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EXISTENCE OF SOLUTIONS FOR FRACTIONAL $p\&q$-KIRCHHOFF SYSTEM IN UNBOUNDED DOMAIN

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Abstract. In this paper, we investigate the fractional $p\&q$-Kirchhoff type system

$$\begin{cases} M_1([u]^p_{s,p})(-\Delta)^s_p u + V_1(x)|u|^{p-2}u \\ \quad = \ell k^{-1}F_u(x,u,v) + \lambda a(x)|u|^{m-2}u, & x \in \Omega, \\ M_2([u]^q_{s,q})(-\Delta)^s_q v + V_2(x)|v|^{q-2}v \\ \quad = \ell k^{-1}F_v(x,u,v) + \mu a(x)|v|^{m-2}v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an unbounded domain with smooth boundary $\partial\Omega$, and $0 < s < 1 < p \le q$ and $sq < N$, $\lambda, \mu > 0$, $1 < m \le k < p^*_s$, $\ell \in \mathbb{R}$, while $[u]^t_{s,t}$ denotes the Gagliardo semi-norm given in (1.2) below. $V_1(x), V_2(x), a(x) : \mathbb{R}^N \to (0,\infty)$ are three positive weights, M_1, M_2 are continuous and positive functions in $\mathbb{R}^+$. Using variational methods, we prove existence of infinitely many high-energy solutions for the above system.

1. Introduction and main results

In this paper, we are interested in the existence of infinitely many solutions to fractional $p\&q$-Kirchhoff type system

$$\begin{cases} M_1([u]^p_{s,p})(-\Delta)^s_p u + V_1(x)|u|^{p-2}u \\ \quad = \ell k^{-1}F_u(x,u,v) + \lambda a(x)|u|^{m-2}u, & x \in \Omega, \\ M_2([u]^q_{s,q})(-\Delta)^s_q v + V_2(x)|v|^{q-2}v \\ \quad = \ell k^{-1}F_v(x,u,v) + \mu a(x)|v|^{m-2}v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an unbounded domain with smooth boundary $\partial\Omega$, and $0 < s < 1 < p \le q$ and $sq < N$, $\lambda, \mu > 0$, $1 < m \le k < p^*_s$, $\ell \in \mathbb{R}$. The functions

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$a(x):\mathbb{R}^N\to\mathbb{R}^+=(0,\infty)$ and $F_u, F_v:\mathbb{R}^N\times\mathbb{R}^2\to\mathbb{R}$ are continuous, and

$$[u]_{s,t}^t=\iint\limits_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^t}{|x-y|^{N+ts}}\mathrm{d}x\mathrm{d}y \ \text{ for } t=p,q. \tag{1.2}$$

The fractional t-Laplacian operator $(-\Delta)_t^s$ with $0<s<1<t$ and $st<N$ is defined along a function $\varphi\in C_0^\infty(\mathbb{R}^N)$ as

$$\begin{aligned}&(-\Delta)_t^s\varphi(x)\\&=2\lim_{\varepsilon\to0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{|\varphi(x)-\varphi(y)|^{t-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ts}}\mathrm{d}y,\quad \forall x\in\mathbb{R}^N,\end{aligned} \tag{1.3}$$

where $B_\varepsilon(x):=\{y\in\mathbb{R}^N:|x-y|<\varepsilon\}$, see [8,25] and the references therein.

When $M_1(t)\equiv1$, the first equation in (1.1) becomes the fractional p-Laplacian equation

$$(-\Delta)_p^s u+V(x)|u|^{p-2}u=f(x,u),\quad x\in\Omega, \tag{1.4}$$

which can been seen as the fractional form of the following classical stationary Schrödinger equation

$$-\Delta_p u+V(x)|u|^{p-2}u=f(x,u),\quad x\in\Omega. \tag{1.5}$$

In recent years, a great interest has been devoted to Kirchhoff equations of the type

$$-\Big(a+b\int_\Omega|\nabla u|^2dx\Big)\Delta u=f(x,u),\quad x\in\Omega, \tag{1.6}$$

where $\Omega\subset\mathbb{R}^N$ is a smooth domain, $a>0, b\ge0$.

