

A NOTE ON THE GENERALIZED HEAT CONTENT FOR LÉVY PROCESSES

WOJCIECH CYGAN AND TOMASZ GRZYWNY

ABSTRACT. Let $\mathbf{X} = \{X_t\}_{t \geq 0}$ be a Lévy process in \mathbb{R}^d and Ω be an open subset of \mathbb{R}^d with finite Lebesgue measure. The quantity $H_\Omega(t) = \int_\Omega \mathbb{P}^x(X_t \in \Omega) dx$ is called the heat content. In this article we consider its generalized version $H_g^\mu(t) = \int_{\mathbb{R}^d} \mathbb{E}^x g(X_t) \mu(dx)$, where g is a bounded function and μ a finite Borel measure. We study its asymptotic behaviour at zero for various classes of Lévy processes.

1. Introduction

Let $\mathbf{X} = (X_t)_{t \geq 0}$ be a Lévy process in \mathbb{R}^d with the distribution \mathbb{P} and such that $X_0 = 0$. We denote by $p_t(dx)$ the distribution of the random variable X_t and we use the standard notation \mathbb{P}^x for the distribution related to the process \mathbf{X} started at $x \in \mathbb{R}^d$.

Let Ω be a non-empty open subset of \mathbb{R}^d such that its Lebesgue measure $|\Omega|$ is finite. We consider the following quantity associated with the process \mathbf{X} ,

$$(1) \quad H_\Omega(t) = \int_\Omega \mathbb{P}^x(X_t \in \Omega) dx = \int_\Omega \int_{\Omega-x} p_t(dy) dx, \quad t \geq 0.$$

Observe that the function $u(t, x) = \int_{\Omega-x} p_t(dy)$ is the weak solution of the initial value problem

$$(2) \quad \begin{aligned} \frac{\partial}{\partial t} u(t, x) &= \mathcal{L} u(t, x), \quad t > 0, x \in \mathbb{R}^d, \\ u(0, x) &= \mathbf{1}_\Omega(x), \end{aligned}$$

where \mathcal{L} is the infinitesimal generator of the process \mathbf{X} , see (4). Therefore, $H_\Omega(t)$ can be interpreted as the amount of *heat* in Ω if its initial temperature is one whereas the initial temperature of Ω^c is zero. The quantity $H_\Omega(t)$ is called the heat content. The asymptotic behaviour - as t goes to zero - of the heat

Received September 27, 2017; Revised December 22, 2017; Accepted January 30, 2018.

2010 *Mathematics Subject Classification.* 60G51, 60J75, 35K05.

Key words and phrases. heat content, isotropic Lévy process, multivariate regular variation.

W. Cygan was supported by National Science Centre (Poland): grant 2015/17/B/ST1/00062 and by Austrian Science Fund project FWF P24028.

content related to the Brownian motion, either on \mathbb{R}^d or on compact manifolds, were studied in many papers among which [18–23]. The heat content for the isotropic stable processes in \mathbb{R}^d was studied in [3], see also [1] and [2]. The direct forerunner of the present paper is article [8] where asymptotic behaviour of (1) were found for numerous examples of Lévy processes.

In this note we study an extended version of the heat content (1). Namely, for a bounded function g and a finite Borel measure μ , we consider the quantity

$$H_g^\mu(t) = \int_{\mathbb{R}^d} \mathbb{E}^x g(X_t) \mu(dx) = \int_{\mathbb{R}^d} v(t, x) \mu(dx), \quad t \geq 0.$$

Here, the function $v(t, x) = \mathbb{E}^x g(X_t)$ is the weak solution to equation (2) with the initial condition $v(0, x) = g(x)$. Thus, $H_g^\mu(t)$ can be interpreted as the amount of *heat* in the set $\text{supp}(\mu)$ if its initial temperature is governed by the function g . On the other hand, the measure μ can be regarded as the initial distribution on \mathbb{R}^d . Notice that taking $g = \mathbf{1}_\Omega$ and $\mu(dx) = \mathbf{1}_\Omega dx$ we obtain that H_g^μ is equal to the heat content defined at (1).

On the basis of the methods developed in [8], we study the asymptotic behaviour of the quantity $H_g^\mu(t)$. We now display the results together with necessary facts and definitions.

Notation

By B_R we denote the closed ball $\{x \in \mathbb{R}^d : \|x\| \leq R\}$ and by \mathbb{S}^{d-1} the unit sphere in \mathbb{R}^d . Positive constants are denoted by c, C, C_1 etc. We write: $f(x) \asymp g(x)$ if there are $c, C > 0$ such that $cg(x) \leq f(x) \leq Cg(x)$; $f(x) = o(g(x))$ at x_0 if $\lim_{x \rightarrow x_0} f(x)/g(x) = 0$, and $f(x) \sim g(x)$, as $x \rightarrow x_0$, if $\lim_{x \rightarrow x_0} f(x)/g(x) = 1$. The generalized inverse V^- of the function V is given by $V^-(u) = \inf\{x \geq 0 : V(x) \geq u\}$. $C_b(\mathbb{R}^d)$ is the set of all bounded and continuous functions in \mathbb{R}^d whereas $C_0(\mathbb{R}^d)$ is the set of all continuous functions which vanish at infinity.

Results and basic facts

The Lévy-Khintchine exponent $\psi(x)$ of the Lévy process \mathbf{X} is given by the formula

$$(3) \quad \psi(x) = \langle x, Ax \rangle - i \langle x, \gamma \rangle - \int_{\mathbb{R}^d} \left(e^{i \langle x, y \rangle} - 1 - i \langle x, y \rangle \mathbf{1}_{\{\|y\| \leq 1\}} \right) \nu(dy), \quad x \in \mathbb{R}^d,$$

where A is a symmetric non-negative definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is a Lévy measure, that is $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (1 \wedge \|y\|^2) \nu(dy) < \infty$.

The heat semigroup $\{T_t\}_{t \geq 0}$ related to the Lévy process \mathbf{X} is

$$T_t f(x) = \int_{\mathbb{R}^d} f(x+y) p_t(dy) \quad \text{for } f \in C_0(\mathbb{R}^d),$$

and the generator \mathcal{L} of the process \mathbf{X} is a linear operator defined by

$$(4) \quad \mathcal{L}f = \lim_{t \rightarrow 0^+} t^{-1} (T_t f - f),$$

with the domain $\text{Dom}(\mathcal{L})$ which is the set of all f such that the right-hand side of (4) exists in the sense of uniform convergence. By [17, Theorem 31.5], we have $C_0^2(\mathbb{R}^d) \subset \text{Dom}(\mathcal{L})$ and for any $f \in C_0^2(\mathbb{R}^d)$ it has the form

$$(5) \quad \begin{aligned} \mathcal{L}f(x) &= \sum_{j,k=1}^d A_{jk} \partial_{jk}^2 f(x) + \langle \gamma, \nabla f(x) \rangle \\ &+ \int_{\mathbb{R}^d} (f(x+z) - f(x) - \mathbf{1}_{\|z\| < 1} \langle z, \nabla f(x) \rangle) \nu(dz), \quad x \in \mathbb{R}^d, \end{aligned}$$

where (A, γ, ν) is the triplet from (3). We refer the reader to [17, Section 31] or [5, Section 3.3] for a detailed discussion on infinitesimal generators of semigroups related to Lévy processes.

To start our discussion on the small time behaviour of $H_g^\mu(t)$ we make an important observation. Let g be a bounded function and μ a finite Borel measure. We set $\check{\mu}(G) = \mu(-G)$ for any Borel set $G \subset \mathbb{R}^d$ and consider the following convolution

$$r(x) = g * \check{\mu}(x) = \int_{\mathbb{R}^d} g(x+y) \mu(dy).$$

We can then write

$$(6) \quad \begin{aligned} H_g^\mu(t) &= \int_{\mathbb{R}^d} \mathbb{E}^x g(X_t) \mu(dx) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(y+x) p_t(dy) \mu(dx) \\ &= \int_{\mathbb{R}^d} g * \check{\mu}(y) p_t(dy) = T_t r(0) \end{aligned}$$

and therefore

$$\lim_{t \rightarrow 0^+} t^{-1} (H_g^\mu(t) - H_g^\mu(0)) = \mathcal{L}r(0),$$

whenever r belongs to $\text{Dom}(\mathcal{L})$. Notice that in front of formula (6), we are rather interested in the pointwise limit in (4) instead of the uniform convergence. Thus, for some special classes of Lévy processes we will weaken the assumption that $r \in \text{Dom}(\mathcal{L})$. This is summarized in Theorem 1.

