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BI-LIPSCHITZ PROPERTY AND DISTORTION THEOREMS FOR PLANAR HARMONIC MAPPINGS WITH M-LINEARLY CONNECTED HOLOMORPHIC PART

Jie Huang and Jian-Feng Zhu

Abstract. Let $f = h + \overline{g}$ be a harmonic mapping of the unit disk $\mathbb{D}$ with the holomorphic part h satisfying that h is injective and $h(\mathbb{D})$ is an M-linearly connected domain. In this paper, we obtain the sufficient and necessary conditions for f to be bi-Lipschitz, which is in particular, quasiconformal. Moreover, some distortion theorems are also obtained.

1. Introduction

A complex-valued function $f(z)$ of class C^2 is said to be a harmonic mapping, if it satisfies $f_{z\bar{z}} = 0$. Assume that $f(z)$ is a harmonic mapping defined in a simply connected domain $\Omega \subseteq \mathbb{C}$. Then $f(z)$ has the canonical decomposition $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in Ω. For more details on planar harmonic mappings we refer to ([6], [13]). Let $\mathbb{D}(a, r) = \{z : |z-a| < r\}$ be the disk center at a with the radius r, $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk, and $\partial\mathbb{D} = \{z : |z| = 1\}$ be the unit circle. Throughout this paper we consider harmonic mappings $f(z)$ in $\mathbb{D}$.

For any $z = re^{i\theta} \in \mathbb{D}$ and $\alpha \in [0, \pi]$, the directional derivative of f is defined by

$$\partial_\alpha f(z) = \lim_{r\to 0^+} \frac{f(z + re^{i\alpha}) - f(z)}{r} = e^{i\alpha} f_z(z) + e^{-i\alpha} f_{\bar{z}}(z). \tag{1}$$

Then, we have

$$\max_{0\leq\alpha<2\pi} |\partial_\alpha f(z)| = \Lambda_f(z) = |f_z(z)| + |f_{\bar{z}}(z)| \tag{2}$$

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and

$$\min_{0\leq\alpha<2\pi}|\partial_\alpha f(z)| = \lambda_f(z) = ||f_z(z)| - |f_{\bar{z}}(z)||\,. \tag{3}$$

It is known from [9] that $f(z)$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if its Jacobian satisfies the following condition

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 > 0 \quad \text{for } z \in \mathbb{D}.$$

For a sense-preserving harmonic mapping $f(z) = h(z) + \overline{g(z)}$ in $\mathbb{D}$, let

$$\omega(z) = \frac{g'(z)}{h'(z)} \tag{4}$$

be the (second) complex dilatation of f. Then $\omega(z)$ is a holomorphic mapping of $\mathbb{D}$ and

$$\|\omega\|_\infty := \sup_{z\in\mathbb{D}} \|\omega(z)\| \leq 1. \tag{5}$$

Throughout this paper we assume that f is sense-preserving.

Given $K \geq 1$ and assume that $f(z)$ is a sense-preserving univalent harmonic mapping of $\mathbb{D}$. Then $f(z)$ is called a harmonic K-quasiconformal mapping if there exists a constant k such that

$$\sup_{z\in\mathbb{D}}\left|\frac{f_{\bar{z}}(z)}{f_z(z)}\right| \leq k = \frac{K-1}{K+1}.$$

A mapping $f(z)$ defined in $\mathbb{D}$ is said to be co-Lipschitz (resp. Lipschitz) in $\mathbb{D}$ if there exists a constant L such that the following inequality

$$\frac{|z_1 - z_2|}{L} \leq |f(z_1) - f(z_2)| \quad (\text{resp.} \quad |f(z_1) - f(z_2)| \leq L|z_1 - z_2|) \tag{6}$$

holds for all z_1, $z_2 \in \mathbb{D}$, where $L \geq 1$ is called the Lipschitz constant. f is said to be bi-Lipschitz if f is co-Lipschitz and Lipschitz.

A sense-preserving harmonic bi-Lipschitz mapping is always quasiconformal, while the converse is not true, in general (cf. [14]).

Denote by S_H the family of all sense-preserving univalent harmonic mappings defined in $\mathbb{D}$ which admit a canonical representation $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{7}$$

are analytic in $\mathbb{D}$. The class S_H^0 is the subclass of S_H with $g'(0) = 0$, see ([4]) for more details.

A domain $\Omega \subset \mathbb{C}$ is said to be *M-linearly connected* if there exists a positive constant $M \in [1, \infty)$ such that for any two points $z, w \in \Omega$ are joined by a path $\gamma \subset \Omega$ with

$$l(\gamma) \leq M|z - w|, \quad \text{where} \quad l(\gamma) = \int_\gamma |dz|.$$

It is easy to see that a 1-linearly connected domain is convex. We remark here that in this paper, we always assume such a path γ mentioned above is *rectifiable* and bounded by $M|z-w|$. We refer to [10] for the definition of *rectifiably M-arcwise connected domain* (see also *properly M-arcwise connected domain*). For extensive discussions on this topic, see the references [1], [2] and [12].

