Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b170825
pISSN: 1015-8634 / eISSN: 2234-3016

POLARITY OF COHOMOGENEITY TWO ACTIONS ON NEGATIVELY CURVED SPACE FORMS

Mojtaba Heidari and Reza Mirzaie

Abstract. We study the polarity of cohomogeneity two isometric actions on Riemannian manifolds of constant negative curvature.

1. Introduction

An action of a Lie group G on a complete Riemannian manifold M is called polar if there exists a complete submanifold W that meets all the orbits and it is perpendicular to the orbits at intersection points. Such a submanifold is called a section and it is a totally geodesic submanifold of M. A polar action is called hyperpolar if the section is a flat Riemannian submanifold of M. This concept was pioneered by Szenthe in [22, 23] and independently by Palais and Terng in [17]. Since in general the smooth structure of the quotient $\frac{M}{G}$ is too complicated to do analysis effectively, polar actions are natural class of group actions where a reduction to a potentially simpler lower dimensional problem along a smooth section is possible. In the present paper we consider polar isometric actions. Classification of polar actions on a given Riemannian manifold and topological or more precisely, geometric characterization of the orbits of such actions on a given Riemannian manifold has been absorbing problems for mathematicians in recent years. Dadok showed in [3] that a linear representation which is polar is (up to orbit equivalence) the isotropy representation of a symmetric space. Polar isometric actions on symmetric spaces have been studied extensively. They were classified for compact rank one symmetric spaces, and for compact irreducible symmetric spaces of higher rank it was shown that a polar action must be hyperpolar ([10, 11, 19]). Kollross ([9]) classified hyperpolar actions on irreducible compact symmetric spaces up to orbit equivalence. For non-compact symmetric spaces, the classification is still open. It is straightforward to see that polar actions on spheres are precisely the restrictions of linear polar actions, and similarly that polar actions on real projective spaces are orbit equivalent to those induced from polar actions on

Received September 24, 2017; Revised January 18, 2018; Accepted February 1, 2018.
2010 *Mathematics Subject Classification.* 53C30, 57S25.
Key words and phrases. Riemannian manifold, Lie group, isometry.

spheres. Polar actions of connected compact Lie groups on Euclidean spheres have been classified in [8]. Polar actions on complex hyperbolic spaces have been classified in [5]. Podesta and Thorbergsson have classified polar actions on the complex quadric making use of their result that in this case the action is coisotropic, meaning that the sections are totally real submanifolds [20]. From the above mentioned classifications one also can deduce many results about orbits of the polar actions. It is shown in [10] that polar actions of cohomogeneity two on simple compact Lie groups of higher rank, endowed with a biinvariant Riemannian metric, are hyperpolar. In the present paper, we study the polarity of isometric cohomogeneity two actions on Riemannian manifolds of constant negative curvature. Topological properties of G-manifolds of low cohomogeneity is an active research area in differential geometry. In the negative curvature case, cohomogeneity one Riemannian manifolds have been classified from topological points of view (see [18]). The present paper is in direction of previous papers [12–14], about cohomogeneity two Riemannian manifolds of negative curvature.

2. Preliminaries

We will use the following notations and assertions in our proofs:

(1) If M is a Riemannian manifold we will denote by $\widetilde{M}$ its universal Riemannian covering manifold with the covering map $\kappa : \widetilde{M} \to M$. We will denote by Δ the decktransformation group which is isomorphic to $\pi_1(M)$ the fundamental group of M.

(2) A Riemannian G-manifold is a Riemannian manifold M equipped with an isometric action of G a connected and closed subgroup of $\mathrm{Iso}(M)$.

(3) If M is a Riemannian G-manifold, then there exists a closed and connected subgroup $\widetilde{G}$ of $\mathrm{Iso}(\widetilde{M})$ such that $\widetilde{G}$ is a covering manifold for G and the covering map $\kappa : \widetilde{M} \to M$ maps $\widetilde{G}$-orbits of $\widetilde{M}$ on to G-orbits of M. Members of Δ and $\widetilde{G}$ commute, so each $\delta \in \Delta$ maps orbits to orbits (see [2], page 63).