Such problems are often referred to as being nonlocal because of the presence of the integral over the entire domain Ω. This problem is analogous to the stationary problem of a model introduced by Kirchhoff [14]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho u_{tt}-\Big(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L u_x^2dx\Big)u_{xx}=0,\quad 0<x<L,\ t>0, \tag{1.7}$$

where ρ,ρ_0,h,E,L are all positive constants. This equation extends the classical D'Alembert wave equation. The study of Kirchhoff-type equation has already been extended to the case involving the p-Laplacian operator

$$-M(\|\nabla u\|_p^p)\Delta_p u=f(x,u),\quad x\in\Omega \tag{1.8}$$

with $M(t)\ge m_0>0$ for any $t\ge0$, see [6,11,13,16,17,26] and therein references.

On the other hand, a great attention has been recently focused on the study of fractional and nonlocal operators of elliptic type. This type of operators is of particular interest in fractional quantum mechanics for the study of particles on stochastic fields modelled by Lévy processes, which occur widely in physics, chemistry and biology. The stable Lévy processes that gives rise to equations with the fractional Laplacian have recently attracted much research interest, see for example [1, 2, 4, 5, 7, 9]. For the basic properties of fractional Sobolev spaces with applications to partial differential equations, we refer the readers to [8, 19].

In [18], Nyamoradi studied a class of Kirchhoff nonlocal fractional equation in a bounded domain and obtained three solutions by using three critical point theorem. Pucci and Saldi [20] established the existence and multiplicity of nontrivial solutions for a Kirchhoff type eigenvalue problem in $\mathbb{R}^N$ involving a critical nonlinearity and the nonlocal fractional Laplacian.

More recently, Pucci et al. [21] studied the existence and multiplicity of entire solutions for the equation

$$\begin{aligned} &M([u]_{s,p}^p)(-\Delta)_p^s u + V(x)|u|^{p-2}u + h(x)|u|^{r-2}u \\ &= \lambda\omega(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N \end{aligned} \tag{1.9}$$

with $\lambda > 0$, $1 < q < r < p_s^*$ and $h(x) > 0$, $\omega(x) \geq 0$ in $\mathbb{R}^N$. Furthermore, $M(t)$ is nondecreasing in $\mathbb{R}^+$ and satisfies: there exists $m_0 > 0$ and $\mu > 1$ such that $M(t) \geq m_0 t^{\mu-1}$ for all $t \geq 0$.

By variational methods and topological degree theory, the authors proved multiplicity results depending on the parameter λ and the integrability properties of the ratio $\frac{\omega(x)}{h(x)}$. Furthermore, when $1 < q < p$, the existence of infinitely many solutions u_n, which the functional sequence $J(u_n) \to 0$ as $n \to \infty$, is also obtained by genus theory. In [24], the authors considered

$$\begin{aligned} &M([u]_{s,p}^p)(-\Delta)_p^s u \\ &= \lambda\omega_1(x)|u|^{q-2}u + \omega_2(x)|u|^{r-2}u + h(x), \quad x \in \mathbb{R}^N \end{aligned} \tag{1.10}$$

with $M(t) = a + bt^{\theta-1}$, $\theta > 1$, $a, b \geq 0$, $a + b > 0$, $\lambda > 0$ and $1 < q < p < \theta p < r < p_s^*$. The functions $\omega_1(x), \omega_2(x)$ and $h(x)$ may change sign on $\mathbb{R}^N$. Under some suitable conditions, they obtained the existence of two nontrivial entire solutions by applying the mountain pass theorem and Ekeland's variational principle.

For the sublinear case $1 < q < p$, it is worth noticing that ones usually proved the existence of small solutions u_n, i.e., $J(u_n) \to 0$ as $n \to \infty$, see [10, 15, 23].

In the present paper, motivated by the above results, we will prove that, if $1 < m \leq k < p_s^*$, system (1.1) admits infinitely many high-energy weak solutions (u_n, v_n) such that $J(u_n, v_n) \to \infty$ as $n \to \infty$, which is an extension of the work in [10, 15, 23]. For this purpose, we apply a version of symmetric mountain pass lemma in [22]. Also, we adapt some ideas developed by Pucci et al. [12, 21], Xiang et al. [25, 27]. It is noted that although the idea was used before for other problems, the adaptation to the procedure to our problem is not trivial at all, since the parameters k, m satisfies $1 < p(1+\alpha) < m \leq k < p_s^*$ and $1 < m < p(1 + \alpha)$, we must consider our problem for suitable space and so we need more delicate estimates and new technique. Let us first state the assumptions on the Kirchhoff functions $M_1(t)$ and $M_2(t)$.