A function $f \in C^1(\mathbb{D})$ is said to be *M-linearly connected* if f is injective and $f(\mathbb{D})$ is an M-linearly connected domain.

In what follows, the notation $L^\infty(\mathbb{D})$ denotes the set of all complex-valued, measurable functions which are *essentially bounded* in $\mathbb{D}$.

In 2007, M. Chuaqui et al. proved the following theorem.

Theorem A ([3, Theorem 1])**.** *Let $h : \mathbb{D} \to \mathbb{C}$ be a holomorphic univalent map. Then there exists $c > 0$ such that every harmonic mapping $f = h + \bar{g}$ with dilatation $\|\omega\|_\infty < c$ is univalent if and only if $h(\mathbb{D})$ is a linearly connected domain.*

The proof of Theorem 1 shows that one can take $c = 1$ when h is convex, an important special case that they state separately as the following corollary.

Corollary 1 ([3, Corollary])**.** *Let h be analytic and convex in $\mathbb{D}$. Then every harmonic mapping of the form $f = h + \bar{g}$ with $\|\omega\|_\infty < 1$ is injective.*

We point out that $f = h + \bar{g}$ is univalent in $\mathbb{D}$ doesn't imply that h is univalent in $\mathbb{D}$. Also, f is quasiconformal in $\mathbb{D}$ then f doesn't need to be co-Lipschitz or Lipschitz in $\mathbb{D}$. It is related to the domain $f(\mathbb{D})$. One can refer to [8] and [11] for the discussion of how can a sense-preserving harmonic mapping f in $\mathbb{D}$ be quasiconformal and bi-Lipschitz, with the image domain $f(\mathbb{D})$ is a bounded convex domain. Based on these facts and motivated by Theorem 1, in this paper assume that $f = h + \bar{g}$ is a harmonic mapping in $\mathbb{D}$ such that its holomorphic part h is M-linearly connected. Then we prove that f is bi-Lipschitz in $\mathbb{D}$ if and only if there exists a constant $0 < c < 1$ such that $\|\omega\|_\infty < c$ and $\log|h'| \in L^\infty(\mathbb{D})$. See Theorem 1 and Remark 1. Moreover, some distortion theorems are also considered in Section 3.

We will first prove some lemmas which are elementally but useful in the section 2 and then give the main results and their proofs in Section 3.

2. Auxiliary results

The following lemmas are useful and will be used in proving our main results.

Lemma 1. *Given $M \geq 1$, let $f \in C^1(\mathbb{D})$ be M-linearly connected. Then $f(z)$ is co-Lipschitz if and only if there exists $c_1 > 0$ such that $\lambda_f(z) \geq c_1$ holds for all $z \in \mathbb{D}$.*

Proof. We first prove the only if part. Since $f(z)$ is co-Lipschitz, then there exists $L > 0$ such that

$$|f(z_1) - f(z_2)| \geq \frac{|z_1 - z_2|}{L}$$

for all $z_1, z_2 \in \mathbb{D}$. For $z_2 = z \in \mathbb{D}$, let r small enough such that $z_1 = z + re^{i\theta} \in \mathbb{D}$. Then we have

$$\left| \frac{f(z+re^{i\theta}) - f(z)}{re^{i\theta}} \right| \geq \frac{1}{L}.$$

By letting $r \to 0$, we obtain

$$\lim_{r\to 0} \left| \frac{f(z+re^{i\theta}) - f(z)}{re^{i\theta}} \right| = |e^{i\theta} f_z(z) + e^{-i\theta} f_{\overline{z}}(z)| \geq \frac{1}{L}. \tag{8}$$

Thus

$$\lambda_f(z) = \min_{\theta\in[0,\pi]} |e^{i\theta} f_z(z) + e^{-i\theta} f_{\overline{z}}(z)| \geq \frac{1}{L}.$$

Now we prove the if part. Assume that there exists $c_1 > 0$ such that $\lambda_f(z) \geq c_1$ holds for all $z \in \mathbb{D}$. Take $z_1, z_2 \in \mathbb{D}$, with $z_1 \neq z_2$. Since $\Omega = f(\mathbb{D})$ is an M-linearly connected domain, we see that there exists a rectifiable path γ in Ω connecting the points $\zeta_1 = f(z_1)$ and $\zeta_2 = f(z_2)$ such that

$$l(\gamma) \leq M|f(z_1) - f(z_2)|. \tag{9}$$

Since $f(z) \in C^1(\mathbb{D})$ is an injective function of $\mathbb{D}$ with $\lambda_f(z) \geq c_1 > 0$, we see that $J_f(z) > 0$ for every $z \in \mathbb{D}$. Therefore, f is a C^1-diffeomorphism of $\mathbb{D}$ onto Ω. Let $g = f^{-1} : \Omega \mapsto \mathbb{D}$ be the inverse function of f. Then $g(\zeta)$ is a C^1-diffeomorphism of Ω onto $\mathbb{D}$ such that the following inequality

