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ON THE LU QI-KENG PROBLEM FOR SLICE MONOGENIC FUNCTIONS

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Abstract. In this note, it is proven that the slice Bergman kernels for some axially symmetric slice domains are zero-free by a simple method.

1. Introduction

Determining whether the Bergman kernel for a domain is zero-free has been a well-known open problem in complex analysis since Lu Qi-Keng in [15] raised the question related to the existence of Bergman representative coordinates. The Lu Qi-Keng conjecture can be formulated as (see e.g., [14])

The Bergman kernel associated to a topologically trivial domain in $\mathbb{C}^n$ is zero-free.

In the complex plane $\mathbb{C}$, the Bergman kernel is zero-free only for a simply connected domain. However, it is not true even for some simply connected domains in several complex variables [1]. As a generalization of complex analysis to the noncommutative setting, the theory of slice regular (and slice monogenic) functions has obtained very rapid developments; see [3, 8, 11–13] and references therein. Function-theoretic properties for Bergman spaces are also generalized from the one complex variable case to the quaternionic and Clifford algebraic settings [2, 4–7, 17]. Very recently, J.-D. Park in [16] has proved that the slice regular (resp. monogenic) Bergman kernels for the half plane and the open unit ball of quaternions $\mathbb{H}$ (resp. $\mathbb{R}^{n+1}$) are zero-free.

Denote by $\mathfrak{R}(\mathbb{R}^{n+1})$ the class of axially symmetric slice domains $\Omega \subset \mathbb{R}^{n+1}$ such that $\Omega_I := \Omega \cap \mathbb{C}_I$ are simply connected for some imaginary unit I. See Section 2 for precise definitions. It is easy to see that the half plane and the open unit ball of $\mathbb{R}^{n+1}$ belong to $\mathfrak{R}(\mathbb{R}^{n+1})$. In this paper, we extend main results in [16] to the more general domain $\mathfrak{R}(\mathbb{R}^{n+1})$ by using new but short methods as follows.

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Theorem 1.1. *Let $\Omega \in \mathfrak{R}(\mathbb{R}^{n+1})$. Then the slice Bergman kernel $\mathcal{K}_\Omega(\cdot,\cdot)$ associated to Ω is zero-free in $\Omega \times \Omega$.*

In fact, Theorem 1.1 still holds for slice Bergman kernels over quaternions. Note that, for the special case $\{q \in \mathbb{H} : \operatorname{Re} q > 0\}$ or $\{q \in \mathbb{H} : |q| < 1\}$, the proof given in [16] is based on the explicit expression for the corresponding slice Bergman kernel. To overcome this obstacle for general domains and circumvent complicated calculations, we first provide a new approach making use of the theory of zero sets for slice regular functions over quaternions. However, this method cannot be used directly in the case of slice monogenic functions in Theorem 1.1. One of the main reasons is that quaternions is a division algebra while Clifford algebras $\mathbb{R}_{0,n}$ for $n \geq 3$ has zero divisors. Fortunately, we also can prove it in the Clifford algebraic setting via a Riemann mapping theorem for the domain in $\mathfrak{R}(\mathbb{R}^{n+1})$ in terms of slice monogenic functions.

The well-known Riemann mapping theorem plays a vital role in complex analysis. The Riemann mapping theorem fails dramatically in several complex variables generally as shown by Poincaré. To prove Theorem 1.1, we shall rewrite the Riemann mapping theorem in the framework of slice monogenic functions based on the complex results in [10] as follows.

Theorem 1.2. *Let $\mathbb{B} = \{x \in \mathbb{R}^{n+1} : |x| < 1\}$ be the open unit ball in $\mathbb{R}^{n+1}$, $\Omega \in \mathfrak{R}(\mathbb{R}^{n+1})$ and $x_0 \in \Omega \cap \mathbb{R}$. Then there exists a unique slice bimonogenic function $f : \Omega \to \mathbb{B}$ such that $f(x_0) = 0$ and $f'(x_0) > 0$.*

Note that the restriction of domains in Theorem 1.2 is legitimate since axially symmetric slice domains are natural domains of definition for the slice monogenic functions from Lemma 2.6 below.

