}(\mathcal{N}_{\leq 10})$ . Only for

polynomials given in Table 1 and Table 3 the computations did not terminate normally due to limits on our server machine.

Similarly, one can omit cyclotomic factors, except for  $\Phi_1(X) = X - 1$ , when searching for Newman multiples: for  $n > 1$ ,  $\Phi_n P \in \mathcal{N}(\mathbb{Z}[X])$  if and only if  $P \in \mathcal{N}(\mathbb{Z}[X])$ . Also, real roots of Newman polynomials must be negative, therefore  $P \in \mathcal{N}(\mathbb{Z}[X])$  implies that  $P$  has no positive real roots. We used SAGE [14] to filter out all such polynomials from  $\mathcal{L}_{\leq 12}$  and then ran the algorithm on the noncyclotomic parts of the remaining polynomials to check whether they have Newman multiples.

Some of the polynomials in sets  $\mathcal{L}_{\leq 12}$  and  $\mathcal{N}_{\leq 10}$  have unimodular roots that are not roots of unity. However, we succeeded in determining whether these polynomials belong to  $\mathcal{L}(\mathcal{B})$  and  $\mathcal{N}(\mathcal{B})$  (see Section 5 above).

Note that in the inequalities (5.1) of Section 5, one has the upper bound  $B = \max\{|a| \mid a \in \mathcal{D}\} = 1$  in case of Littlewood and Newman multiples. To fasten the search, a new variable  $\delta$ ,  $0 \leq \delta < 1$  was introduced, replacing the bound  $B$  by  $B - \delta$ . This eliminates some of the vertices in the original graph  $\mathcal{G}(P, \mathcal{D})$ . We start with the initial value  $\delta = 0.95$ . If a Littlewood (or Newman) multiple is found, then we are done. Otherwise we decrease  $\delta$  by 0.05 and try again. For polynomials in  $\mathcal{N} \setminus \mathcal{L}(\mathcal{N})$  and  $\mathcal{L} \setminus \mathcal{N}(\mathcal{L})$  the variable  $\delta$  always reaches the value  $\delta = 0$  in order to construct the full graph  $\mathcal{G}(P, \mathcal{D})$ .

All computations took approximately 49 hours of CPU time. The maximum recursion depth reached by the algorithm was 98 936. It took about 90 seconds of CPU time to run our algorithm to find the Littlewood multiple  $L(X)$  of  $P(X) = X^{15} + X^{10} + X^8 + 1$  of smallest possible degree  $\deg L = \deg_2 \tilde{P} - 1 = 32\,765$ ; 283 501 vertices of the graph  $\mathcal{G}(P, \{-1, 1\})$  were constructed. On the other hand, it took 5960 seconds of CPU time to find a Littlewood multiple  $L(X)$  of  $P(X) = X^{14} - X^{12} + X^3 + 1$  of smallest possible degree  $\deg L = \deg_2 \tilde{P} - 1 = 16\,381$ ; more than one million vertices of the graph  $\mathcal{G}(P, \{-1, 1\})$  were constructed. For every polynomial in Table 1 and Table 3 it took 2 hours of CPU time before the computation was interrupted due to limitations of computational resources.

## References

- [1] S. Akiyama, J. M. Thuswaldner, and T. Zaïmi, *Comments on the height reducing property II*, *Indag. Math. (N.S.)* **26** (2015), no. 1, 28–39.
- [2] P. Borwein and K. G. Hare, *Some computations on the spectra of Pisot and Salem numbers*, *Math. Comp.* **71** (2002), no. 238, 767–780.
- [3] P. Drungilas, J. Jankauskas, and J. Šiurys, *On Littlewood and Newman polynomial multiples of Borwein polynomials*, *Math. Comp.* **87** (2018), no. 311, 1523–1541.
- [4] A. Dubickas and J. Jankauskas, *On Newman polynomials which divide no Littlewood polynomial*, *Math. Comp.* **78** (2009), no. 265, 327–344.
- [5] C. Frougny, *Representations of numbers and finite automata*, *Math. Systems Theory* **25** (1992), no. 1, 37–60.
- [6] K. G. Hare and M. J. Mossinghoff, *Negative Pisot and Salem numbers as roots of Newman polynomials*, *Rocky Mountain J. Math.* **44** (2014), no. 1, 113–138.

- [7] F. Johansson, *Arb: efficient arbitrary-precision midpoint-radius interval arithmetic*, IEEE Trans. Comput. **66** (2017), no. 8, 1281–1292.
- [8] K.-S. Lau, *Dimension of a family of singular Bernoulli convolutions*, J. Funct. Anal. **116** (1993), no. 2, 335–358.
- [9] OpenMP Architecture Review Board, *OpenMP*, Application Program Interface Version 4.0, 2013, <http://www.openmp.org/mp-documents/OpenMP4.0.0.pdf>.
- [10] R. Salem, *A remarkable class of algebraic integers. Proof of a conjecture of Vijayaraghavan*, Duke Math. J. **11** (1944), 103–108.
- [11] ———, *Power series with integral coefficients*, Duke Math. J. **12** (1945), 153–172.
- [12] ———, *Algebraic Numbers and Fourier Analysis*, D. C. Heath and Co., Boston, MA, 1963.
- [13] D. Stankov, *On spectra of neither Pisot nor Salem algebraic integers*, Monatsh. Math. **159** (2010), no. 1–2, 115–131.
- [14] The Sage Developers, *SageMath*, the Sage Mathematics Software System (Version 7.6), 2017; <http://www.sagemath.org>.
- [15] T. Zaimi, *On real parts of powers of complex Pisot numbers*, Bull. Aust. Math. Soc. **94** (2016), no. 2, 245–253.

PAULIUS DRUNGILAS  
DEPARTMENT OF MATHEMATICS AND INFORMATICS  
VILNIUS UNIVERSITY  
NAUGARDUKO 24, VILNIUS LT-03225, LITHUANIA  
*Email address:* pdrungilas@gmail.com

JONAS JANKAUSKAS  
MATHEMATIK UND STATISTIK, MONTANUNIVERSITÄT LEOBEN  
FRANZ JOSEF STRASSE 18, A-8700 LEOBEN, AUSTRIA  
*Email address:* jonas.jankauskas@gmail.com

GRINTAS JUNEVIČIUS  
DEPARTMENT OF MATHEMATICS AND INFORMATICS  
VILNIUS UNIVERSITY  
NAUGARDUKO 24, VILNIUS LT-03225, LITHUANIA  
*Email address:* grintas@gmail.com

LUKAS KLEBONAS  
DEPARTMENT OF MATHEMATICS AND INFORMATICS  
VILNIUS UNIVERSITY  
NAUGARDUKO 24, VILNIUS LT-03225, LITHUANIA  
*Email address:* lukasklebonas@gmail.com

JONAS ŠIURYS  
DEPARTMENT OF MATHEMATICS AND INFORMATICS  
VILNIUS UNIVERSITY  
NAUGARDUKO 24, VILNIUS LT-03225, LITHUANIA  
*Email address:* jonas.siurys@mif.vu.lt