$$|g(\zeta_1) - g(\zeta_2)| \leq \int_{g(\gamma)} |dg(\zeta)| \leq \int_{\gamma} \Lambda_g(\zeta)|d\zeta|$$

holds for all ζ_1, $\zeta_2 \in \Omega$. Elementary calculations lead to $g_\zeta = \frac{\overline{f_z}}{J_f}$ and $g_{\bar{\zeta}} = \frac{-f_{\bar{z}}}{J_f}$. This shows that $\Lambda_g(\zeta) = \frac{1}{\lambda_f} \leq \frac{1}{c_1}$. By using (9), we have

$$|g(\zeta_1) - g(\zeta_2)| \leq \frac{1}{c_1} l(\gamma) \leq \frac{M}{c_1} |\zeta_1 - \zeta_2|.$$

Therefore,

$$|f(z_1) - f(z_2)| \geq \frac{c_1}{M} |z_1 - z_2|.$$

This shows that $f(z)$ is co-Lipschitz. □

Lemma 2. *Let $f \in C^1(\mathbb{D})$. Then $f(z)$ is Lipschitz if and only if there exists a constant $c_2 > 0$ such that $\Lambda_f(z) \leq c_2$ holds for all $z \in \mathbb{D}$.*

Proof. We first prove the only if part. According to the assumption, we know that f is Lipschitz. Therefore there exists $L > 0$ such that

$$|f(z_1) - f(z_2)| \leq L|z_1 - z_2|$$

holds for all $z_1, z_2 \in \mathbb{D}$. Let $z_2 = z \in \mathbb{D}$ for r small enough such that $z_1 = z + re^{i\theta} \in \mathbb{D}$. Then

$$\left| \frac{f(z+re^{i\theta}) - f(z)}{re^{i\theta}} \right| \leq L.$$

Letting $r \to 0$, we obtain

$$\lim_{r\to 0}\left|\frac{f(z+re^{i\theta})-f(z)}{re^{i\theta}}\right| = |e^{i\theta}f_z(z)+e^{-i\theta}f_{\overline{z}}(z)| \le L. \tag{10}$$

Thus $\Lambda_f(z) = \max_{\theta\in[0,\pi]} |e^{i\theta}f_z(z)+e^{-i\theta}f_{\overline{z}}(z)| \le L.$

Now we prove the if part. Assume that there exists $c_2 > 0$ such that $\Lambda_f(z) \le c_2$ holds for all $z \in \mathbb{D}$. Take $z_1, z_2 \in \mathbb{D}$, let $C : z = z(t) = z_1 + t(z_2 - z_1)$ be the segment line which joining z_1 and z_2, and $\gamma = f(C)$. Then

$$\begin{aligned}|f(z_1)-f(z_2)| &\le \int_\gamma |df(z)|\\ &= \int_C |f_z(z(t))e^{i\alpha}+f_{\overline{z}}(z(t))e^{-i\alpha}||dz(t)|\\ &\le |z_1-z_2|\int_0^1 \Lambda_f dt\\ &\le c_2|z_1-z_2|,\end{aligned}$$

where $\alpha = \arg(z_1 - z_2)$. This implies that $f(z)$ is Lipschitz. □

Lemma 3. *Given $M \ge 1$, let $f = h+\overline{g}$ be a harmonic mapping of $\mathbb{D}$ such that h is M-linearly connected. Then the inequality*

$$|h(z_1)-h(z_2)| \ge M\|\omega\|_\infty |g(z_1)-g(z_2)| \tag{11}$$

holds for all z_1, $z_2 \in \mathbb{D}$. If additionally $M\|\omega\|_\infty < 1$, then f is univalent in $\mathbb{D}$.

Proof. Let $\Omega = h(\mathbb{D})$. For any two points $\zeta_1, \zeta_2 \in \Omega$, since Ω is an M-linearly connected domain, we see that there exists a path $\Gamma : [0,1] \mapsto \Omega$ connecting the points $\zeta_1 = \Gamma(0)$ and $\zeta_2 = \Gamma(1)$ such that $l(\Gamma) \le M|\zeta_1 - \zeta_2|$.