The remaining part of this paper is organized as follows. In Section 2, we set up basic notation and give some preliminary results from the theory of slice monogenic functions. Section 3 is devoted to the proof of Theorem 1.2. Theorem 1.1 in the cases of slice regular functions and of slice monogenic functions will be proved in Section 4.

2. Preliminaries

Now we recall some preliminary definitions and results from the theory of slice monogenic functions (see e.g., [8]).

The Clifford algebra $\mathbb{R}_{0,n}$ is a real universal associative algebra generated by $e_1, e_2, \ldots, e_n$, subject to the relations

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0, \quad i, j = 1, \ldots, n.$$

As a vector space, the dimension of $\mathbb{R}_{0,n}$ is 2^n. Each element in $\mathbb{R}_{0,n}$ can be represented as

$$b = \sum_{|A|=0}^{n} b_A e_A,$$

where $b_A \in \mathbb{R}$, $e_A = e_{h_1} \dots e_{h_r}$ with $A = \{h_1, \dots, h_r\} \subseteq \{1, \dots, n\}$ such that $1 \le h_1 < \dots < h_r \le n$. Notice that when $|A| = 0$, we have $A = \varnothing$ so that we take the convention that $e_\varnothing = e_0 = 1$. The modulus of the Clifford number b is defined by

$$|b| = \Big(\sum_{|A|=0}^{n} |b_A|^2 \Big)^{\frac{1}{2}}.$$

This modulus is associated to a scalar product defined as $\langle a, b\rangle = \mathrm{Re}\,(\overline{a}b) = \mathrm{Re}\,(\overline{b}a)$, where $\overline{a}$ denotes the conjugate of a.

A paravector x in $\mathbb{R}^{n+1}$ can be taken as a Clifford number $x = \sum_{i=0}^{n} x_i e_i$, so that it has inverse $x^{-1} = \overline{x}|x|^{-2}$ provided $x \neq 0$.

We denote by $\mathbb{S}$ the unit $(n-1)$-sphere of $\mathbb{R}^n$, i.e.,

$$\mathbb{S} = \big\{ x = x_1 e_1 + \dots + x_n e_n \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 = 1 \big\},$$

whose elements I are characterized by the property $I^2 = -1$. Now we denote by $\mathbb{C}_I = \mathbb{R} + \mathbb{R}I$, a complex plane passing through the origin, 1, and I. It inherits a natural complex structure.

Notice that any element $x \in \mathbb{R}^{n+1}$ can be expressed as $z = u + vI$, where $u, v \in \mathbb{R}$ and $I \in \mathbb{S}$. For a function $f : \Omega \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_{0,n}$, denote by f_I the restriction of f on $\Omega_I := \Omega \cap \mathbb{C}_I$.

Definition 2.1. Let Ω be a domain in $\mathbb{R}^{n+1}$ and $f \in C^1(\Omega, \mathbb{R}_{0,n})$. We say that f is a (left) slice monogenic function on Ω if, for any $I \in \mathbb{S}$, we have

$$\overline{\partial}_I f(u+vI) := \frac{1}{2}\Big(\frac{\partial}{\partial u} + I\frac{\partial}{\partial v}\Big) f_I(u+vI) = 0$$

on Ω_I.

From now on, we will denote by $\mathcal{SM}(\Omega)$ the set of slice monogenic functions in Ω.