Recall that according to [17, Theorem 21.9] a Lévy process \mathbf{X} has finite variation on any interval $(0, t)$ if and only if $A = 0$ and $\int_{\|x\| \leq 1} \|x\| \nu(dx) < \infty$. In this case the Lévy-Khintchine exponent has the simplified form

$$(7) \quad \psi(x) = i \langle x, \gamma_0 \rangle + \int_{\mathbb{R}^d} (1 - e^{i \langle x, y \rangle}) \nu(dy), \quad \text{with } \gamma_0 = \int_{\|y\| \leq 1} y \nu(dy) - \gamma.$$

Notice that for symmetric Lévy processes with finite variation we have

$$\int_{\|y\| \leq 1} y \nu(dy) = 0.$$

Theorem 1. *Let \mathbf{X} be a Lévy process in \mathbb{R}^d with the triplet $(0, \gamma, \nu)$ and such that*

$$\int_{\|x\| < 1} \|x\|^\beta \nu(dx) < \infty \quad \text{for some } 0 \leq \beta < 2.$$

*Assume that g is bounded, μ is finite Borel measure and that the function $r = g * \check{\mu}$ is in $C_b(\mathbb{R}^d)$. We distinguish the following cases.*

1. $0 \leq \beta \leq 1$. *In this case we assume that $\gamma_0 = 0$ and $|r(x) - r(0)| \leq C\|x\|^\beta$ for $\|x\| < 1$. Then*

$$\lim_{t \rightarrow 0^+} t^{-1} (H_g^\mu(t) - H_g^\mu(0)) = \int_{\mathbb{R}^d} (r(x) - r(0)) \nu(dx).$$

2. $1 \leq \beta < 2$. *We consider two cases*

(i) *If \mathbf{X} is symmetric (i.e., $\gamma = 0$ and ν is symmetric) we assume that function r satisfies $|r(x) + r(-x) - 2r(0)| \leq C\|x\|^\beta$ for $\|x\| < 1$. Then*

$$\lim_{t \rightarrow 0^+} t^{-1} (H_g^\mu(t) - H_g^\mu(0)) = \frac{1}{2} \int_{\mathbb{R}^d} (r(x) + r(-x) - 2r(0)) \nu(dx).$$

(ii) *If \mathbf{X} is arbitrary, assume that r is differentiable at 0 and such that $|r(x) - r(0) - \langle x, \nabla r(0) \rangle| \leq C\|x\|^\beta$ for $\|x\| < 1$. Then*

$$\begin{aligned} & \lim_{t \rightarrow 0^+} t^{-1} (H_g^\mu(t) - H_g^\mu(0)) \\ &= \langle \gamma, \nabla r(0) \rangle + \int_{\mathbb{R}^d} (r(x) - r(0) - \langle x, \nabla r(0) \rangle \mathbf{1}_{\{\|x\| \leq 1\}}) \nu(dx). \end{aligned}$$

We emphasize that Theorem 1 was recently obtained by Kühn and Schilling [14] for a wider range of stochastic processes, namely for the class of *rich Lévy-type* processes, cf. Theorem 3.5 and Theorem 4.1 in [14]. Our result is expressed in terms of the heat content and extends slightly an admissible classes of functions for Lévy processes.

The next theorem provides the asymptotic behaviour of the generalized heat content under the assumption that the Lévy-Khintchine exponent ψ is a multivariate regularly varying function, see condition (10) and [16, Chapter 6] for an elaborate approach. Recall that in the one-variable case a function $f(r)$ is regularly varying of index α at infinity, denoted by $f \in \mathcal{R}_\alpha$, if for any $\lambda > 0$, $\lim_{r \rightarrow \infty} \frac{f(\lambda r)}{f(r)} = \lambda^\alpha$. The following property, so-called *Potter bounds*, of regularly varying functions appears to be very useful, see [6, Theorem 1.5.6]. For every $C > 1$ and $\varepsilon > 0$ there is $x_0 = x_0(C, \varepsilon) > 0$ such that for all $x, y \geq x_0$

$$(8) \quad \frac{f(x)}{f(y)} \leq C \max \{ (x/y)^{\alpha-\varepsilon}, (x/y)^{\alpha+\varepsilon} \}.$$

For a given function ψ we define the related non-decreasing function ψ^* by

$$\psi^*(u) = \sup_{\|x\| \leq u} \psi(x).$$

Theorem 2. Let $\beta \in [1, 2)$ be fixed. Let \mathbf{X} be a symmetric Lévy process in \mathbb{R}^d with the Lévy-Khintchine exponent ψ . We assume that

$$(9) \quad \psi(x) \asymp \psi^*(x) \quad \text{for } \|x\| \geq 1.$$

Suppose that there is a function $V \in \mathcal{R}_\alpha$ with $\alpha \in (\beta, 2]$ and a continuous function $\Lambda: \mathbb{S}^{d-1} \rightarrow (0, \infty)$ such that

$$(10) \quad \lim_{s \rightarrow \infty} \frac{\psi(s\theta)}{V(s)} = \Lambda(\theta), \quad \theta \in \mathbb{S}^{d-1}.$$

Let g be a bounded function and μ a finite Borel measure. Set $r = g * \check{\mu}$ and assume that the below limit exists

$$\lim_{t \rightarrow 0^+} t^{-\beta} (r(t\theta) + r(-t\theta) - 2r(0)) = R_\beta(\theta) \quad \text{for all } \theta \in \mathbb{S}^{d-1}.$$

Moreover, suppose that r satisfies

$$(11) \quad |r(x) + r(-x) - 2r(0)| \leq L\|x\|^\beta \quad \text{for } L > 0.$$

Then

$$\lim_{t \rightarrow 0^+} [V^-(1/t)]^\beta (H_g^\mu(t) - H_g^\mu(0)) = \frac{1}{2} \int_{\mathbb{R}^d} R_\beta(x/\|x\|) \|x\|^\beta p_\Lambda(x) dx,$$

where the density function $p_\Lambda(x)$ is uniquely determined by the formula

$$(12) \quad e^{-\Lambda(\frac{x}{\|x\|})\|x\|^\alpha} = \int_{\mathbb{R}^d} e^{i\langle x, y \rangle} p_\Lambda(y) dy.$$

The particular choice $g = \mathbf{1}_\Omega$ and $\mu(dx) = \mathbf{1}_\Omega dx$ leads to the result for the classical heat content defined at (1). Then the function $r(x) = g * \check{\mu}(x) = |\Omega \cap (\Omega + x)|$ is the covariance function of the set Ω and the function R_β for $\beta = 1$ is determined in terms of the related directional derivative, cf. [8, Subsection 2.1] for more details.

The following corollary gives the asymptotic behaviour when the Lévy process \mathbf{X} is isotropic and its (radial) Lévy-Khintchine exponent $\psi(r)$ is a regularly varying function at infinity with index greater than one. Let us recall that a Lévy process \mathbf{X} is isotropic if the measure $p_t(dx)$ is radial (rotationally invariant) for each $t > 0$, which is equivalent to saying that the matrix $A = \lambda I$ for some $\lambda \geq 0$, the Lévy measure ν is rotationally invariant and $\gamma = 0$. For isotropic processes the Lévy-Khintchine exponent has the specific form

$$\psi(x) = \int_{\mathbb{R}^d} (1 - \cos\langle x, y \rangle) \nu(dy) + \lambda\|x\|^2$$

for some $\lambda \geq 0$. We usually abuse notation by setting $\psi(r)$ to be equal to $\psi(x)$ for any $x \in \mathbb{R}^d$ with $\|x\| = r > 0$.

By ψ^- we denote the generalized inverse of ψ^* . Using [6, Theorem 1.5.3], if $\psi \in \mathcal{R}_\alpha$ for some $\alpha > 0$, then $\psi^* \in \mathcal{R}_\alpha$ and thus $\psi^- \in \mathcal{R}_{1/\alpha}$, which implies that $\lim_{t \rightarrow 0} \psi^-(1/t) = \infty$.

The precise constant in the below formula is found by an application of a variant of [17, Eq. (25.6)].