Consider the holomorphic mapping $\varphi(\zeta) = g \circ h^{-1}(\zeta)$, where $\zeta = h(z) \in \Omega$ and $z \in \mathbb{D}$. Then we have

$$|\varphi'(\zeta)| = \left|\frac{g'(z)}{h'(z)}\right| \le \|\omega\|_\infty. \tag{12}$$

Therefore we have

$$\begin{aligned}|\varphi(\zeta_1)-\varphi(\zeta_2)| &= \left|\int_\Gamma d\varphi\right|\\ &\le \int_\Gamma |d\varphi| \le \|\omega\|_\infty \int_\Gamma |d\zeta|\\ &\le \|\omega\|_\infty M|\zeta_1-\zeta_2|.\end{aligned}$$

This shows that

$$\sup_{\zeta_1,\zeta_2\in\Omega}\left|\frac{\varphi(\zeta_1)-\varphi(\zeta_2)}{\zeta_1-\zeta_2}\right| \le M\|\omega\|_\infty. \tag{13}$$

Thus
$$\frac{|g\circ h^{-1}(\zeta_1)-g\circ h^{-1}(\zeta_2)|}{|\zeta_1-\zeta_2|}\le M\|\omega\|_\infty.$$
Using $z=h^{-1}(\zeta)$, then
$$|g(z_1)-g(z_2)|\le M\|\omega\|_\infty|h(z_1)-h(z_2)|. \tag{14}$$
If additionally $M\|\omega\|_\infty<1$, then we have
$$\begin{aligned}|f(z_1)-f(z_2)|&\ge|h(z_1)-h(z_2)|-|g(z_1)-g(z_2)|\\&\ge(1-M\|\omega\|_\infty)|h(z_1)-h(z_2)|>0\end{aligned}$$
hold for all z_1, $z_2\in\mathbb{D}$. This shows that f is univalent in $\mathbb{D}$. □

3. Main results

Theorem 1. *For $M\ge1$, let $f=h+\overline{g}$ be a harmonic mapping in $\mathbb{D}$. If h is M-linearly connected, then the following statements hold.*

(I) *If $\|\omega\|_\infty<\frac{1}{M}$ and $\log|h'|\in L^\infty(\mathbb{D})$, then f is a bi-Lipschitz mapping in $\mathbb{D}$ and its Lipschitz constant L is related to M and $\|\omega\|_\infty$.*

(II) *Let f be a bi-Lipschitz mapping of $\mathbb{D}$ with its Lipschitz constant $L\ge1$. Then*
$$\|\omega\|_\infty\le\frac{L^2-1}{L^2+1}\quad\text{and}\quad\log|h'|\in L^\infty(\mathbb{D}).$$

Furthermore, we have $f(\mathbb{D})$ is an M_1-linearly connected domain with $M_1=ML^2\frac{1+\|\omega\|_\infty}{1-\|\omega\|_\infty}$.

Proof. (I) Since $\log|h'|\in L^\infty(\mathbb{D})$, this shows that there exist constants $0<c_1\le c_2<+\infty$ such that $c_1\le|h'(z)|\le c_2$ hold for all $z\in\mathbb{D}$. For any $z_1,z_2\in\mathbb{D}$, with $z_1\ne z_2$, let $\zeta_1=h(z_1)$ and $\zeta_2=h(z_2)$. Since h is an injective, analytic function in $\mathbb{D}$ (and therefore $h\in C^1(\mathbb{D})$), with $|h'|\ge c_1$ and $h(\mathbb{D})$ is an M-linearly connected domain, it follows from the proof of the "if" part in Lemma 1 that
$$|h(z_1)-h(z_2)|\ge\frac{c_1|z_1-z_2|}{M}.$$
Applying (11), we have
$$|f(z_1)-f(z_2)|\ge(1-M\|\omega\|_\infty)|h(z_1)-h(z_2)|\ge\frac{c_1(1-M\|\omega\|_\infty)}{M}|z_1-z_2|.$$
This shows that $f(z)$ is co-Lipschitz.

On the other hand, assume that $C: z=z(t)=z_1+t(z_2-z_1)$, $0\le t\le1$, be the line segment which joining z_1 and z_2. Let $\Gamma=f(C)$. Then
$$\begin{aligned}|f(z_1)-f(z_2)|&\le\int_\Gamma|df(z)|=\int_C|f_z(z(t))dz(t)+f_{\overline{z}}(z(t))d\overline{z(t)}|\\&=|z_1-z_2|\int_0^1|f_z(z(t))e^{i\alpha}+f_{\overline{z}}(z(t))e^{-i\alpha}|dt\end{aligned}$$

$$\begin{aligned}
&\le |z_1 - z_2| \int_0^1 |f_z(z(t))| \left(1 + \left|\frac{f_{\overline{z}}(z(t))}{f_z(z(t))}\right|\right) dt \\
&\le |z_1 - z_2| \int_0^1 |h'(z(t))| \left(1 + \|\omega\|_\infty\right) dt \\
&= |z_1 - z_2|(1 + \|\omega\|_\infty) \int_0^1 |h'(z(t))| dt \\
&\le |z_1 - z_2|(1 + \|\omega\|_\infty) c_2,
\end{aligned}$$

where $\alpha = \arg(z_1 - z_2)$. Let $L = \max\{(1 + \|\omega\|_\infty)c_2,\ \frac{M}{c_1(1 - M\|\omega\|_\infty)}\}$, then

$$\frac{1}{L} \le \left|\frac{f(z_1) - f(z_2)}{z_1 - z_2}\right| \le L$$

hold for all $z_1,\ z_2 \in \mathbb{D}$.