For $f \in \mathcal{SM}(\Omega)$, its slice derivative f' at $x = u + vI$ is given by

$$f'(x) = f'(u+vI) = \frac{\partial}{\partial u} f_I(u+vI).$$

Lemma 2.2 (Splitting Lemma)**.** *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be a domain and let $f \in \mathcal{SM}(\Omega)$. For every $I = I_1 \in \mathbb{S}$, let $I_2, \dots, I_n$ be a completion to a basis of $\mathbb{R}_{0,n}$ satisfying the defining relations $I_r I_s + I_s I_r = -2\delta_{rs}$. Then there exist 2^{n-1} holomorphic functions $F_A : \Omega_I \to \mathbb{C}_I$ such that, for any $z = u + Iv \in \Omega_I$,*

$$f_I(z) = \sum_{|A|=0}^{n-1} F_A(z) I_A,$$

where $I_A = I_{i_1} \cdots I_{i_s}$ with subscript $A = \{i_1, \dots, i_s\} \subset \{2, \dots, n\}$ and $i_1 < \dots < i_s$. We interpret $I_A = I_\emptyset = 1$ when $|A| = 0$.

Definition 2.3. Let Ω be a domain in $\mathbb{R}^{n+1}$.

1. Ω is called a *slice domain* if it intersects the real axis and if, for all $I \in \mathbb{S}$, Ω_I is a domain of $\mathbb{C}_I$.

2. Ω is called *axially symmetric* if for any point $x + yI \in \Omega$, with $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$, the entire sphere $x + y\mathbb{S}$ is contained in Ω.

Lemma 2.4 (Identity Principle)**.** *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be a slice domain and $f, g \in \mathcal{SM}(\Omega)$. If, for some $I \in \mathbb{S}$, f and g coincide on a subset of Ω_I having an accumulation point in Ω_I, then $f = g$ in Ω.*

This fact implies that:

Lemma 2.5 (Representation Formula)**.** *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be an axially symmetric slice domain and $f \in \mathcal{SM}(\Omega)$. Let $u + vI \in \Omega$ with $I \in \mathbb{S}$. Then for all $J \in \mathbb{S}$, the following equality holds*

$$f(u + vJ) = \frac{1}{2}\Big(f(u + vI) + f(u - vI)\Big) + \frac{1}{2}JI\Big(f(u - vI) - f(u + vI)\Big).$$

The Representation formula allows one to construct a slice monogenic function from a holomorphic function defined on a (complex) conjugate invariant domain.

Lemma 2.6 (Extension Lemma)**.** *Let G be a domain in $\mathbb{C}_I$ symmetric with respect to the real axis and such that $G \cap \mathbb{R} \neq \emptyset$. Let Ω_G be the axially symmetric slice domain defined by*

$$\Omega_G = \bigcup_{u+vI \in G, J \in \mathbb{S}} u + vJ.$$

If $f : G \to \mathbb{R}_{0,n}$ is holomorphic (i.e. $\overline{\partial_I} f = 0$), then the function

$$\operatorname{ext}(f)(u + vJ) = \frac{1}{2}\Big(f(u + vI) + f(u - vI)\Big) + \frac{1}{2}JI\Big(f(u - vI) - f(u + vI)\Big)$$

is the unique slice monogenic extension of f to Ω_G.

3. Proof of Theorem 1.2

In this section, we first recall the following version of the Riemann mapping theorem (see e.g., [9, 10]) to prove Theorem 1.2.

Theorem 3.1. *Let $G \subset \mathbb{C}$ be a nonempty simply connected domain such that $G \cap \mathbb{R} \neq \emptyset$. For a fixed $x_0 \in G \cap \mathbb{R}$, there exists a unique biholomorphic function $f : G \to \mathbb{D}$ with $f(x_0) = 0, f'(x_0) > 0$; then f is such that f^{-1} is typically real if and only if G is symmetric with respect to the real axis if and only if f is complex intrinsic.*

Here the complex-valued function defined on the open set $G \subseteq \mathbb{C}$ symmetric with respect to the real axis and such that $\overline{f(z)} = f(\overline{z})$ ia called intrinsic [19]. The function is called typically real if this function defined on the open unit disc $\mathbb{D}$ is univalent and takes real values just on the real line [9]. Typically real functions have real coefficients when expanded into power series at the origin and so they are complex intrinsic. The image of such mappings is symmetric with respect to the real line.