(4) In (2), $(\widetilde{M}, \widetilde{G})$ is called the universal action cover of (M, G). M is called universally polar if the action of $\widetilde{G}$ on $\widetilde{M}$ is polar.

(5) If M is G-manifold, then the dimension of the orbit space $\frac{M}{G}$ is called the cohomogeneity of the action of G on M and is denoted by $\mathrm{coh}(M, G)$. It is clear that if $G(x)$ is an orbit with maximum dimension among the orbits, then $\mathrm{coh}(M, G) = \dim M - \dim G(x)$.

Remark 2.1. We recall that if M^n is a simply connected Riemannian manifold of negative curvature, then the infinity $M(\infty)$ of M is the set of classes of the asymptotic geodesics in M. If γ is a geodesic in M and $z = [\gamma]$ is its asymptotic class, then there is a foliation of M by hypersurfaces which all of them intersect the geodesics in $z = [\gamma]$ perpendicularly. Hypersurfaces of the mentioned foliation are called horospheres centered at z. $M \cup M(\infty)$ is a topological space homeomorphic to closed disk $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ (see [6]).

If M is a Riemannian manifold of negative curvature we denote the isometry group of M by $\mathrm{Iso}(M)$. If $\phi \in \mathrm{Iso}(M)$, then the following map is called the squared displacement function related to ϕ

$$f_\phi : M \to \mathbb{R}, \quad f_\phi(x) = d^2(x, \phi(x)).$$

Remark 2.2 (see [1]). If M is a Riemannian manifold of negative curvature, $\phi \in \mathrm{Iso}(M)$ and C is the minimum point set of f_ϕ, then one of the following is true:

(1) ϕ has fixed point and C is equal to the fixed point set.

(2) ϕ translates a geodesic (i.e., there is a geodesic γ such that $\phi(\gamma) = \gamma$) and C is equal to the image of γ.

(3) f_ϕ has no minimum point.

The isometries satisfying (1), (2) and (3) are called elliptic, axial and parabolic, respectively.

Definition 2.3. A nonsimply connected Riemannian manifold M of negative curvature is called axial (parabolic) if all elements of the decktransformation group Δ are axial (parabolic).

Remark 2.4. If M^n, $n \geq 2$, is an axial Riemannian manifold of negative curvature, then M is a vector bundle over a circle (see [7], Corollary 6.16). So, it is diffeomorphic to $S^1 \times \mathbb{R}^{n-1}$ or $B^2 \times \mathbb{R}^{n-2}$, where B^2 is the Moebius band.

Fact 2.5. If M is a simply connected Riemannian manifold of negative curvature and $\phi \in \mathrm{Iso}(M)$, then ϕ can be extended to a homeomorphism on $\overline{M} = M \cup M(\infty)$ which we denote it also by ϕ. By Brouwer's fixed point theorem ϕ has fixed point on $\overline{M}$. If ϕ is non-elliptic and has two different fixed points $x, y \in \overline{M}$, and γ is a geodesic joining x to y, then γ will be an axis for ϕ (see [7], Proposition 6.4).

Remark 2.6. We know that if M is a simply Riemannian manifold of constant negative curvature, then any pair of different points in M can be joined by a unique geodesic. This criterion is not true for $\overline{M}$ in general, but if M has constant negative curvature (or in general case, if M has strictly negative curvature), then for any different points $x, y \in \overline{M}$ there exists a unique geodesic joining x to y (see [7], page 61).

Remark 2.7 (see [7], pages 47, 58). Let S be a horosphere in a simply connected Riemannian manifold M of negative curvature, related to asymptotic class of geodesics $[\gamma]$. If γ is a unit speed geodesic, then the function $f : M \to \mathbb{R}$, defined by $f(p) = \lim_{t\to\infty} d(p, \gamma(t)) - t$, is called a Bussmann function. For each point $p \in M$ there is a point $\eta_S(p)$ in S which is the unique point of S nearest p, and the following map is a homeomorphism:

$$\phi : M \to S \times \mathbb{R}, \quad \phi(p) = (\eta_S(p), f(p)).$$

Corollary 2.8 (see [15], Corollary 2.4)**.** *If M is a simply connected Riemannian manifold of negative curvature and G is a closed and connected subgroup of $Iso(M)$ such that $M^G = \emptyset$, then there exists at most one geodesic orbit.*