(II) According to the assumption, we have

$$\frac{1}{L} \le \left|\frac{f(z_1) - f(z_2)}{z_1 - z_2}\right| \le L$$

hold for all $z_1,\ z_2 \in \mathbb{D}$, where $L \ge 1$. By using (8) and (10), we have

$$\Lambda_f(z) = \max_{\theta \in [0,\pi]} |e^{i\theta} f_z(z) + e^{-i\theta} f_{\overline{z}}(z)| \le L$$

and

$$\lambda_f(z) = \min_{\theta \in [0,\pi]} |e^{i\theta} f_z(z) + e^{-i\theta} f_{\overline{z}}(z)| \ge \frac{1}{L}$$

hold true for all $z \in \mathbb{D}$. This implies that

$$\frac{\Lambda_f(z)}{\lambda_f(z)} = \frac{|h'(z)| + |g'(z)|}{|h'(z)| - |g'(z)|} = \frac{1 + \left|\frac{g'(z)}{h'(z)}\right|}{1 - \left|\frac{g'(z)}{h'(z)}\right|} \le L^2.$$

Hence $\left|\frac{g'(z)}{h'(z)}\right| \le \frac{L^2-1}{L^2+1}$ holds for all $z \in \mathbb{D}$. Therefore, we obtain that

$$\|\omega\|_\infty = \sup_{z \in \mathbb{D}} \left|\frac{g'(z)}{h'(z)}\right| \le \frac{L^2 - 1}{L^2 + 1} < 1.$$

Furthermore, since

$$L \ge \Lambda_f(z) \ge \lambda_f(z) = |h'(z)| \left(1 - \left|\frac{g'(z)}{h'(z)}\right|\right) \ge |h'(z)|(1 - \|\omega\|_\infty)$$

and

$$\frac{1}{L} \le \lambda_f(z) \le \Lambda_f(z) = |h'(z)| \left(1 + \left|\frac{g'(z)}{h'(z)}\right|\right) \le |h'(z)|(1 + \|\omega\|_\infty)$$

we have

$$|h'(z)| \le \frac{L}{1 - \|\omega\|_\infty} \tag{15}$$

and

$$|h'(z)| \geq \frac{1}{L(1+\|\omega\|_\infty)} \tag{16}$$

hold true. This shows that

$$\log|h'| \in L^\infty(\mathbb{D}) \tag{17}$$

as desired. Now we prove $f(\mathbb{D})$ is an M_1-linearly connected domain. For any $w_1, w_2 \in f(\mathbb{D})$, let Γ be arbitrary curve in $f(\mathbb{D})$ which joining w_1 and w_2. $l = f^{-1}(\Gamma)$ is the curve in $\mathbb{D}$ with the end points $z_1 = f^{-1}(w_1)$ and $z_2 = f^{-1}(w_2)$. $\widetilde{\gamma} = h(l)$ is the curve in $h(\mathbb{D})$ with the end points $\zeta_1 = h(z_1)$ and $\zeta_2 = h(z_2)$. Note that $h(\mathbb{D})$ is an M-linearly connected domain, then

$$\begin{aligned}
l(\Gamma) &= \int_\Gamma |df(z)| = \int_l |f_z(z(t))e^{i\beta} + f_{\overline{z}}(z(t))e^{-i\beta}||dz(t)| \\
&\leq \int_l |f_z(z(t))| \left(1 + \left|\frac{f_{\overline{z}}(z(t))}{f_z(z(t))}\right|\right) |dz(t)| \\
&\leq (1+\|\omega\|_\infty) \int_l |h'(z(t))||dz(t)| \\
&= (1+\|\omega\|_\infty) l_{\widetilde{\gamma}} \\
&\leq M(1+\|\omega\|_\infty)|\zeta_1 - \zeta_2|,
\end{aligned}$$

where $\beta = \arg dz(t)$ for $l : z = z(t)$.

Let $C : z = z(t) = z_1 + t(z_2 - z_1)$ be the line segment which joining z_1 and z_2, $\gamma = h(C)$ is the curve in $h(\mathbb{D})$ with the end points $\zeta_1 = h(z_1)$ and $\zeta_2 = h(z_2)$. Then (15) yields that

$$\begin{aligned}
|\zeta_1 - \zeta_2| &\leq \int_\gamma |dh(z)| \\
&\leq \int_C |h'(z(t))||dz(t)| \\
&= |z_1 - z_2| \int_0^1 |h'(z(t))|dt \\
&\leq |z_1 - z_2| \frac{L}{1-\|\omega\|_\infty}.
\end{aligned}$$

Therefore,

$$l(\Gamma) \leq ML\frac{1+\|\omega\|_\infty}{1-\|\omega\|_\infty}|z_1 - z_2| \leq ML^2\frac{1+\|\omega\|_\infty}{1-\|\omega\|_\infty}|f(z_1) - f(z_2)|.$$

This shows that $f(\mathbb{D})$ is an M_1-linearly connected domain, where

$$M_1 = ML^2\frac{1+\|\omega\|_\infty}{1-\|\omega\|_\infty}.$$

The proof is completed. □

Remark 1. (1) Under the assumptions of Theorem 1, by using Lemma 1 and Lemma 2, we know that $\log|h'| \in L^\infty(\mathbb{D})$ is equivalent to h is bi-Lipschitz.