(H_1) The functions $M_1(t)$ and $M_2(t)$ are continuous in $\mathbb{R}$, and there exists $\alpha \in (0, \frac{sp}{N-sp}), \beta \in (0, \frac{sq}{N-sq})$ such that $M_1(t) \geq a_0 + a_1 t^\alpha, M_2(t) \geq b_0 + b_1 t^\beta$ and $\mu_1(t) = \int_0^t M_1(\tau)d\tau \geq (1+\alpha)^{-1} M_1(t)t,\ \mu_2(t) = \int_0^t M_2(\tau)d\tau \geq (1+\beta)^{-1} M_2(t)t,\ t \in \mathbb{R}^+$, where $a_0, b_0, a_1, b_1 > 0$ are some constants.

A typical example of M_1 is given by $M_1(t) = a_0 + a_1 t^\alpha$ with $\alpha > 0, a_0 > 0, a_1 \geq 0$ for all $t \geq 0$. Under assumption (H_1), we can also deal with cases in which M_1 is not monotone as $M_1(t) = (1+t)^\alpha + (1+t)^{-1}$ for $t \geq 0$ with $0 < \alpha < 1$. Then the condition (H_1) holds provided that $\alpha \in (0,1)$ is so small that $\alpha \in (0, \frac{sp}{N-sp})$ and then $p(1+\alpha) < p_s^*$.

Throughout this paper, we make the following assumptions:

(H_2) Let $\lambda, \mu > 0$ in (1.1). The parameters p, q, m, α, β satisfies $1 < p \leq q < p_0 < m \leq k < p_s^* = pN/(N-sp), sq < N,$, where α, β are given in (H_1) and $p_0 = \max\{p(1+\alpha), q(1+\beta)\}$.

(H_3) The function $a(x) > 0$ in Ω and $a(x) \in L^{\mu_0}_{loc}(\Omega)$ with $\mu_0 = \frac{N}{qs}$. Moreover, $\limsup_{|x|\to\infty} a(x) < \infty$.

(H_4) The potential functions $V_1(x), V_2(x) \in C(\Omega)$ satisfy $V_1(x), V_2(x) \geq V_0 > 0$ in $\mathbb{R}^N$ with some constant V_0. In addition, for every $d > 0$, $meas(\{x \in \Omega : \frac{V_i(x)}{a(x)} \leq d\}) < \infty$ with $i = 1, 2$.

(H_5) The functions $F, F_u, F_v : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, $F_u(x,u,v)$ is even in v and odd in u and $F_v(x,u,v)$ is even in u and odd in v. In addition, there exists $k \in (q, p_s^*)$ such that $F(x, tu, tv) = t^k F(x,u,v)$ for any $t > 0$ and $(x,u,v) \in \Omega \times \mathbb{R}^2$. Moreover,

$$
\begin{aligned}
&0 \leq |F_u(x,u,v)| + |F_v(x,u,v)| \leq c_1 a(x)(|u|^{k-1} + |v|^{k-1}), \\
&\quad \forall (x,u,v) \in \Omega \times \mathbb{R}^2, \\
&0 \leq F(x,u,v) \leq c_1 a(x)(|u|^k + |v|^k),\ \forall (x,u,v) \in \Omega \times \mathbb{R}^2, \\
&\quad \lim_{|(u,v)|\to+\infty} \Big(|(u,v)|^{-p_0} F(x,u,v)\Big) = +\infty,\ \text{uniformly in } \in x \in \Omega
\end{aligned}
\tag{1.11}
$$

with some constant $c_1 > 0$, $p_0 = \max\{p(1+\alpha), q(1+\beta)\}$ and $|(u,v)| = \sqrt{u^2+v^2}, (u,v) \in \mathbb{R}^2$.

(H_6) $F(x,u,v) = a(x)|u|^\sigma |v|^\tau, \sigma, \tau > 1, \sigma + \tau = m = k > p_0$ and $\ell > -\ell_0 \equiv -m(\frac{\lambda}{\sigma})^{\frac{\sigma}{m}}(\frac{\mu}{\tau})^{\frac{\tau}{m}}$.

(H_7) Let $\lambda, \mu > 0,\ \ell > 0,\ 1 < p = q$ with $sp < N$ and $M_1(t) = M_2(t) = a_0 + a_1 t^\alpha$ with $a_0, a_1 > 0$. Moreover, assume $1 < m < p_0 = p(1+\alpha) < p_s^*, m \neq p$ and $0 < a(x) \in L^\delta(\Omega)$ with $\delta = \frac{p_s^*}{p_s^* - m}$.