Corollary 1. *Let $\beta \in [1, 2)$ be fixed. Let \mathbf{X} be an isotropic Lévy process in \mathbb{R}^d with the Lévy-Khintchine exponent ψ such that $\psi \in \mathcal{R}_\alpha$, $\alpha \in (\beta, 2]$. Let g be a bounded function and μ a finite Borel measure. Assume that $r = g * \tilde{\mu}$ satisfies the assumptions of Theorem 2. Then*

$$\begin{aligned} & \lim_{t \rightarrow 0^+} [\psi^-(1/t)]^\beta (H_g^\mu(t) - H_g^\mu(0)) \\ &= \pi^{-d/2} 4^{\beta/2-1} \Gamma\left(\frac{d+\beta}{2}\right) \frac{\Gamma\left(1-\frac{\beta}{\alpha}\right)}{\Gamma\left(1-\frac{\beta}{2}\right)} \int_{\mathbb{S}^{d-1}} R_\beta(\theta) \sigma(d\theta). \end{aligned}$$

The next theorem treats about the assumption on the Lévy measure, that is we require it is regularly varying according to the presentation by Resnick [16].

Theorem 3. *Let $\beta \in [1, 2)$ be fixed. Let \mathbf{X} be a symmetric Lévy process in \mathbb{R}^d with the triplet $(0, 0, \nu)$. Suppose that there is a measure η on $\mathbb{R}^d \setminus \{0\}$ such that*

$$(13) \quad \lim_{s \rightarrow 0^+} \frac{\nu(sG)}{\nu(B_s^c)} = \eta(G) \quad \text{for } G \subset \mathbb{R}^d \setminus \{0\} \text{ with } \eta(\partial G) = 0,$$

where $V(t) = \nu(B_{1/t}^c)$ is regularly varying at infinity of index $\alpha \in (\beta, 2)$. Let g be a bounded function, μ a finite Borel measure and $r = g * \tilde{\mu}$. Assume that there is a real function R_β defined on the sphere \mathbb{S}^{d-1} such that

$$\lim_{t \rightarrow 0^+} \sup_{\theta \in \mathbb{S}^{d-1}} \left| \frac{r(t\theta) + r(-t\theta) - 2r(0)}{t^\beta} - R_\beta(\theta) \right| = 0.$$

Moreover, suppose that r satisfies (11). Then

$$\lim_{t \rightarrow 0^+} [V^-(1/t)]^\beta (H_g^\mu(t) - H_g^\mu(0)) = \frac{1}{2} \int_{\mathbb{R}^d} R_\beta(x/\|x\|) \|x\|^\beta p_\eta(x) dx,$$

where the density function p_η is uniquely determined by the formula

$$e^{-\int_{\mathbb{R}^d} (1 - \cos\langle \xi, y \rangle) \eta(dy)} = \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} p_\eta(x) dx.$$

It is worth pointing out that Theorem 3 (as well as Theorem 2) applies in the case when \mathbf{X} is the stable process in \mathbb{R}^d . We also emphasize that the support of the measure η in (13) may be contained in some hyperplane of \mathbb{R}^d , see Example 5. Further, condition (13) forces the scaling property of the measure η , that is there exists some $\alpha \geq 0$ such that $\eta(tG) = t^{-\alpha} \eta(G)$ for all $t > 0$ and sets G with $\eta(\partial G) = 0$. This in turn implies that η in Theorem 3 is the Lévy measure of the α -stable law.

In the rest of the paper we first present a list of examples and concluding Section 3 is devoted to the proofs of the aforementioned results.

2. Examples

Let Ω be a non-empty open subset of \mathbb{R}^d such that its Lebesgue measure $|\Omega|$ is finite.

Example 1. Let \mathbf{X} be a Lévy process with finite variation and \mathcal{L}^0 be the generator of the process $X_t^0 = X_t + t\gamma_0$. Let $g(x) = \mathbf{1}_\Omega(x)$ and $\mu(dx) = f(x)dx$, where $f \in C_0^1(\mathbb{R}^d)$ with ∇f bounded. In particular, f is Lipschitz and $\lim_{\|x\| \rightarrow \infty} f(x) = 0$. In this case $r(x) = g * \check{f}(x)$ and it is Lipschitz with $\lim_{\|x\| \rightarrow \infty} r(x) = 0$, belongs to $\text{Dom}(\mathcal{L}^0)$ and $\nabla r = g * \nabla \check{f}$. Moreover, we claim that $\mathcal{L}^0 r(0) = \int_\Omega \mathcal{L}^0 f(x)dx$. Indeed, applying [8, Lemma 2] we obtain that

$$\begin{aligned} \mathcal{L}^0 r(0) &= \int_{\mathbb{R}^d} (r(y) - r(0)) \nu(dy) \\ &= \int_{\mathbb{R}^d} \nu(dy) \int_{\mathbb{R}^d} f(-x) (\mathbf{1}_\Omega(y-x) - \mathbf{1}_\Omega(-x)) dx \\ &= \int_{\mathbb{R}^d} \nu(dy) \int_\Omega (f(x+y) - f(x)) dx \\ &= \int_\Omega \int_{\mathbb{R}^d} (f(x+y) - f(x)) \nu(dy) dx \\ &= \int_\Omega \mathcal{L}^0 f(x) dx. \end{aligned}$$

Hence, by Theorem 1,

$$\lim_{t \rightarrow 0^+} t^{-1} (H_g^\mu(t) - H_g^\mu(0)) = \int_\Omega \mathcal{L}^0 f(x) dx + \|\gamma_0\| \nabla_{\frac{\gamma_0}{\|\gamma_0\|}} r(0) \mathbf{1}_{\mathbb{R}^d \setminus \{0\}}(\gamma_0).$$

Example 2. Let \mathbf{X} be a Lévy process in \mathbb{R}^d . Let $g(x) = \mathbf{1}_\Omega(x)$ and $\mu(dx) = \mathbf{1}_{\Omega_0} dx$, for some $\Omega_0 \subset \mathbb{R}^d$ with $|\Omega_0| < \infty$. We have $r(x) = |\Omega \cap (\Omega_0 + x)|$ and it is bounded, uniformly continuous and vanishes at infinity. We consider two cases:

Case 1. Let $\Omega \cap \Omega_0 = \emptyset$ with $\text{dist}(\Omega, \Omega_0) = D > 0$. Then $r(x) = 0$, for $\|x\| < D$ and, applying [17, Corollary 8.9], we obtain

$$t^{-1} H_g^\mu(t) = t^{-1} \int_{\mathbb{R}^d} r(x) p_t(dx) \longrightarrow \int_\Omega \nu(y - \Omega_0) dy \quad \text{as } t \rightarrow 0^+.$$

Case 2. If $\Omega \subset \Omega_0$ with $\text{dist}(\Omega, \Omega_0^c) > 0$, we similarly get that, as $t \rightarrow 0^+$,

$$t^{-1} (H_g^\mu(t) - H_g^\mu(0)) = t^{-1} \int_{\mathbb{R}^d} (r(x) - r(0)) p_t(dx) \longrightarrow - \int_\Omega \nu(y - \Omega_0^c) dy.$$

Example 3. Let \mathbf{X} be a Lévy process in \mathbb{R}^d . Let $\mu(dx) = f(x)dx$ with the function $f(x) = (2\pi)^{-d/2} e^{-\|x\|^2/2}$ and suppose that $g \in L^\infty(\mathbb{R}^d)$. Then $\check{f} = f$ and $g * f \in C_0^\infty(\mathbb{R}^d)$, and whence it also belongs to $\text{Dom}(\mathcal{L})$. Since ∇f is bounded, we deduce that f is Lipschitz. By (6) we obtain that

$$\lim_{t \rightarrow 0^+} t^{-1} \left(H_g^\mu(t) - \int_{\mathbb{R}^d} f(x) g(x) dx \right) = \mathcal{L}(g * f)(0).$$

Similarly we can apply Theorem 1.