(2) If $f = h + \bar{g}$ is quasiconformal (not bi-Lipschitz) in $\mathbb{D}$, then $\log|h'| < \infty$ does not need to hold. We show this by using the following function

$$f(z) = h(z) + \overline{g(z)} = (1-z)^\alpha + k(1-\overline{z})^\alpha,$$

where $0 < \alpha < 1$ and $0 < k < \frac{1}{M} \leq 1$.

Theorem 2. *Given $M \geq 1$, $f = h + \overline{g}$ is a harmonic mapping of $\mathbb{D}$ such that h is M-linearly connected. If $\|\omega\|_\infty < \frac{1}{M}$, then*

(I) *$T_\theta = h + e^{i\theta} g$ is univalent in $\mathbb{D}$, for all $\theta \in [0, 2\pi]$. Moreover, $T_\theta(\mathbb{D})$ is an M_1-linearly connected domain, where $M_1 = \frac{M(1+\|\omega\|_\infty)}{1-M\|\omega\|_\infty}$.*

(II) *If f can be extended continuously to the boundary, then there exist positive constants c_2 and $c_3 < 2$ such that for $\zeta_1,\ \zeta_2 \in \partial\mathbb{D}$,*

$$|f(\zeta_1) - f(\zeta_2)| \geq c_2|\zeta_1 - \zeta_2|^{c_3},$$

where c_2 depends on M.

Proof. (I) Take arbitrary two points $z_1, z_2 \in \mathbb{D}$. According to (14) we see that

$$\begin{aligned}|T_\theta(z_1) - T_\theta(z_2)| &\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\ &\geq (1 - M\|\omega\|_\infty)|h(z_1) - h(z_2)|.\end{aligned}$$

Since $M\|\omega\|_\infty < 1$ and $h(z)$ is injective, we know that

$$|T_\theta(z_1) - T_\theta(z_2)| \geq (1 - M\|\omega\|_\infty)|h(z_1) - h(z_2)| > 0.$$

This shows that $T_\theta(z)$ is univalent in $\mathbb{D}$ for all $\theta \in [0, 2\pi]$.

For $w \in h(\mathbb{D})$, let

$$H(w) = T_\theta(h^{-1}(w)) = w + e^{i\theta} g \circ h^{-1}(w). \tag{18}$$

Then we have $H(w)$ is holomorphic in $h(\mathbb{D})$ with $H'(w) = 1 + e^{i\theta}\omega(w)$.

Fixed two points $\xi_1 = T_\theta(z_1)$ and $\xi_2 = T_\theta(z_2) \in T_\theta(\mathbb{D})$ and let $\gamma \subset T_\theta(\mathbb{D})$ be the curve which joining ξ_1 and ξ_2. Since $h(\mathbb{D})$ is an M-linearly connected domain, we know that for any two points $w_1,\ w_2 \in h(\mathbb{D})$, there is a curve $\Gamma \subset h(\mathbb{D})$ joining w_1 and w_2 such that $l(\Gamma) \leq M|w_1 - w_2|$. Now we set $\gamma = H(\Gamma)$. Then

$$\begin{aligned}l(\gamma) &= \int_\gamma |dH(w)| \\ &\leq \int_\Gamma (1 + \|\omega\|_\infty)|dw| \\ &= (1 + \|\omega\|_\infty) l(\Gamma) \\ &\leq (1 + \|\omega\|_\infty) M|w_1 - w_2|.\end{aligned}$$

Applying (11) we know that

$$|\xi_1 - \xi_2| = |T_\theta(z_1) - T_\theta(z_2)| \tag{19}$$

$$\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)|$$
$$\geq (1 - M(\|\omega\|_\infty))|h(z_1) - h(z_2)|$$
$$= (1 - M\|\omega\|_\infty)|w_1 - w_2|.$$

This shows that

$$l(\gamma) \leq \frac{M(1 + \|\omega\|_\infty)}{1 - M\|\omega\|_\infty}|\xi_1 - \xi_2|.$$

Thus $T_\theta(\mathbb{D})$ is an M_1-linearly connected domain, where $M_1 = \frac{M(1+\|\omega\|_\infty)}{1-M\|\omega\|_\infty}$.