3. Results

Consider the Lorentzian space $\mathbb{R}^{n,1}(= \mathbb{R}^{n+1})$ with a non-degenerate scalar product $\langle , \rangle$ given by:

$$\langle x, y\rangle = -x_1y_1 + \sum_{i=2}^{n+1} x_iy_i.$$

It is well known that any simply connected Riemannian manifold of constant negative curvature $c < 0$, is isometric to the hyperbolic space of curvature c defined by:

$$H^n(c) = \{x \in \mathbb{R}^{n,1} : \langle x, x\rangle = -r^2\}, \quad c = \frac{1}{r^2}.$$

It is well known that each horosphere in $H^n(c)$ is isometric to $\mathbb{R}^{n-1}$. By using the statements and remarks on pages 201 and 202 of [21], we have the following fact:

Fact 3.1 (see [21])**.** Let S_0 be a horosphere centered at a point $z \in H^n(\infty)$. The point z determines a unique unit vector field ζ on H^n which is also a parallel normal field to any submanifold of H^n which is contained in a horosphere centered at z. For each $t \in \mathbb{R}$, put

$$S_t = \{\exp(t\zeta(q)) : q \in S_0\}.$$

The 1-parameter family S_t, $t \in \mathbb{R}$, coincides with the foliation by horospheres centered at z. let G be a connected and closed Lie subgroup of the isometries of $H^n(c)$ such that $G(S_0) = S_0$. Then

(a) We have $G(S_t) = S_t$, $t \in \mathbb{R}$, and for each orbit $G(p)$ in S_0,

$$G(\exp(t\zeta(p))) = \{\exp(t\zeta(q)) : q \in G(p)\}.$$

(b) If V_0 is a totally geodesic submanifold of S_0, and $V_t = \{\exp(t\zeta(q) : q \in V_0\}$, then V_t is a totally geodesic submanifold of S_t and $\bigcup_t V_t$ is a totally geodesic submanifold of H^n.

Remark 3.2. The open disk model of $M = H^n(c)$ is diffeomorphic to open disk D^n in $\mathbb{R}^n$ and $M(\infty)$ can be considered as the boundary of the open disc which is homeomorphic to S^{n-1}. Similarly, in the general case, if M is a simply connected Riemannian manifold of negative curvature, then $M \cup M(\infty)$ is homeomorphic to $\overline{D^n} = D^n \cup S^{n-1}$.

Proposition 3.3 (see [13])**.** *If M^n is a cohomogeneity two Riemannian G-manifold of negative curvature and $M^G \neq \emptyset$, then either M is simply connected or it is diffeomorphic to $S^1 \times \mathbb{R}^{n-1}$ or $B^2 \times \mathbb{R}^{n-2}$ (B^2 is the Moebius band).*

Proposition 3.4. *Let M^n, $n \geq 3$, be a non-simply connected Riemannian G-manifold of constant negative curvature, and $(\widetilde{M} = H^n(c), \widetilde{G})$ be its universal action cover. Then one of the following is true*:

(a) $M^G \neq \emptyset$.

(b) *M is a parabolic manifold, all $\widetilde{G}$-orbits are included in horospheres centered at the same point at infinity, and M is homeomorphic to a product $M_1 \times \mathbb{R}$ such that M_1 is a flat G-manifold of cohomogeneity* $\mathrm{coh}(M, G) - 1$.

(c) *M is axial and diffeomorphic to $S^1 \times \mathbb{R}^{n-1}$ or $B^2 \times \mathbb{R}^{n-2}$, where B^2 is the Moebius band.*

Proof. We show that if (a) is not true, then (b) or (c) is true. Consider the following two cases separately.

Case 1. For all non-identity $\delta \in \Delta$, f_δ has no minimum.

Case 2. There exists a non-identity $\delta \in \Delta$ such that f_δ has minimum point.