Example 4. Let $S^{(\alpha)}$ be the α -stable process in \mathbb{R}^d with $0 < \alpha < 2$ and with the Lévy measure $\nu^{(\alpha)}$ given by the formula

$$\nu^{(\alpha)}(B) = \int_{\mathbb{S}^{d-1}} m^{(\alpha)}(d\theta) \int_0^\infty \mathbf{1}_B(r\theta) \frac{dr}{r^{1+\alpha}} \quad \text{for } B \in \mathcal{B}(\mathbb{R}),$$

where $m^{(\alpha)}$ is a finite measure on the sphere \mathbb{S}^{d-1} , cf. [17, Theorem 14.3]. We abuse notation and we write just m for the measure $m^{(\alpha)}$. We additionally assume that there is no hyperplane \mathcal{V} of \mathbb{R}^d such that m is supported in \mathcal{V} . The corresponding Lévy-Khintchine exponent $\psi^{(\alpha)}$ takes the form

$$\psi^{(\alpha)}(x) = \int_{\mathbb{S}^{d-1}} \int_0^\infty (1 - \cos\langle x, r\theta \rangle) \frac{dr}{r^{1+\alpha}} m(d\theta).$$

Consider a symmetric Lévy process \mathbf{X} of which the Lévy-Khintchine exponent ψ equals

$$\psi(x) = \int_{\mathbb{S}^{d-1}} \int_0^\infty (1 - \cos\langle x, r\theta \rangle) \frac{f(1/r)}{r} dr m(d\theta)$$

for a given function $f \in \mathcal{R}_\alpha$. The corresponding Lévy measure is

$$\nu_f(B) = \int_{\mathbb{S}^{d-1}} m(d\theta) \int_0^\infty \mathbf{1}_B(r\theta) \frac{f(1/r)}{r} dr \quad \text{for } B \in \mathcal{B}(\mathbb{R}).$$

It follows, cf. [17, Remark 14.4], that for any non-negative and measurable function F we have

$$(14) \quad \int_{\mathbb{R}^d} F(x) \nu_f(dx) = \int_{\mathbb{S}^{d-1}} \int_0^\infty F(r\theta) \frac{f(1/r)}{r} dr m(d\theta).$$

Our aim is to apply Theorem 2 to the process \mathbf{X} . For that reason we need to verify condition (9) for the function ψ . We first show that there are some $R > 0$ and positive constants $c = c(f)$, $C = C(f)$ such that for all $s \geq R$ we have

$$(15) \quad cf(s) \leq \psi(s\theta_0) \leq Cf(s) \quad \text{for each } \theta_0 \in \mathbb{S}^{d-1}.$$

Indeed, for the upper bound, by a suitable change of variable we write

$$\frac{\psi(s\theta_0)}{f(s)} = \int_{\mathbb{S}^{d-1}} \int_0^\infty (1 - \cos(\rho\langle \theta_0, \theta \rangle)) \frac{f(s/\rho)}{f(s)} \frac{d\rho}{\rho} m(d\theta).$$

By (8), for any $\varepsilon > 0$ there is $R > 0$ such that

$$\frac{f(s/\rho)}{f(s)} \leq 2 \max\{(1/\rho)^{\alpha+\varepsilon}, (1/\rho)^{\alpha-\varepsilon}\}, \quad s \geq \rho R, \quad s \geq R.$$

Thus we get for $s \geq R$

$$\begin{aligned} \frac{\psi(s\theta_0)}{f(s)} &\leq 2 \int_{\mathbb{S}^{d-1}} \int_0^1 (1 - \cos(\rho\langle \theta_0, \theta \rangle)) \frac{d\rho}{\rho^{\alpha+\varepsilon+1}} m(d\theta) \\ &\quad + 2 \int_{\mathbb{S}^{d-1}} \int_1^{s/R} (1 - \cos(\rho\langle \theta_0, \theta \rangle)) \frac{d\rho}{\rho^{\alpha-\varepsilon+1}} m(d\theta) \end{aligned}$$

$$+ \int_{\mathbb{S}^{d-1}} \int_{s/R}^{\infty} (1 - \cos(\rho\langle\theta_0, \theta\rangle)) \frac{f(s/\rho) d\rho}{f(s) \rho} m(d\theta).$$

Since $1 - \cos(\rho\langle\theta_0, \theta\rangle) \leq 2 \min\{1, \rho^2\}$, we can estimate the two first integrals by the quantity $Km(\mathbb{S}^{d-1})$ for some $K > 0$. To the last integral we use formula (14) which yields

$$\begin{aligned} & \int_{\mathbb{S}^{d-1}} \int_{s/R}^{\infty} (1 - \cos(\rho\langle\theta_0, \theta\rangle)) \frac{f(s/\rho) d\rho}{f(s) \rho} m(d\theta) \\ &= \frac{1}{f(s)} \int_{\|x\| > \frac{1}{R}} (1 - \cos\langle\theta_0, x\rangle) \nu_f(dx) \leq \frac{2}{f(s)} \nu_f(B_{1/R}^c). \end{aligned}$$

As f diverges to infinity, the upper bound independent of θ_0 is found. For the lower bound we use again Potter bounds (8). For $\varepsilon > 0$ there is $R > 0$ such that

$$\frac{f(s/\rho)}{f(s)} \geq \frac{1}{2} \min\{(1/\rho)^{\alpha+\varepsilon}, (1/\rho)^{\alpha-\varepsilon}\}, \quad s \geq \rho R, \quad s \geq R.$$

Taking $\varepsilon = \alpha/2$ we obtain

$$\frac{\psi(s\theta_0)}{f(s)} \geq \frac{1}{2} \int_{\mathbb{S}^{d-1}} \int_0^1 (1 - \cos(\rho\langle\theta_0, \theta\rangle)) \frac{d\rho}{\rho^{1+\frac{\alpha}{2}}} m(d\theta) = \frac{1}{2} \tilde{\psi}^{(\alpha/2)}(\theta_0),$$

where $\tilde{\psi}^{(\alpha/2)}$ is the Lévy-Khintchine exponent of the pure-jump truncated $(\alpha/2)$ -stable process with the Lévy measure $\nu^{(\alpha/2)}|_{B_1}$. In particular, the function $\tilde{\psi}^{(\alpha/2)}$ is continuous. We finally observe that $\inf_{\theta \in \mathbb{S}^{d-1}} \tilde{\psi}^{(\alpha/2)}(\theta) > 0$. Indeed, suppose *a contrario* that this infimum is zero. Then there must exist $\theta_1 \in \mathbb{S}^{d-1}$ such that $\tilde{\psi}^{(\alpha/2)}(\theta_1) = 0$. But then the support of the measure m is contained in the subspace $\{\lambda\theta_1 : \lambda \in \mathbb{R}\}^\perp$. This contradicts our assumption and hence we have also found the lower bound in (15).

Since $f \in \mathcal{R}_\alpha$ with $\alpha > 0$, we have $\sup_{R \leq r \leq s} f(r) \asymp f(s)$, cf. [6, Theorem 1.5.3]. With the aid of (15) we conclude that $\sup_{\theta \in \mathbb{S}^{d-1}, R \leq r \leq s} \psi(r\theta) \asymp f(s)$ and thus condition (9) is satisfied for all $\|x\| \geq R$. But clearly continuity of ψ allows us to deduce the result also for $1 \leq \|x\| \leq R$ and the proof of (9) is finished.

We next observe that

$$\lim_{s \rightarrow \infty} \frac{\psi(s\theta)}{f(s)} = \psi^{(\alpha)}(\theta), \quad \theta \in \mathbb{S}^{d-1}.$$

This follows by an application of the Lebesgue dominated convergence theorem which is justified by an analogous argument to that one used for the upper bound in (15).

Finally, let μ be a finite measure and g be a bounded function. Assume that the function $r = g * \check{\mu}$ satisfies all the conditions of Theorem 2. We obtain

$$\lim_{t \rightarrow 0^+} [f^-(1/t)]^\beta (H_g^\mu(t) - H_g^\mu(0)) = \frac{1}{2} \int_{\mathbb{R}^d} R_\beta(x/\|x\|) \|x\|^\beta p^{(\alpha)}(x) dx,$$

where the density function $p^{(\alpha)}(x)$ is uniquely determined by $e^{-\psi^{(\alpha)}(x)} = \int_{\mathbb{R}^d} e^{i\langle x, y \rangle} p^{(\alpha)}(y) dy$.

Example 5. Let $\mathbf{X} = (S^{(\alpha)}, S^{(\rho)})$, where $S^{(\alpha)}$ and $S^{(\rho)}$ are two independent symmetric stable processes in \mathbb{R} with indexes α and ρ respectively and such that $0 < \alpha < \rho < 2$. The Lévy measure ν of \mathbf{X} is supported on axes OX and OY . Condition (13) forces that the same holds for the limit measure η . Moreover, since in this case $V(t) = \nu(B_{1/t}^c)$ belongs to \mathcal{R}_ρ , we conclude that $\eta(OX) = 0$ and thus η is the symmetric ρ -stable measure supported on OY .