(II) By [12, Proposition 5.6] we know that T_θ is continuous in $\overline{\mathbb{D}}$ with values in $\mathbb{C} \cup \{\infty\}$. Applying [12, Proposition 5.7(5)] to T_θ, we see that there are constants $c_2 > 0$ and $c_3 < 2$ such that for ζ_2, $\zeta_2 \in \partial\mathbb{D}$,

$$|T_\theta(\zeta_1) - T_\theta(\zeta_2)| \geq c_2|\zeta_1 - \zeta_2|^{c_3}. \tag{20}$$

Inequality (20) and the arbitrary taking of θ shows that

$$|f(\zeta_1) - f(\zeta_2)| \geq c_2|\zeta_1 - \zeta_2|^{c_3}.$$

This completes the proof. □

Remark 2. The following lemma easily follows from [7, Proposition 2.1].

Lemma B. *If for any ϵ with $|\epsilon| = 1$, the function $h + \epsilon g$ is univalent in $\mathbb{D}$, then $f = h + \overline{g}$ is univalent in $\mathbb{D}$, where h and g are holomorphic in $\mathbb{D}$.*

Therefore, one can easily obtain that $T_\theta(z)$ is univalent in $\mathbb{D}$ (one of the results in Theorem 2) implies that $f(z)$ is univalent in $\mathbb{D}$ (the result in Lemma 3).

Furthermore, under the assumption of Theorem 2 we have $f(\mathbb{D})$ is also an M_1-linearly connected domain.

Theorem 3. *Given $M \geq 1$, and assume that $f = h + \overline{g}$ is a sense-preserving harmonic mapping of $\mathbb{D}$ such that h is M-linearly connected with*

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \textit{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{21}$$

If $\|\omega\|_\infty < \frac{1}{M}$, then we have the results as follows.

(I) *The coefficients of* (21) *satisfying*

$$|a_n| + |b_n| \leq n \quad \textit{for all} \ \ n \geq 2.$$

(II) *The inequalities*

$$\Lambda_f(z) \leq \frac{1 + |z|}{(1 - |z|)^3}, \tag{22}$$

$$\lambda_f(z) \geq \frac{1 - |z|}{(1 + |z|)^3}, \tag{23}$$

and

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2} \tag{24}$$

hold for all $z \in \mathbb{D}$.

Proof. (I) According to Theorem 2, we see that $T_\theta(z) = h(z) + e^{i\theta}g(z)$ is univalent in $\mathbb{D}$ for all $\theta \in [0, 2\pi)$. Since h and g are normalized by (21), we know that

$$\begin{aligned} h(z) + e^{i\theta}g(z) &= z + \sum_{n=2}^{\infty} a_n z^n + e^{i\theta} \sum_{n=2}^{\infty} b_n z^n \\ &= z + \sum_{n=2}^{\infty} (a_n + e^{i\theta} b_n) z^n \in S. \end{aligned}$$

Therefore, using the Bieberbach coefficients conjecture (see [5]) we obtain

$$|a_n + e^{i\theta} b_n| \le n, \quad \text{for} \;\; \theta \in [0, 2\pi) \quad \text{and} \quad n \ge 2.$$

Therefore,

$$|a_n| + |b_n| = \max_{\theta \in [0,2\pi)} |a_n + e^{i\theta} b_n| \le n \quad \text{for} \;\; n \ge 2.$$

(II) Since $T_\theta(z) \in S$, it follows from the distortion theorem in S that

$$\frac{1-|z|}{(1+|z|)^3} \le |T'_\theta(z)| \le \frac{1+|z|}{(1-|z|)^3}, \quad z \in \mathbb{D}.$$

This shows in particular that

$$|h'(z)| - |g'(z)| = \min_{\theta \in [0,2\pi)} |T'_\theta(z)| \ge \frac{1-|z|}{(1+|z|)^3}, \quad z \in \mathbb{D} \tag{25}$$

and

$$|h'(z)| + |g'(z)| = \max_{\theta \in [0,2\pi)} |T'_\theta(z)| \le \frac{1+|z|}{(1-|z|)^3}, \quad z \in \mathbb{D}. \tag{26}$$

Fix $z \in \mathbb{D}$. The last inequality (26) shows that

$$\begin{aligned} |f(z)| &= \left| \int_\Gamma f_\zeta(\zeta) d\zeta + f_{\bar{\zeta}}(\zeta) d\bar{\zeta} \right| \\ &\le \int_\Gamma (|h'(\zeta)| + |g'(\zeta)|) |d\zeta| \\ &\le \int_0^{|z|} \frac{(1+\rho)}{(1-\rho)^3} d\rho \\ &= \frac{|z|}{(1-|z|)^2}, \end{aligned}$$

where Γ is the radial line segment from 0 to z. Next let γ be the preimage under f of the radial segment from 0 to $f(z)$. Then

$$\begin{aligned}|f(z)| &= \int_\gamma \left|f_\zeta(\zeta)d\zeta + f_{\bar\zeta}(\zeta)d\bar\zeta\right| \\ &\geq \int_\gamma (|h'(\zeta)| - |g'(\zeta)|)|d\zeta| \\ &\geq \int_0^{|z|} \frac{(1-\rho)}{(1+\rho)^3} d\rho \\ &= \frac{|z|}{(1+|z|)^2},\end{aligned}$$

which completes the proof. □

Theorem 4. *Let $f = h + \overline{g}$ denote a sense-preserving harmonic mapping in the unit disk $\mathbb{D}$ such that h is injective and $h(\mathbb{D})$ is a convex domain. Then for all $z_1, z_2 \in \mathbb{D}$, $z_1 \neq z_2$ we have*

$$|g(z_1) - g(z_2)| < |h(z_1) - h(z_2)|$$

and f is a univalent harmonic close-to-convex mapping.