In analogy with the concept of complex biholomorphism, we introduce the following definition.

Definition 3.2. Let Ω, Ξ be axially symmetric slice domains and let $f \in \mathcal{SM}(\Omega)$ be such that $f(\Omega) = \Xi$. Then f is called slice bimonogenic if there exists $f^{-1} \in \mathcal{SM}(\Xi)$ such that $f^{-1} \circ f = I_\Omega$ and $f \circ f^{-1} = I_\Xi$.

Let us introduce two subclasses of slice monogenic functions. Let Ω be a domain in $\mathbb{R}^{n+1}$. Denote

$$\mathcal{N}(\Omega) = \{f \in \mathcal{SM}(\Omega) : f(\Omega_I) \subseteq \mathbb{C}_I \text{ for all } I \in \mathbb{S}\}.$$

Another subclass of slice monogenic functions is defined as

$$\mathcal{V}(\Omega) = \{f \in \mathcal{SM}(\Omega) : f(\Omega_I) \subseteq \mathbb{C}_I \text{ for some } I \in \mathbb{S}\}.$$

Those two function classes are very useful in the sequel.

Remark 3.3. The composition of slice monogenic functions, when defined, does not give generally a slice monogenic function. However, if $f \in \mathcal{N}(\Omega)$, $g \in \mathcal{SM}(\Xi)$ be such that $f(\Omega) \subseteq \Xi$, then the composition $g \circ f$ is slice monogenic. Similarly, the point-wise product of two slice monogenic functions is not slice monogenic in general. But it holds that $fg \in \mathcal{SM}(\Omega)$ for any $f \in \mathcal{N}(\Omega)$ and $g \in \mathcal{SM}(\Omega)$.

Now we are in a position to prove Theorem 1.2. Although the coming proof follows the pattern as in [10], we include it for the sake of completeness.

Proof of Theorem 1.2. Let $\Omega \in \mathfrak{R}(\mathbb{R}^{n+1})$. Consider the domain Ω_I for some $I \in \mathbb{S}$. Then $\Omega_I \subset \mathbb{C}_I$ is simply connected by definition and symmetric with respect to the real axis since Ω is axially symmetric. By Theorem 3.1, there exists a unique complex intrinsic biholomorphic function $f_I : \Omega_I \to \mathbb{B}_I$ such that $f_I(x_0) = 0, f_I'(x_0) > 0$. From Lemma 2.6, we obtain the unique function $f = \mathrm{ext}(f_I) \in \mathcal{SM}(\Omega)$. Furthermore, it can be easily proved from Lemma 2.5 that $f \in \mathcal{N}(\Omega)$, which implies that $f_J : \Omega_J \to \mathbb{B}_J$ for all $J \in \mathbb{S}$. Note that $\mathbb{B} = \bigcup_{J\in\mathbb{S}} \mathbb{B}_J$ and then we have $f(\Omega) = \mathbb{B}$. By again applying the fact that $f \in \mathcal{N}(\Omega)$, there exists $f^{-1} \in \mathcal{SM}(\mathbb{B})$ such that $f^{-1} \circ f = I_\Omega$ and $f \circ f^{-1} = I_\mathbb{B}$. This completes the proof. □

4. Proof of Theorem 1.1

In this section, we first recall some necessary definitions and preliminary results on slice Bergman modules and slice Bergman kernels for slice monogenic functions [5].

Let Ω be an axially symmetric slice domain in $\mathbb{R}^{n+1}$. For any $I \in \mathbb{S}$, the Bergman modules associated to Ω_I for slice monogenic functions f is defined as

$$\mathcal{A}(\Omega_I) = \Big\{ f \in \mathcal{SM}(\Omega) : \int_{\Omega_I} |f_I|^2 d\sigma < +\infty \Big\},$$

endowed with the scalar product

$$\langle f, g \rangle_{\mathcal{A}(\Omega_I)} = \int_{\Omega_I} \overline{f_I} g_I d\sigma,$$

where $d\sigma$ denotes the area element in the complex plane $\mathbb{C}_I$ and it does not depend on $I \in \mathbb{S}$.