Let $\beta = 1$, $1 < \rho < 2$ and set $g = \mathbf{1}_\Omega$, $d\mu = \mathbf{1}_\Omega dx$ for a radial set Ω . We shall apply Theorem 3. With this choice we have that $r(x) = g * \check{\mu}(x) = |\Omega \cap (\Omega + x)|$ is the so-called covariance function of the set Ω for which $\lim_{t \rightarrow 0^+} t^{-1}(r(0) - r(t\theta)) = V_\theta(\Omega)/2$, where $V_\theta(\Omega)$ is the directional derivative of $\mathbf{1}_\Omega$ in the direction $\theta \in \mathbb{S}^1$. For sets of finite perimeter the following relation holds

$$(16) \quad \text{Per}(\Omega) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d-1)/2}} \int_{\mathbb{S}^{d-1}} V_\theta(\Omega) \sigma(d\theta), \quad \Omega \subset \mathbb{R}^d.$$

For all of this we refer the reader to [8, Subsection 2.1]. In particular, see eg. [8, Eq. (4)] for the precise definition of the perimeter $\text{Per}(\Omega)$ of the set Ω , cf. also [4] and [10].

By our choice of the function g and the measure μ , we have $H_g^\mu(0) - H_g^\mu(t) = |\Omega| - H_\Omega(t)$, where $H_\Omega(t)$ is the heat content defined at (1). Moreover, since for radial sets $V_\theta(\Omega)$ is constant, setting $e_2 = (0, 1)$ we obtain that

$$\lim_{t \rightarrow 0^+} V^-(1/t) (|\Omega| - H_\Omega(t)) = \frac{V_{e_2}(\Omega)}{2} \int_0^\infty x p_\eta^{(\rho)}(x) dx = \pi^{-2} \Gamma\left(1 - \frac{1}{\rho}\right) \text{Per}(\Omega),$$

where for the last equality we used (16) together with the formula for the expectation of the stable random variable, cf. [17, Eq. (25.6)]. We mention that for non-radial sets the last equality in the above formula is not valid. Indeed, if we take Ω to be the rectangle centered at $(0, 0)$ and with sides of length $0 < a < b$, then one easily computes that $V_{e_2}(\Omega) = V_{-e_2}(\Omega) = 4a$ and whence

$$\begin{aligned} \lim_{t \rightarrow 0^+} V^-(1/t) (|\Omega| - H_\Omega(t)) &= \frac{V_{e_2}(\Omega) + V_{-e_2}(\Omega)}{4} \int_0^\infty x p_\eta^{(\rho)}(x) dx \\ &= 2\pi^{-1} \Gamma\left(1 - \frac{1}{\rho}\right) a. \end{aligned}$$

3. Proofs

We start with an auxiliary lemma which is closely related to the small-time moment behaviour of Lévy processes studied in [9, 12, 14]. In particular, we extend an admissible class of functions from [12, Section 5.2] in the case $\beta = 1$, and the result [14, Theorem 3.5] for Lévy processes.

After [15] we consider the following function related to the Lévy process \mathbf{X} , for any $r > 0$,

$$(17) \quad h(r) = \|A\|r^{-2} + r^{-1} \left\| \gamma + \int_{\mathbb{R}^d} (\mathbf{1}_{\|y\| < r} - \mathbf{1}_{\|y\| < 1}) y \nu(dy) \right\| + \int_{\mathbb{R}^d} (1 \wedge \|y\|^2 r^{-2}) \nu(dy),$$

where (A, γ, ν) is the triplet from (3) and $\|A\| = \max_{\|x\|=1} \|Ax\|$. We shall repeatedly use the estimate [15, Formula (3.2)]; there is some positive constant $C = C(d)$ such that for any $r > 0$,

$$(18) \quad \mathbb{P}(\|X_t\| \geq r) \leq \mathbb{P}\left(\sup_{0 \leq s \leq t} \|X_s\| \geq r\right) \leq Cth(r).$$

We also recall that for symmetric Lévy processes, see [11, Corollary 1],

$$(19) \quad \frac{1}{2} \psi^*(r^{-1}) \leq h(r) \leq 8(1 + 2d) \psi^*(r^{-1}).$$

Lemma 1. *Let \mathbf{X} be a Lévy process in \mathbb{R}^d with the triplet $(0, \gamma, \nu)$ and such that*

$$\int_{\|x\| < 1} \|x\|^\beta \nu(dx) < \infty, \quad 0 \leq \beta \leq 2.$$

For $\beta \in [0, 1]$ we additionally assume that $\gamma_0 = 0$. Let $F \in C_b(\mathbb{R}^d)$ satisfy $|F(x) - F(0)| \leq C\|x\|^\beta$ for $\|x\| < 1$. Then

$$\lim_{t \rightarrow 0^+} t^{-1} \int_{\mathbb{R}^d} (F(x) - F(0)) p_t(dx) = \int_{\mathbb{R}^d} (F(x) - F(0)) \nu(dx).$$

Proof. We choose $0 < \varepsilon < 1$ and a function $\chi_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \chi_\varepsilon \leq 1$, $\chi_\varepsilon(x) = 1$ for $\|x\| < \varepsilon/2$, and $\chi_\varepsilon(x) = 0$ for $\|x\| > \varepsilon$. We write

$$\begin{aligned} \int_{\mathbb{R}^d} (F(x) - F(0)) p_t(dx) &= \int_{\mathbb{R}^d} (F(x) - F(0)) \chi_\varepsilon(x) p_t(dx) \\ &\quad + \int_{\mathbb{R}^d} (F(x) - F(0)) (1 - \chi_\varepsilon(x)) p_t(dx) \\ &= I_\varepsilon(t) + II_\varepsilon(t). \end{aligned}$$

By [17, Corollary 8.9],

$$\lim_{t \rightarrow 0^+} t^{-1} II_\varepsilon(t) = \int_{\mathbb{R}^d} (F(x) - F(0)) (1 - \chi_\varepsilon(x)) \nu(dx).$$

Using our assumption we estimate the first integral as follows

$$|I_\varepsilon(t)| \leq C \int_{\mathbb{R}^d} \phi_\beta(x) p_t(dx), \quad \text{where } \phi_\beta = \chi_\varepsilon(x) \|x\|^\beta.$$

We observe that for $\beta > 1$ the gradient $\nabla\phi_\beta(0) = 0$ and thus applying [14, Theorem 4.1] we obtain that

$$t^{-1}|I_\varepsilon(t)| \leq C t^{-1} \int_{\mathbb{R}^d} \phi_\beta(x) p_t(dx) \xrightarrow{t \rightarrow 0^+} C \int_{\mathbb{R}^d} \phi_\beta(x) \nu(dx) \leq C_1 \varepsilon.$$

The proof is finished. \square

Proof of Theorem 1. We start with the case $0 \leq \beta \leq 1$. Applying formula (6) and Lemma 1 we obtain that

$$\lim_{t \rightarrow 0^+} t^{-1} (H_g^\mu(t) - H_g^\mu(0)) = \int_{\mathbb{R}^d} (r(x) - r(0)) \nu(dx).$$

Suppose that $1 \leq \beta < 2$ and \mathbf{X} is symmetric. We set $F(x) = r(x) + r(-x)$ and then $|F(x) - F(0)| \leq C \|x\|^\beta$ for all $\|x\| < 1$. We also observe that by symmetry $\gamma_0 = 0$ when $\beta = 1$. Thus, by Lemma 1 we conclude that

$$\lim_{t \rightarrow 0^+} t^{-1} \int_{\mathbb{R}^d} (F(x) - F(0)) p_t(dx) = \int_{\mathbb{R}^d} (F(x) - F(0)) \nu(dx),$$

and symmetry implies $t^{-1} \int_{\mathbb{R}^d} (F(x) - F(0)) p_t(dx) = 2 t^{-1} (H_g^\mu(t) - r(0))$, which gives the result.