Case 1. Consider a non-identity $\delta \in \Delta$ and consider the continuous extension of δ as a map $\delta : \widetilde{M} \cup \widetilde{M}(\infty) \to \widetilde{M} \cup \widetilde{M}(\infty)$. Since $\widetilde{M} \cup \widetilde{M}(\infty)$ is homeomorphic to $\overline{D^n}$, by Brouwer's fixed point theorem, $\delta : \widetilde{M} \cup \widetilde{M}(\infty) \to \widetilde{M} \cup \widetilde{M}(\infty)$ has fixed point. Since δ has non fixed point in $\widetilde{M}$ (from general theory of covering spaces in topology, it is known that the members of Δ have no fixed points in $\widetilde{M}$), then the fixed point of δ belongs to $\widetilde{M}(\infty)$. If there exists two fixed points for δ on $\widetilde{M}(\infty)$, then by Remark 2.2(2) and Fact 2.5, f_δ has minimum point, which is not true by assumption. So, there is a unique fixed point $z \in \widetilde{M}(\infty)$ for δ. Let $[\gamma]$ be the asymptotic class of the geodesics such that $[\gamma] = z$. This means that δ is parabolic and leaves invariant the horosphere foliation centered at z, then by [2], Lemma 3, for all horospheres S centered at z, $\delta(S) = S$. Since the elements of Δ and $\widetilde{G}$ commute, we can get easily from the uniqueness of z that for all $g \in \widetilde{G}$, $g(z) = z$.

Now consider a $g \in \widetilde{G}$. If g is parabolic, then by [2], Lemma 3, for all horospheres S centered at S, $g(S) = S$.

If g is axial and λ is its axes, then we get from the uniqueness of λ and $\delta g = g\delta$ that $\delta\lambda = \lambda$ which is contradiction (since λ must be minimum point set of f_δ). So, $\widetilde{G}$ has no axial element. If g is elliptic and for a point $x \in \widetilde{M}$, $g(x) = x$, then g leaves invariant the geodesic λ joining x and z, which is impossible as before. So there is no elliptic element in $\widetilde{G}$. Therefore, all elements of $\widetilde{G}$ are parabolic such that for all horospheres S centered at z and all $g \in \widetilde{G}$, $gS = S$.

Thus, all $\widetilde{G}$-orbits are included in horospheres centered at z. If δ' is another member of Δ, in a similar way as for δ, there is a point z' at infinity fixed by δ' such that all $\widetilde{G}$-orbits must be included in horospheres centered at z'. We show that $z = z'$. If $z \neq z'$, then each $\widetilde{G}$ orbit is included in intersection of two different horospheres. But intersection of two different horospheres in H^n is a compact set (for proof, consider horospheres in the Poincare model of the hyperbolic space), so all $\widetilde{G}$ orbits are compact. Then $\widetilde{M}^{\widetilde{G}} \neq \emptyset$, so $M^G \neq \emptyset$, which is contradiction. Therefore, $z = z'$. Since δ' is arbitrary in Δ, we get that

$\Delta(z) = z$, so $\Delta(S) = S$. Now, by using of Remark 2.7, and since $\widetilde{G}(S) = S$, we get that M is homeomorphic to $\frac{S}{\Delta} \times \mathbb{R}$, and $\frac{S}{\Delta}$ is a G-manifold of cohomgeneity $\mathrm{coh}(M, G) - 1$. This is part (b) of the theorem.

Case 2. Since δ is not elliptic then by Remark 2.2, the minimum point set of f_δ is the image of a geodesic γ. For each $g \in \widetilde{G}$ we have $\delta g = g\delta$, so $g\gamma$ is also a minimum point set for f_δ. But γ with the mentioned property is unique, so $g\gamma = \gamma$, and $\widetilde{G}(\gamma) = \gamma$. Since $\widetilde{M}^{\widetilde{G}}$ is empty by assumption, then γ must be a $\widetilde{G}$-orbit. Since by Remark 2.8, the geodesic orbit is unique and all members of Δ map orbits to orbits, we have $\Delta(\gamma) = \gamma$. Therefore, M is axial and by Remark 2.4, we get part (c) of the theorem. □

Remark 3.5. If $\mathbb{R}^n$ is of cohomogeneity one under the action of G a closed and connected subgroup of the isometries, then all orbits are perpendicular to some lines which are called normal geodesics (see [16]).