Remark 1.1. (1) When $a(x) = 1$, assumption (H_4) was originally introduced by Bartsch and Wang in [3] to overcome the lack of compactness.

(2) By the assumption (H_4), we know that $F(x,u,v)$ is even in u, v and have the Euler identity

$$
F_u(x,u,v)u + F_v(x,u,v)v = kF(x,u,v), \quad \forall (x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2. \tag{1.12}
$$

Obviously, the functions $F_1(x,u,v) = a(x)|u|^\sigma|v|^\tau$ and $F_2(x,u,v) = a(x)(|u|^k + |v|^k)$ satisfy (H_4), where $\sigma, \tau > 1$, $\sigma+\tau = k \in (q, p_s^*)$. In general, the function $F(x,u,v) = \sum_{i=1}^n h_i(x)|u|^{\sigma_i}|v|^{\tau_i}$ with $\sigma_i, \tau_i > 1$, $\sigma_i + \tau_i = k_i \in (q, p_s^*)$ also verifies (H_4), where $a(x), h_i(x)$ satisfy (H_2).

(3) Condition (1.11) implies that the functions $F(x,u,v), F_u(x,u,v)$ and $F_v(x,u,v)$ do not have to be bounded in $x \in \mathbb{R}^N$ since $a(x)$ must not bounded in $\mathbb{R}^N$. For example, let $a(x) = |x|^{-\theta}$ for $0 < |x| < 1$ and $a(x) = 1$ for $|x| \geq 1$, in which $\theta \in (0, N/\mu_0)$.

In order to state our main results, we recall some fractional Sobolev spaces and norms [8, 13]. We define the fractional Sobolev space $W_0^{s,p}(\Omega)$ $(0 < s < 1 < t, st < N)$ as follows:

$$W_0^{s,t}(\Omega) = \{u \in W^{s,t}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \bar{\Omega}\}. \tag{1.13}$$

This space is endowed with the natural norm

$$\|u\|_{W_0^{s,t}} = \Big(\|u\|_{L^t(\Omega)}^t + [u]_{s,t}^t\Big)^{1/t}, \tag{1.14}$$

while $[u]_{s,p}$ denotes the Gagliardo semi-norm given in (1.2), that is,

$$[u]_{s,t} = \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^t}{|x-y|^{N+ts}} dxdy\Big)^{1/t}. \tag{1.15}$$

We define the subspaces $X \subset W_0^{s,p}(\Omega)$ and $Y \subset W_0^{s,q}(\Omega)$ with the norms as follows:

$$X = \Big\{u \in W_0^{s,p}(\Omega) : \int_\Omega V_1(x)|u|^p dx < \infty\Big\} \tag{1.16}$$

with $\|u\|_X = \Big([u]_{s,p}^p + \|u\|_{L^p(\Omega,V_1)}^p\Big)^{1/p}$, and

$$Y = \Big\{u \in W_0^{s,q}(\Omega) : \int_\Omega V_2(x)|u|^q dx < \infty\Big\} \tag{1.17}$$

with $\|u\|_Y = \Big([u]_{s,q}^q + \|u\|_{L^q(\Omega,V_2)}^q\Big)^{1/q}$, where $\|u\|_{L^t(\Omega,h)}^t = \|u\|_{t,h} = \int_\Omega h|u|^t dx$ and $h \geq 0$ in Ω.

For reader's convenience, we recall the main embedding results for $W_0^{s,p}(\Omega)$.

Lemma 1.2 ([8])**.** *Let $s \in (0,1)$ and $t \geq 1$ such that $st < N$. Then there exists a positive constant $S_0 = S_0(N,t,s)$ such that, for any measurable and compactly supported function $u : \Omega \to \mathbb{R}$, we have*

$$\|u\|_{t_s^*} \leq S_0[u]_{s,t}, \tag{1.18}$$

where $t_s^* = \frac{tN}{N-ts}$ is the fractional critical exponent. Consequently, the space $W_0^{s,p}(\Omega)$ is continuously embedded in $L^r(\Omega)$ for any $r \in [t, t_s^*]$. Moreover, the embedding $W_0^{s,t}(\Omega) \hookrightarrow L^r(\Omega)$ is locally compact whenever $1 < r < t_s^*$.