Next, we consider the case when \mathbf{X} is a general Lévy process. We set

$$F(x) = r(x) - \langle x, \nabla r(0) \rangle \chi(x),$$

where χ is a compactly supported smooth function such that $0 \leq \chi \leq 1$ and it is one for $\|x\| \leq 1$ and zero for $\|x\| > 2$. Then our assumption implies that $|F(x) - F(0)| \leq C \|x\|^\beta$ for $\|x\| < 1$. Thus for $1 < \beta < 2$, by Lemma 1, we conclude that

(20)

$$\begin{aligned} \lim_{t \rightarrow 0^+} t^{-1} \int_{\mathbb{R}^d} (F(x) - F(0)) p_t(dx) &= \int_{\mathbb{R}^d} (F(x) - F(0)) \nu(dx) \\ &= \int_{\mathbb{R}^d} (r(x) - r(0) - \langle x, \nabla r(0) \rangle \chi(x)) \nu(dx). \end{aligned}$$

To get the same limit in the case when $\beta = 1$ we proceed as in Lemma 1. For $0 < \varepsilon < 1$ we pick a function $\chi_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \chi_\varepsilon \leq 1$, $\chi_\varepsilon(x) = 1$ for $\|x\| < \varepsilon/2$, and $\chi_\varepsilon(x) = 0$ for $\|x\| > \varepsilon$. Then we have

$$\begin{aligned} \lim_{t \rightarrow 0^+} t^{-1} \int_{\mathbb{R}^d} (F(x) - F(0)) (1 - \chi_\varepsilon(x)) p_t(dx) \\ = \int_{\mathbb{R}^d} (F(x) - F(0)) (1 - \chi_\varepsilon(x)) \nu(dx). \end{aligned}$$

We observe that $\nabla F(0) = 0$ and whence $F(x) - F(0) = o(\|x\|)$, which allows us to estimate the remaining integral as follows

$$\frac{1}{t} \left| \int_{\mathbb{R}^d} (F(x) - F(0)) \chi_\varepsilon(x) p_t(dx) \right| \leq \frac{\varepsilon}{t} \int_{\mathbb{R}^d} \|x\| \chi_\varepsilon(x) p_t(dx)$$

$$\begin{aligned}
&= \frac{\varepsilon}{t} \int_{\mathbb{R}^d} \|x - t\gamma_0\| \chi_\varepsilon(x - t\gamma_0) p_t^0(dx) \\
&\leq \frac{\varepsilon}{t} \int_{\mathbb{R}^d} \|x\| \chi_{2\varepsilon}(x) p_t^0(dx) + \varepsilon \|\gamma_0\|,
\end{aligned}$$

where $p_t^0(dx)$ is the transition probability of the process \mathbf{X}^0 which is shifted by γ_0 , i.e., $X_t^0 = X_t + t\gamma_0$. Since the function $x \mapsto \|x\| \chi_{2\varepsilon}(x)$ is Lipschitz, we apply [14, Theorem 4.1] and deduce that the last integral tends to $\int_{\mathbb{R}^d} \|x\| \chi_{2\varepsilon}(x) \nu(dx)$, which implies (20). Finally, we write

$$\begin{aligned}
&t^{-1} \int_{\mathbb{R}^d} (F(x) - F(0)) p_t(dx) \\
&= t^{-1} (H_g^\mu(t) - r(0)) - \left\langle t^{-1} \int_{\mathbb{R}^d} x \chi(x) p_t(dx), \nabla r(0) \right\rangle.
\end{aligned}$$

The function $x\chi(x) \in C_0^\infty(\mathbb{R}^d)$ and thus (5) yields

$$\begin{aligned}
&\lim_{t \rightarrow 0^+} \left\langle t^{-1} \int_{\mathbb{R}^d} x \chi(x) p_t(dx), \nabla r(0) \right\rangle \\
&= \left\langle \gamma + \int_{\mathbb{R}^d} x (\chi(x) - \mathbf{1}_{\{\|x\| \leq 1\}}) \nu(dx), \nabla r(0) \right\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
&\lim_{t \rightarrow 0^+} t^{-1} (H_g^\mu(t) - H_g^\mu(0)) \\
&= \langle \gamma, \nabla r(0) \rangle + \int_{\mathbb{R}^d} (r(x) - r(0) - \langle x, \nabla r(0) \rangle \mathbf{1}_{\{\|x\| \leq 1\}}) \nu(dx)
\end{aligned}$$

and the proof is finished. \square

Before we prove Theorem 2 we state an auxiliary lemma.

Lemma 2. *Let \mathbf{X} be a symmetric Lévy process in \mathbb{R}^d with the transition probability $p_t(dx)$. Assume that its Lévy-Khintchine exponent ψ satisfies (10) with functions $V \in \mathcal{R}_\alpha$, $\alpha \in (0, 2]$ and continuous $\Lambda: \mathbb{S}^{d-1} \rightarrow (0, \infty)$, and that condition (9) holds. Then $p_t(dx) = p_t(x)dx$ and*

$$(21) \quad \lim_{t \rightarrow 0^+} \frac{p_t\left(\frac{x}{V^-(1/t)}\right)}{(V^-(1/t))^d} = p_\Lambda(x),$$

where p_Λ is the density defined at (12).

Proof. Conditions (9) and (10) imply that

$$\lim_{x \rightarrow \infty} \frac{\psi(x)}{\log(1 + \|x\|)} = \infty,$$

and whence $p_t(dx) = p_t(x)dx$ with the density $p_t \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$, see e.g. [13, Theorem 1]. By the Fourier inversion formula, see [5, Section 3.3],

$$(22) \quad \frac{p_t\left(\frac{x}{V^-(1/t)}\right)}{(V^-(1/t))^d} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \cos\langle x, \xi \rangle e^{-t\psi(V^-(1/t)\xi)} d\xi.$$

By [6, Theorem 1.5.12], $tV(V^-(1/t)) \rightarrow 1$. Set $\theta = \xi/\|\xi\|$ and then we get that

$$\begin{aligned} \frac{\psi(V^-(1/t)\xi)}{1/t} &= \frac{\psi(V^-(1/t)\xi)}{V(V^-(1/t))} \cdot \frac{V(V^-(1/t))}{1/t} \\ &\sim \frac{\psi(V^-(1/t)\|\xi\|\theta)}{V(V^-(1/t)\|\xi\|)} \cdot \frac{V(V^-(1/t)\|\xi\|)}{V(V^-(1/t))} \rightarrow \Lambda(\theta)\|\xi\|^\alpha, \quad t \rightarrow 0^+, \end{aligned}$$

and this leads to

$$\lim_{t \rightarrow 0^+} e^{-t\psi(V^-(1/t)\xi)} = e^{-\Lambda(\theta)\|\xi\|^\alpha}.$$

Therefore, to finish the proof we apply the Dominated convergence theorem. First observe that equation (9) followed by (10) implies that

$$(23) \quad \psi^*(r) \asymp V(r), \quad r \geq 1.$$

Now we split the integral in (22) into two parts. According to Potter bounds (8) applied for the function V , there is $r_0 > 0$ such that, for t small enough and $\|\xi\| \geq r_0$,

$$t\psi(V^-(1/t)\xi) \geq \frac{1}{2} \frac{\psi(V^-(1/t)\|\xi\|\theta)}{V(V^-(1/t)\|\xi\|)} \cdot \frac{V(V^-(1/t)\|\xi\|)}{V(V^-(1/t))} \geq C\|\xi\|^{\alpha/2}$$

for some $C > 0$ which does not depend on ξ . Here we used the fact that Λ is bounded from below and (9) followed by (23). This implies that $e^{-t\psi(V^-(1/t)\xi)} \leq e^{-C\|\xi\|^{\alpha/2}}$ for $\|\xi\| \geq r_0$ and t small enough. For $\|\xi\| < r_0$ we bound $e^{-t\psi(V^-(1/t)\xi)}$ by one. The Dominated convergence theorem followed by the Fourier inversion formula proves (21). \square