The function $\mathcal{K}_{\Omega_I}(\cdot, \cdot) : \Omega_I \times \Omega_I \to \mathbb{R}_{0,n}$ will be called the slice Bergman kernel associated to Ω_I, for $I \in \mathbb{S}$. The kernel $\mathcal{K}_{\Omega_I}$ has the reproducing property

$$f_I(z) = \int_{\Omega_I} \mathcal{K}_{\Omega_I}(z, \xi) f_I(\xi) d\sigma(\xi).$$

As a consequence of Lemma 2.5, the following result holds.

Proposition 4.1. *The Bergman kernel $\mathcal{K}_{\Omega_I}$ associated to Ω_I coincides with the complex Bergman kernel for Ω_I for any $I \in \mathbb{S}$.*

Proposition 4.1 implies that the slice Bergman kernel $\mathcal{K}_{\mathbb{B}_I}(\cdot, \cdot)$ associated to $\mathbb{B}_I$ is given by

$$\mathcal{K}_{\mathbb{B}_I}(z, \xi) = \frac{1}{\pi} \frac{1}{(1 - z\bar{\xi})^2}.$$

The slice Bergman kernel associated to Ω is defined by the function $\mathcal{K}_{\Omega}(\cdot, \cdot) : \Omega \times \Omega \to \mathbb{R}_{0,n}$ satisfying

$$\mathcal{K}_{\Omega}(x, y) = \frac{1}{2}(1 - JI) K_{\Omega_I}(u + vI, y) + \frac{1}{2}(1 + JI) K_{\Omega_I}(u - vI, y)$$

for all $x = u + vJ$, $y \in \Omega$, $J \in \mathbb{S}$.

Hence the slice Bergman kernel on the unit ball $\mathbb{B}$ is given by

$$\mathcal{K}_{\mathbb{B}}(x, y) = \frac{1}{\pi}(1 - 2x\mathrm{Re}\,(y) + x^2|y|^2)^{-2}(1 - 2xy + x^2y^2).$$

The corresponding definitions can be repeated in the framework of Bergman spaces for slice regular functions but we omit them here.

Denote by $\mathfrak{R}(\mathbb{H})$ the class of axially symmetric slice domains $\Omega \subset \mathbb{H}$ such that $\Omega_I = \Omega \cap \mathbb{C}_I$ are simply connected for some $I \in \mathbb{S}^2 = \{q \in \mathbb{H} : q^2 = -1\}$. We now formulate Theorem 1.1 in the quaternionic setting as follows.

Theorem 4.2. *Let $\Omega \in \mathfrak{R}(\mathbb{H})$. Then the slice Bergman kernel $\mathcal{K}_{\Omega}(\cdot, \cdot)$ associated to Ω is zero-free in $\Omega \times \Omega$.*

Proof. Let $\mathcal{K}_{\Omega}(\cdot, \cdot)$ be the slice Bergman kernel associated to $\Omega \in \mathfrak{R}(\mathbb{H})$. Assume that $\mathcal{K}_{\Omega}(q, r) = 0$ for some $(q, r) \in \Omega \times \Omega$. Set $q = u + vJ$, $r = x + yI$ for some $J, I \in \mathbb{S}^2$. Then, the symmetrization of $\mathcal{K}_{\Omega}(\cdot, r)$ is $\mathcal{K}^s_{\Omega}(\cdot, r)$ which vanishes identically on $[q]=\{u+vK : K\in\mathbb{S}^2\}$ (see e.g., [13, Proposition 3.9]). Specially, we have $\mathcal{K}^s_\Omega(z,r)=0$ for $z\in[q]\cap\mathbb{C}_I$. Due to the fact $\mathcal{K}_{\Omega_I}(\cdot,r)\subseteq\mathbb{C}_I\setminus\{0\}$ and [13, Theorem 3.4], there holds, for $z\in[q]\cap\mathbb{C}_I$,