Theorem 3.6. *If M is a complete parabolic cohomogeneity two Riemannian G-manifold of constant negative curvature and $M^G = \emptyset$, then M is universally polar.*

Proof. By the proof of Theorem 3.4, Case 1, all $\widetilde{G}$-orbits of $\widetilde{M} = H^n(c)$ are included in the horospheres centered at the same point z at infinity. We know that the collection of all horospheres centered at z is in fact a one parameter family $\{S_t\}$ of horospheres. Consider a horosphere S_0 centered at z. The action of $\widetilde{G}$ on S_0 is of cohomogeneity one. Since S_0 is isometric to $\mathbb{R}^{n-1}$, then by Remark 3.5, it is polar and each section is a normal geodesic (a geodesic normal to the orbits at intersection points). Let λ_0 be the image of a normal geodesic in S_0. Keeping the symbols of Fact 3.1, consider the vector field ζ on $\widetilde{M}$ which is parallel normal field to any submanifold of $\widetilde{M}$ which is contained in a horosphere centered at z. Now, put $\lambda_t = \{\exp(t\zeta(q)) : q \in \lambda_0\}$ and let $W = \bigcup_t \lambda_t$. By Fact 3.1, W is a totally geodesic surface in $\widetilde{M}$. Let D_0 be an orbit in S_0. Since λ_0 is a normal geodesic in S_0, D_0 is orthogonal to λ_0 at a point $q \in S_0$. Fix q and consider the curve $\alpha(s) = \exp(s\zeta(q))$ in $\widetilde{M}$. By definition of W, α is a curve in W and $\alpha'(0) = \zeta(q)$. Since $\zeta(q)$ is orthogonal to D_0, D_0 would be orthogonal to the curves α and λ_0 of the surface W at their intersection point q. This means that D_0 is orthogonal to W. In a similar way we can show that each orbit D_t in S_t, $t \in \mathbb{R}$, is orthogonal to W. All orbits are included in horospheres centered at z, so all orbits are orthogonal to W. Thus, W is a section and $\widetilde{G}$-action on $\widetilde{M}$ is polar. □

By combination of Proposition 3.3, Proposition 3.4, and Theorem 3.6, we get the following theorem.

Theorem 3.7. *Let M^n, $n \geq 3$, be a nonsimply connected Riemannian G-manifold of constant negative curvature and of cohomogeneity two. Then either M is universally polar or it is diffeomorphic to $S^1 \times \mathbb{R}^{n-1}$ or $B^2 \times \mathbb{R}^{n-2}$ (B^2 is the Moebius band).*

Remark 3.8. Polar actions on the hyperbolic space H^n, $n \geq 2$ have been classified (see [4], page 328). Thus, our Theorem 3.7, can be useful to reduce the classification problem of cohomogeneity two actions on Riemannian manifolds of constant negative curvature to simpler cases.

Albeit classification of cohomogeneity two actions on Riemannian manifolds of constant negative curvature is open yet, but there is a topological description of the orbits of this kind of actions in [14]. To improve the main result of [14], as an application of our theorems, we mention the following corollary.

Corollary 3.9. *If M^n, $n \geq 3$, is a complete and nonsimply connected orientable cohomogeneity two Riemannian manifold of constant negative curvature, then either $\pi_1(M) = Z^p$ for some positive integer p or M is a fiber bundle over S^1.*

Proof. By Theorems 3.3 and 3.5 in [16], if M_1 is an orientable nonsimply connected cohomogeneity one flat Riemannian manifold, then one of the following is true:

(a) there is one singular orbit in M_1 and $\pi_1(M) = Z^p$ for some positive integer p.

(b) There is no singular orbit and M_1 is a fiber bundle over S^1 or $\mathbb{R}$. Clearly, if M_1 is not simply connected and it is fiber bundle over $\mathbb{R}$, then by [16], Proposition 3.4(c), it is also a fiber bundle over S^1.

Now, by Proposition 3.4 and its proof and using Proposition 3.3, either M is diffeomorphic to one of the spaces $S^1 \times \mathbb{R}^{n-1}$ or $B^2 \times \mathbb{R}^{n-2}$, or it is homeomorphic to the product $M_1 \times \mathbb{R}$, where M_1 is flat cohomogeneity one Riemannian manifold. Since M is supposed to be orientable, then M_1 is orientable and we get the result. □

Acknowledgement. We would like to thank the referee for the valuable suggestions and remarks.