Proof of Theorem 2. The proof is based on that of [8, Theorem 2] but it requires numerous adjustments and improvements. We split the integral in (6) into two parts

$$\begin{aligned} H_g^\mu(t) - H_g^\mu(0) &= \int_{\|x\| \leq \frac{M}{V^-(1/t)}} p_t(x) (r(x) - r(0)) dx \\ &\quad + \int_{\|x\| > \frac{M}{V^-(1/t)}} p_t(x) (r(x) - r(0)) dx \\ &= I_1(t) + I_2(t) \end{aligned}$$

for some fixed $M > 1$. We estimate $I_2(t)$ as follows

$$(24) \quad \left| \int_{\|x\| > \frac{M}{V^-(1/t)}} p_t(x) (r(x) - r(0)) dx \right|$$

$$\begin{aligned}
&\leq C \int_{\|x\| > \frac{M}{V^-(1/t)}} \left(1 \wedge \|x\|^\beta\right) p_t(dx) \\
&= C \int_{\|x\| > \frac{M}{V^-(1/t)}} \int_0^{1 \wedge \|x\|^\beta} du p_t(dx) \\
&= C \int_0^1 \mathbb{P} \left(\|X_t\| > \frac{M}{V^-(1/t)} \vee u^{1/\beta} \right) du.
\end{aligned}$$

This yields that

$$\begin{aligned}
&(V^-(1/t))^\beta |I_2| \\
&\leq CM^\beta \mathbb{P} \left(\|X_t\| > \frac{M}{V^-(1/t)} \right) + C (V^-(1/t))^\beta \int_{(M/V^-(1/t))^\beta}^1 \mathbb{P} \left(\|X_t\| > u^{1/\beta} \right) du.
\end{aligned}$$

Thus using (18) followed by (19), (23) and Potter bounds (8) for V , we get that for t small enough and for $0 < \varepsilon < \alpha - \beta$,

$$\begin{aligned}
M^\beta \mathbb{P} \left(\|X_t\| > M/V^-(1/t) \right) &\leq M^\beta t \psi^* \left(V^-(1/t)/M \right) \\
&\leq C_1 M^\beta \frac{V(V^-(1/t)/M)}{V(V^-(1/t))} \leq C_2 M^{\beta-\alpha+\varepsilon}.
\end{aligned}$$

We proceed similarly with the second term. Applying Karamata's theorem [6, Proposition 1.5.8] and Potter bounds we obtain that for t small enough

$$\begin{aligned}
&(V^-(1/t))^\beta \int_{(M/V^-(1/t))^\beta}^1 \mathbb{P} \left(\|X_t\| > u^{1/\beta} \right) du \\
&\leq Ct (V^-(1/t))^\beta \int_{M/V^-(1/t)}^1 V(u^{-1}) u^{\beta-1} du \\
&\leq C_1 \frac{M^\beta}{\alpha - \beta} t V(V^-(1/t)/M) \\
&\leq C_2 \frac{M^\beta}{\alpha - \beta} \frac{V(V^-(1/t)/M)}{V(V^-(1/t))} \leq C_3 \frac{M^{\beta-\alpha+\varepsilon}}{\alpha - \beta}.
\end{aligned}$$

We are left to study the integral $I_1(t)$: by a change of variable we get

$$[V^-(1/t)]^\beta I_1(t) = \frac{1}{2} \int_{\|x\| < M} \frac{p_t \left(\frac{x}{V^-(1/t)} \right)}{(V^-(1/t))^d} K_\beta(x, t) \|x\|^\beta dx,$$

where

$$K_\beta(x, t) = \frac{\left(r \left(\frac{x}{V^-(1/t)} \right) + r \left(-\frac{x}{V^-(1/t)} \right) - 2r(0) \right)}{\|x\|^\beta / (V^-(1/t))^\beta}.$$

We claim that for any fixed $M > 0$,

$$(25) \quad \begin{aligned} & \lim_{t \rightarrow 0^+} \int_{\|x\| < M} \frac{p_t \left(\frac{x}{V^-(1/t)} \right)}{(V^-(1/t))^d} K_\beta(x, t) \|x\|^\beta dx \\ &= \int_{\|x\| < M} R_\beta \left(\frac{x}{\|x\|} \right) \|x\|^\beta p_\Lambda(x) dx, \end{aligned}$$

where $p_\Lambda(x)$ is given by (12). To show the claim we use the Dominated convergence theorem. By (11),

$$|K_\beta(x, t)| \leq L \quad \text{and} \quad \lim_{t \rightarrow 0^+} K_\beta(x, t) = R_\beta \left(\frac{x}{\|x\|} \right) \quad \text{for any } x.$$

Next, by [7, Formula (23)], for t small enough,

$$\frac{p_t \left(\frac{x}{V^-(1/t)} \right)}{(V^-(1/t))^d} \leq \frac{p_t(0)}{(V^-(1/t))^d} \leq C,$$

and, by Lemma 2,

$$\lim_{t \rightarrow 0^+} \frac{p_t \left(\frac{x}{V^-(1/t)} \right)}{(V^-(1/t))^d} = p_\Lambda(x).$$

The Dominated convergence theorem implies (25).

If we let M tend to infinity we finally conclude that.

$$\lim_{t \rightarrow 0^+} [V^-(1/t)]^\beta I_1(t) = \frac{1}{2} \int_{\mathbb{R}^d} R_\beta \left(\frac{x}{\|x\|} \right) \|x\|^\beta p_\Lambda(x) dx$$

and the result follows. \square

Proof of Theorem 3. Take a smooth function χ_ε such that $0 \leq \chi_\varepsilon \leq 1$ and it is one for $\varepsilon < \|x\| < 1/\varepsilon$, and zero for $2/\varepsilon > \|x\|$ or $\|x\| < \varepsilon/2$. By (6) we can write

$$\begin{aligned} 2(H_g^\mu(t) - H_g^\mu(0)) &= \int_{\mathbb{R}^d} (r(x) + r(-x) - 2r(0)) (1 - \chi_\varepsilon(V^-(1/t)x)) p_t(dx) \\ &\quad + \int_{\mathbb{R}^d} (r(x) + r(-x) - 2r(0)) \chi_\varepsilon(V^-(1/t)x) p_t(dx). \end{aligned}$$

We have

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} (r(x) + r(-x) - 2r(0)) (1 - \chi_\varepsilon(V^-(1/t)x)) p_t(dx) \right| \\ & \leq C_1 \left(\frac{\varepsilon}{V^-(1/t)} \right)^\beta + C_2 \int_{\|x\| > 1/(\varepsilon V^-(1/t))} (1 \wedge \|x\|)^\beta p_t(dx). \end{aligned}$$

We show that the last integral is small similarly as in the proof of Theorem 2, cf. (24). This and a change of variable yield that

$$2[V^-(1/t)]^\beta (H_g^\mu(t) - r(0)) = o(1) + \int_{\mathbb{R}^d} K_\beta(x, t) \|x\|^\beta \chi_\varepsilon(x) \tilde{p}_t(dx),$$

where $\tilde{p}_t(G) = p_t(G/V^-(1/t))$ for any Borel set $G \subset \mathbb{R}^d$ and

$$K_\beta(x, t) = \frac{r\left(\frac{x}{V^-(1/t)}\right) + r\left(-\frac{x}{V^-(1/t)}\right) - 2r(0)}{\|x\|^\beta / (V^-(1/t))^\beta}.$$

Further we write

$$\begin{aligned} \int_{\mathbb{R}^d} K_\beta(x, t) \|x\|^\beta \chi_\varepsilon(x) \tilde{p}_t(dx) &= \int_{\mathbb{R}^d} \left(K_\beta(x, t) - R_\beta\left(\frac{x}{\|x\|}\right) \right) \|x\|^\beta \chi_\varepsilon(x) \tilde{p}_t(dx) \\ &\quad + \int_{\mathbb{R}^d} R_\beta\left(\frac{x}{\|x\|}\right) \|x\|^\beta \chi_\varepsilon(x) \tilde{p}_t(dx) \end{aligned}$$

and the first integral we estimate as follows

$$\left| \int_{\mathbb{R}^d} \left(K_\beta(x, t) - R_\beta\left(\frac{x}{\|x\|}\right) \right) \|x\|^\beta \chi_\varepsilon(x) \tilde{p}_t(dx) \right| \leq C\varepsilon \int_{\mathbb{R}^d} \|x\|^\beta \chi_\varepsilon(x) \tilde{p}_t(dx).$$

To finish the proof we need the following observation: the measures $\tilde{p}_t(dx)$, as t goes to zero, converge weakly to the measure \tilde{p}_η which is uniquely determined by the formula

$$e^{-\int_{\mathbb{R}^d} (1 - \cos\langle \xi, y \rangle) \eta(dy)} = \int_{\mathbb{R}^d} e^{i\langle \xi, y \rangle} \tilde{p}_\eta(dy).$$