Furthermore, if f is a harmonic quasiconformal mapping, then the inequality

$$|g(z_1) - g(z_2)| \leq \|\omega\|_\infty |h(z_1) - h(z_2)|$$

holds for all $z_1, z_2 \in \mathbb{D}$.

Proof. For all $z_1, z_2 \in \mathbb{D}, z_1 \neq z_2$. Since $h(\mathbb{D})$ is a convex domain, there exists a line $\Gamma : t \mapsto th(z_2) + (1-t)h(z_1)$, $t \in [0,1]$ satisfies $\Gamma([0,1]) \subset h(\mathbb{D})$. Let $\zeta = h(z)$. Then

$$\begin{aligned}|g(z_1) - g(z_2)| &= \left|g \circ h^{-1}(h(z_1)) - g \circ h^{-1}(h(z_2))\right| \\ &= \left|\int_\Gamma \frac{d(g \circ h^{-1})(\zeta)}{d\zeta} d\zeta\right| \\ &< \int_\Gamma |d\zeta| = |h(z_1) - h(z_2)|\end{aligned}$$

the above inequality holds because $\left|\frac{d(g\circ h^{-1})(\zeta)}{d\zeta}\right| = \left|\frac{g'(z)}{h'(z)}\right| < 1$. Thus

$$\begin{aligned}|f(z_1) - f(z_2)| &= |h(z_1) - h(z_2) + \overline{g(z_1) - g(z_2)}| \\ &\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| > 0.\end{aligned}$$

According to Clunie Sheil-Small's result [4], we know that $f(z)$ is a close-to-convex mapping. If $f(z)$ is a harmonic quasiconformal mapping, then

$$\|\omega\|_\infty = \sup_{z\in\mathbb{D}} \left|\frac{g'(z)}{h'(z)}\right| < 1,$$

therefore

$$|g(z_1) - g(z_2))| \le \|\omega\|_\infty |h(z_1) - h(z_2)|.$$

This completes the proof. □

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References

[1] S. Chen, S. Ponnusamy, and X. Wang, *Stable geometric properties of pluriharmonic and biholomorphic mappings, and Landau-Bloch's theorem*, Monatsh. Math. **177** (2015), no. 1, 33–51.

[2] ______, *Linear connectivity, Schwarz-Pick lemma and univalency criteria for planar harmonic mapping*, Acta Math. Sin. (Engl. Ser.) **32** (2016), no. 3, 297–308.

[3] M. Chuaqui and R. Hernández, *Univalent harmonic mappings and linearly connected domains*, J. Math. Anal. Appl. **332** (2007), no. 2, 1189–1194.

[4] J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **9** (1984), 3–25.

[5] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), no. 1-2, 137–152.

[6] P. Duren, *Harmonic Mappings in the Plane*, Cambridge Tracts in Mathematics, **156**, Cambridge University Press, Cambridge, 2004.

[7] R. Hernández and M. J. Martín, *Stable geometric properties of analytic and harmonic functions*, Math. Proc. Cambridge Phil. Soc. **155** (2013), no. 2, 343–359.

[8] D. Kalaj, *Quasiconformal and harmonic mappings between Jordan domains*, Math. Z. **260** (2008), no. 2, 237–252.

[9] H. Lewy, *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. **42** (1936), no. 10, 689–692.

[10] R. Näkki and B. Palka, *Lipschitz conditions, b-arcwise connectedness and conformal mappings*, J. Analyse Math. **42** (1982/83), 38–50.

[11] M. Pavlović, *Boundary correspondence under harmonic quasiconformal homeomorphisms of the unit disk*, Ann. Acad. Sci. Fenn. Math. **27** (2002), no. 2, 365–372.

[12] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Grundlehren der Mathematischen Wissenschaften, **299**, Springer-Verlag, Berlin, 1992.

[13] T. Sheil-Small, *Constants for planar harmonic mappings*, J. London Math. Soc. (2) **42** (1990), no. 2, 237–248.

[14] J.-F. Zhu, *Some estimates for harmonic mappings with given boundary function*, J. Math. Anal. Appl. **411** (2014), no. 2, 631–638.

Jie Huang
School of Mathematical Sciences
Huaqiao University
Quanzhou 362021, P. R. China
Email address: 479772308@qq.com

Jian-Feng Zhu
School of Mathematical Sciences
Huaqiao University
Quanzhou 362021, P. R. China
Email address: flandy@hqu.edu.cn