$$\begin{aligned}\mathcal{K}^s_\Omega(z,r)&=\mathcal{K}_\Omega(z,r)\mathcal{K}^c_\Omega\big(\mathcal{K}_\Omega(z,r)^{-1}z\mathcal{K}_\Omega(z,r)\big)\\&=\mathcal{K}_\Omega(z,r)\mathcal{K}^c_\Omega(z,r)=\mathcal{K}_\Omega(z,r)\overline{\mathcal{K}_\Omega(\overline{z},r)}=0,\end{aligned}$$

which implies $\mathcal{K}_\Omega(z,r)=0$ or $\mathcal{K}_\Omega(\overline{z},r)=0$. From the version of Proposition 4.1 in quaternionic setting, the complex Bergman kernel $\mathcal{K}_{\Omega_I}(\cdot,\cdot)$ is zero-free in $\Omega_I\times\Omega_I$. It is a contradiction, as desired. □

In order to prove Theorem 1.1 in the Clifford algebraic setting, we establish the following result.

Proposition 4.3. *Let $\Omega\in\mathfrak{R}(\mathbb{R}^{n+1})$. Then there exists a unique slice bimonogenic function $f\in\mathcal{N}(\Omega)$ with $f(\Omega)=\mathbb{B}$ such that*

$$\mathcal{K}_\Omega(x,y)=f'(x)\mathcal{K}_{\mathbb{B}}(f(x),f(y))\overline{f'(y)}. \tag{4.1}$$

Proof. The proof goes as that of Proposition 4.2 in [6]. Let $\Omega\in\mathfrak{R}(\mathbb{R}^{n+1})$. From the proof in Theorem 1.2, there exists a unique slice bimonogenic function $f\in\mathcal{N}(\Omega)$ such that $f(\Omega)=\mathbb{B}$. It remains to check formula (4.1).

Let $y\in\Omega_I$ be fixed for some $I\in\mathbb{S}$. Note that $f\in\mathcal{N}(\Omega)$ implies that $f'\in\mathcal{N}(\Omega)$. Then $f'(\cdot)\mathcal{K}_{\mathbb{B}}(f(\cdot),f(y))\overline{f'(y)}\in\mathcal{SM}(\Omega)$ by Remark 3.3. In view of Lemma 2.4, we need only to show

$$\mathcal{K}_{\Omega_I}(\cdot,y)=f'_I(\cdot)\mathcal{K}_{\mathbb{B}_I}(f(\cdot),f(y))\overline{f'(y)}. \tag{4.2}$$

Note again that $f\in\mathcal{N}(\Omega)$, so we have $f(\Omega_I)=\mathbb{B}_I$. Then (4.2) follows immediately from the theory of classical holomorphic Bergman spaces (see e.g., [14, Proposition 1.1.14]). The proof is completed. □

To prove Theorem 1.1, one needs only to show $\mathcal{K}_{\mathbb{B}}(\cdot,\cdot)$ is zero-free by Proposition 4.3. This result has been proved in [16]. For the reader's convenience, another new proof is presented here. We resort to the so-called convex combination identity [18]:

$$|f(u+vJ)|^2=\frac{1+\langle I,J\rangle}{2}|f(u+vI)|^2+\frac{1-\langle I,J\rangle}{2}|f(u-vI)|^2$$

for $f\in\mathcal{V}(\Omega)$, where Ω is an axially symmetric slice domain.

Hence $\mathcal{K}_{\mathbb{B}}(\cdot,\cdot)$ is zero-free in $\mathbb{B}\times\mathbb{B}$ when we notice that $\mathcal{K}_{\mathbb{B}}(\cdot,y)\in\mathcal{V}(\Omega)$ for any fixed $y\in\Omega$ and $\mathcal{K}_{\mathbb{B}_I}(\cdot,\cdot)$ is zero-free in $\mathbb{B}_I\times\mathbb{B}_I$.

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