We work with characteristic functions and, since $\tilde{p}_t(dx)$ is the distribution of the random variable $V^-(1/t)X_t$, it suffices to show that

$$\lim_{t \rightarrow 0^+} t\psi(V^-(1/t)\xi) = \int_{\mathbb{R}^d} (1 - \cos\langle \xi, y \rangle) \eta(dy).$$

We set $s = 1/V^-(1/t)$. Since $V \in \mathcal{R}_\alpha$ with $\alpha > 0$, we can investigate the above limit at zero along the new variable s for which we have $t \sim 1/V(1/s)$ as t tends to 0. Then in terms of the new variable we obtain

$$\begin{aligned} \frac{1}{V(1/s)} \psi(s^{-1}\xi) &= \int_{\mathbb{R}^d} (1 - \cos\langle \xi, y \rangle) \frac{\nu_s(dy)}{V(1/s)} \\ &= \int_{\mathbb{R}^d} (1 - \chi_\varepsilon(y)) (1 - \cos\langle \xi, y \rangle) \frac{\nu_s(dy)}{V(1/s)} \\ &\quad + \int_{\mathbb{R}^d} \chi_\varepsilon(y) (1 - \cos\langle \xi, y \rangle) \frac{\nu_s(dy)}{V(1/s)}, \end{aligned}$$

where $\nu_s(G) = \nu(sG)$, for any Borel set $G \subset \mathbb{R}^d$. Condition (13) forces that the last integral converges, as s goes to zero, to $\int \chi_\varepsilon(x) (1 - \cos\langle \xi, y \rangle) \eta(dy)$. Thus we are left to prove that the first integral approaches zero. We have

$$\int_{\mathbb{R}^d} (1 - \chi_\varepsilon(y)) (1 - \cos\langle \xi, y \rangle) \frac{\nu_s(dy)}{V(1/s)} \leq C \left(\int_{B_\varepsilon} + \int_{B_{1/\varepsilon}^c} \right) (1 - \cos\langle \xi, y \rangle) \frac{\nu_s(dy)}{V(1/s)}.$$

For the second integral we apply Potter bounds: for s small enough we have

$$\left| \int_{B_{1/\varepsilon}^c} (1 - \cos\langle \xi, y \rangle) \frac{\nu_s(dy)}{V(1/s)} \right| \leq \frac{2}{V(1/s)} \nu_s(B_{1/\varepsilon}^c) = \frac{2}{V(1/s)} V(\varepsilon/s) \leq C\varepsilon^{\alpha/2}.$$

The first integral is bounded by

$$\int_{B_\varepsilon} \|\xi\|^2 \|y\|^2 \frac{\nu_s(dy)}{V(1/s)} = \frac{\|\xi\|^2 \varepsilon^2}{V(1/s)} \int_{B_{s\varepsilon}} \frac{\|y\|^2}{(s\varepsilon)^2} \nu(dy) \leq \|\xi\|^2 \varepsilon^2 \frac{h(s\varepsilon)}{V(1/s)} \leq \|\xi\|^2 \varepsilon^{2-\alpha},$$

where in the last inequality we used the fact that for $s < 1$, $h(s) \asymp V(1/s)$. To obtain these two inequalities we write

$$\begin{aligned} h(s) &= \nu(B_s^c) + \frac{1}{s^2} \int_{B_s} \|y\|^2 \nu(dy) = V(1/s) + \frac{2}{s^2} \int_{B_s} \int_0^{\|y\|} u \, du \, \nu(dy) \\ &\leq V(1/s) + \frac{2}{s^2} \int_0^s u \nu(B_u^c) du = V(1/s) + \frac{2}{s^2} \int_0^s u V(1/u) du. \end{aligned}$$

The function $V(s) = \nu(B_{1/s}^c)$ is regularly varying at infinity of index $\alpha \in (\beta, 2)$ and thus the Karamata's theorem [6, Section 1.6] implies that the last integral behaves like $s^2 V(1/s)$, as s goes to zero, and we conclude the result. \square

References

- [1] L. Acuña Valverde, *Heat content estimates over sets of finite perimeter*, J. Math. Anal. Appl. **441** (2016), no. 1, 104–120.
- [2] ———, *On the one dimensional spectral heat content for stable processes*, J. Math. Anal. Appl. **441** (2016), no. 1, 11–24.
- [3] ———, *Heat content for stable processes in domains of \mathbb{R}^d* , J. Geom. Anal. **27** (2017), no. 1, 492–524.
- [4] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.
- [5] D. Applebaum, *Lévy Processes and Stochastic Calculus*, second edition, Cambridge Studies in Advanced Mathematics, **116**, Cambridge University Press, Cambridge, 2009.
- [6] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Encyclopedia of Mathematics and its Applications, **27**, Cambridge University Press, Cambridge, 1987.
- [7] K. Bogdan, T. Grzywny, and M. Ryznar, *Density and tails of unimodal convolution semigroups*, J. Funct. Anal. **266** (2014), no. 6, 3543–3571.
- [8] W. Cygan and T. Grzywny, *Heat content for convolution semigroups*, J. Math. Anal. Appl. **446** (2017), no. 2, 1393–1414.
- [9] J. E. Figueroa-López, *Small-time asymptotics for Lévy processes*, Statistics & Probability Letters **78** (2008), 3355–3365.
- [10] B. Galerne, *Computation of the perimeter of measurable sets via their covariogram: Applications to random sets*, Image Analysis & Stereology **30** (2011), no. 1, 39–51.
- [11] T. Grzywny, *On Harnack inequality and Hölder regularity for isotropic unimodal Lévy processes*, Potential Anal. **41** (2014), no. 1, 1–29.
- [12] J. Jacod, *Asymptotic properties of power variations of Lévy processes*, ESAIM Probab. Stat. **11** (2007), 173–196.
- [13] V. Knopova and R. L. Schilling, *A note on the existence of transition probability densities of Lévy processes*, Forum Math. **25** (2013), no. 1, 125–149.

- [14] F. Kühn and R. L. Schilling, *On the domain of fractional Laplacians and related generators of Feller processes*, Preprint: arXiv:1610.08197, 2016.
- [15] W. E. Pruitt, *The growth of random walks and Lévy processes*, Ann. Probab. **9** (1981), no. 6, 948–956.
- [16] S. I. Resnick, *Heavy-Tail Phenomena*, Springer Series in Operations Research and Financial Engineering, Springer, New York, 2007.
- [17] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, translated from the 1990 Japanese original, Cambridge Studies in Advanced Mathematics, **68**, Cambridge University Press, Cambridge, 1999.
- [18] M. van den Berg, *Heat content and Brownian motion for some regions with a fractal boundary*, Probab. Theory Related Fields **100** (1994), no. 4, 439–456.
- [19] ———, *Heat flow and perimeter in \mathbb{R}^m* , Potential Anal. **39** (2013), no. 4, 369–387.
- [20] M. van den Berg, E. B. Dryden, and T. Kappeler, *Isospectrality and heat content*, Bull. Lond. Math. Soc. **46** (2014), no. 4, 793–808.
- [21] M. van den Berg and P. Gilkey, *Heat flow out of a compact manifold*, J. Geom. Anal. **25** (2015), no. 3, 1576–1601.
- [22] M. van den Berg and K. Gittins, *Uniform bounds for the heat content of open sets in Euclidean space*, Differential Geom. Appl. **40** (2015), 67–85.
- [23] M. van den Berg and J.-F. Le Gall, *Mean curvature and the heat equation*, Math. Z. **215** (1994), no. 3, 437–464.

WOJCIECH CYGAN
INSTYTUT MATEMATYCZNY
UNIwersytet WROCLAWSKI
PL. GRUNWALDZKI 2/4
50-384 WROCLAW, POLAND
AND
INSTITUT FÜR DISKRETE MATHEMATIK
TECHNISCHE UNIVERSITÄT GRAZ
STEYRERGASSE 30
8010 GRAZ, AUSTRIA
Email address: wojciech.cygan@uwr.edu.pl

TOMASZ GRZYWNY
WYDZIAŁ MATEMATYKI
POLITECHNIKA WROCLAWSKA
WYB. WYSPIAŃSKIEGO 27
50-370 WROCLAW, POLAND
Email address: tomasz.grzywny@pwr.edu.pl