Clearly, from assumption (H_4), we have

$$\begin{aligned} &\min\{1, V_0\}\|u\|_{W_0^{s,p}} \le \|u\|_X, \forall u \in W_0^{s,p}(\Omega); \\ &\min\{1, V_0\}\|u\|_{W_0^{s,q}} \le \|u\|_Y, \forall u \in W_0^{s,q}(\Omega). \end{aligned} \tag{1.19}$$

Furthermore, it follows from (1.19) that for any $p \le r \le p_s^*$ and $q \le s \le q_s^*$, there exist $S_r, T_s > 0$ such that

$$\|u\|_r \le S_r\|u\|_X, \quad \forall u \in X \quad \text{and} \quad \|u\|_s \le T_s\|u\|_Y, \quad \forall u \in Y. \tag{1.20}$$

For convenience, let $S_0 = S_{p_s^*}, T_0 = T_{q_s^*}$.

Remark 1.3. By the density of the compactly supported functions in $W_0^{s,p}(\Omega)$, we know that (1.18) holds for any $u \in W_0^{s,p}(\Omega)$.

For the product space $E = X \times Y$, we introduce the norm

$$\|(u, v)\|_E = \|u\|_X + \|v\|_Y, \quad \forall (u, v) \in E. \tag{1.21}$$

Then E is a reflexive Banach space endowed with the norm $\|(u, v)\|_E$.

Definition 1.1. A function $(u, v) \in E$ is said to be a pair of (weak) solution of system (1.1) if

$$\begin{aligned} &M_1([u]_{s,p}^p)T_u(\varphi) + M_2([v]_{s,q}^q)S_v(\psi) \\ &+ \int_\Omega (V_1|u|^{p-2}u\varphi + V_2|v|^{q-2}v\psi)dx \\ = &\frac{\iota}{k}\int_\Omega (F_u(x, u, v)\varphi + F_v(x, u, v)\psi)ds \\ &+ \int_\Omega a(x)(\lambda|u|^{m-2}u\varphi + \mu|v|^{m-2}v\psi)dx, \ \forall (\varphi, \psi) \in E, \end{aligned} \tag{1.22}$$

where

$$\begin{aligned} T_u(\varphi) &= \int_{\Omega^2} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}(\varphi(x)-\varphi(y))dxdy, \ \forall \varphi \in X, \\ S_v(\psi) &= \int_{\Omega^2} \frac{|v(x)-v(y)|^{q-2}(v(x)-v(y))}{|x-y|^{N+qs}}(\psi(x)-\psi(y))dxdy, \ \forall \psi \in Y. \end{aligned} \tag{1.23}$$

By the assumptions (H_1)–(H_5), all the integrals in (1.22) and (1.23) are well defined and converge.

Let $J(u, v) : E \to \mathbb{R}$ be the corresponding energy functional of problem (1.1), which is defined by

$$\begin{aligned} J(u, v) = &\frac{1}{p}\Big(\mu_1([u]_{s,p}^p) + \|u\|_{p,V_1}^p\Big) + \frac{1}{q}\Big(\mu_2([v]_{s,q}^q) + \|v\|_{q,V_2}^q\Big) \\ &- \frac{\ell}{k}\int_\Omega F(x, u, v)dx - R_{\lambda,\mu}(u, v) \end{aligned} \tag{1.24}$$

with

$$R_{\lambda,\mu}(u,v) = \frac{1}{m}\int_\Omega a(x)(\lambda|u|^m + \mu|v|^m)dx \\ = \frac{1}{m}(\lambda\|u\|_{m,a}^m + \mu\|v\|_{m,a}^m), \quad \forall (u,v) \in E. \tag{1.25}$$

Then, by (H_1)–(H_5), we see that $J \in C^1(E,\mathbb{R})$ and for $\forall(\phi,\psi) \in E$, there holds

$$\begin{aligned} \langle J'(u,v),(\phi,\psi)\rangle = {} & M_1([u]_{s,p}^p)T_u(\varphi) + M_2([v]_{s,q}^q)S_v(\psi) \\ & + \int_\Omega (V_1|u|^{p-2}u\varphi + V_2|v|^{q-2}v\psi)dx \\ & - \frac{\iota}{k}\int_\Omega (F_u(x,u,v)\varphi + F_v(x,u,v)\psi)dx \\ & - \int_\Omega a(x)(\lambda|u|^{m-2}u\varphi + \mu|v|^{m-2}v\psi)dx. \end{aligned} \tag{1.26}$$

It is well known that the weak solution of problem (1.1) is the critical point of the energy functional $J(u,v)$ in E. Thus, to prove the existence of weak solutions for problem (1.1), it is sufficient to show that $J(u,v)$ admits a sequence of critical points. Our main conclusions in this paper are as follows.