References

[1] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, 1972.

[2] W. Byers, *Isometry groups of manifolds of negative curvature*, Proc. Amer. Math. Soc. **54** (1976), 281–185.

[3] J. Dadok, *Polar coordinates induced by actions of compact Lie groups*, Trans. Amer. Math. Soc. **288** (1985), no. 1, 125–137.

[4] J. C. Diaz Ramos, *Polar actions on symmetric spaces*, in Recent trends in Lorentzian geometry, 315–334, Springer Proc. Math. Stat., **26**, Springer, New York, 2013.

[5] J. C. Diaz Ramos, M. Domingues Vazquez, and A. Kollross, *Polar actions on complex hyperbolic spaces*, Math. Z. **287** (2017), no. 3-4, 1183–1213.

[6] P. Eberlein, *Geodesic flows in manifolds of nonpositive curvature*, http://math.unc.edu/Faculty/pbe/AMS_Summer.pdf.

[7] P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math. **46** (1973), 45–109.

[8] C. Gorodski and A. Lytchak, *Isometric actions on spheres with an orbifold quotient*, Math. Ann. **365** (2016), no. 3-4, 1041–1067.

[9] A. Kollross, *A classification of hyperpolar and cohomogeneity one actions*, Trans. Amer. Math. Soc. **354** (2002), no. 2, 571–612.
[10] A. Kollross and A. Lytchak, *Polar actions on symmetric spaces of higher rank*, Bull. Lond. Math. Soc. **45** (2013), no. 2, 341–350.
[11] A. Lytchak, *Polar foliations of symmetric spaces*, Geom. Funct. Anal. **24** (2014), no. 4, 1298–1315.
[12] P. W. Michor, *Isometric actions of Lie groups and invariants*, Lecture course at the university of Vienna, 1996.
[13] R. Mirzaie, *On negatively curved G-manifolds of low cohomogeneity*, Hokkaido Math. J. **38** (2009), no. 4, 797–803.
[14] ______, *On Riemannian manifolds of constant negative curvature*, J. Korean Math. Soc. **48** (2011), no. 1, 23–31.
[15] ______, *On homogeneous submanifolds of negatively curved Riemannian manifolds*, Publ. Math. Debrecen **82** (2013), no. 2, 267–275.
[16] R. Mirzaie and S. M. B. Kashani, *On cohomogeneity one flat Riemannian manifolds*, Glasg. Math. J. **44** (2002), no. 2, 185–190.
[17] R. S. Palais and C.-L. Terng, *A general theory of canonical forms*, Trans. Amer. Math. Soc. **300** (1987), no. 2, 771–789.
[18] F. Podestà and A. Spiro, *Some topological properties of cohomogeneity one manifolds with negative curvature*, Ann. Global Anal. Geom. **14** (1996), no. 1, 69–79.
[19] F. Podestà and G. Thorbergsson, *Polar actions on rank-one symmetric spaces*, J. Differential Geom., **53** (1999), 131–175,
[20] ______, *Polar and coisotropic actions on Kähler manifolds*, Trans. Amer. Math. Soc. **354** (2002), no. 5, 1759–1781.
[21] A. J. Di Scala and C. Olmos, *The geometry of homogeneous submanifolds of hyperbolic space*, Math. Z. **237** (2001), no. 1, 199–209.
[22] J. Szenthe, *A generalization of the Weyl group*, Acta Math. Hungar. **41** (1983), no. 3-4, 347–357.
[23] ______, *Orthogonally transversal submanifolds and the generalizations of the Weyl group*, Period. Math. Hungar. **15** (1984), no. 4, 281–299.

Mojtaba Heidari
Department of Mathematics
Faculty of Science
Imam Khomeini International University (IKIU)
Qazvin, Iran
Email address: heidari.geometry@gmail.com

Reza Mirzaie
Department of Mathematics
Faculty of Science
Imam Khomeini International University (IKIU)
Qazvin, Iran
Email address: r.mirzaei@sci.ikiu.ac.ir