Theorem 1.4. *Let (H_1)–(H_4) hold. If either (H_5) with $\ell > 0$ or (H_6) is satisfied, then system* (1.1) *admits infinitely many high-energy weak solutions $(u_n,v_n) \in E$ such that $J(u_n,v_n) \to \infty$ as $n \to \infty$.*

Theorem 1.5. *Let (H_1), (H_3)–(H_5) and (H_7) hold. Assume $\ell > 0$ in (H_5). then system* (1.1) *admits infinitely many high-energy weak solutions $(u_n,v_n) \in E$ such that $J(u_n,v_n) \to \infty$ as $n \to \infty$.*

The paper is organized as follows. In Section 2, we will establish a series of lemmas and prove that the functional associated to system (1.1) satisfies the $(PS)_c$ condition. Section 3 is devoted to the proof of Theorems 1.4 and 1.5. The main ingredients of the paper is that we establish the compactness of the embeddings $X \hookrightarrow L^r(\Omega,a)$ and $Y \hookrightarrow L^r(\Omega,a)$ under the assumptions (H_3) and (H_4) and we prove some important limits that will be used for the proof of the main results. We believe that these limits can be applied to other quasilinear elliptic problems in Ω.

2. Preliminaries

In this Section we will establish a series lemmas to prove our main theorem.

Lemma 2.1. *Let (H_2) and (H_3) be satisfied and $p \le r < p_s^*$. Assume that $\{u_n\}$ is a bounded sequence in X. Then there exists $u \in X \cap L^r(\Omega,a)$ such that up to a subsequence, $u_n \to u$ strongly in $L^r(\Omega,a)$.*

Proof. Since $\{u_n\}$ is bounded in X, there exist a subsequence, still denoted by $\{u_n\}$, and $u \in X$ such that

$$(2.1)\qquad \begin{aligned} &u_n \rightharpoonup u \text{ weakly in } X,\ u_n \to u \text{ strongly in } L^r_{loc}(\Omega), \forall r \in [1, p^*),\\ &u_n(x) \to u(x) \text{ a.e. in } \Omega \end{aligned}$$

Without loss of generality, let $\Omega \cap B_1 \neq \emptyset$ and, for all $n \geq 1$, $\|u_n\|_X + \|u\|_X \leq d_1$ with some $d_1 > 0$. Then we claim that for every small $\varepsilon > 0$, there exists $R > 1$, such that

$$(2.2)\qquad \int_{\Omega^c_R} a(x)|u_n|^p dx < \varepsilon \quad \forall n \geq 1,$$

where $\Omega^c_R = \Omega \cap B^c_R$. In fact, we fix the small $\varepsilon > 0$ and choose constants $t \in (1, \frac{N}{N-p})$ and c_0, d, where

$$(2.3)\qquad c_0 = \sup_{u \in X\setminus\{0\}} \frac{\|u_n\|^p_{pt}}{\|u_n\|^p_X},\quad \frac{2}{\varepsilon^m} \sup_{n\geq 1} \|u_n\|^p_X \leq \frac{2}{\varepsilon^m} d_1^p < d,$$

in which $m > 1$ will be given below. By the embedding inequality (1.18), one sees that $0 < c_0 < \infty$.

On the other hand, let $S_j = \{x \in \Omega \cap B_j : \frac{V_1(x)}{a(x)} \leq d\}$, where and in the sequel, $B_j = \{x \in \mathbb{R}^N : |x| < j\}$, $B^c_j = \{x \in \mathbb{R}^N : |x| \geq j\}$. Then $S_j \subset S_{j+1}$ for $j = 1, 2, \ldots$, and $S_\infty = \{x \in \Omega : \frac{V_1(x)}{a(x)} \leq d\} = \cup_{j=1}^\infty S_j$. By the property of Lebesgue measure, we have $meas(S_\infty) = \lim_{j\to\infty} meas(S_j)$.

By (H_3), there exist $j_0 \geq 1$ and $A_0 > 0$ such that $0 \leq a(x) \leq A_0$ for $x \in \Omega \cap B^c_{j_0}$. Now, (H_4) implies that $meas(S_\infty)$ is bounded and for every $\varepsilon > 0$ there exists $j_1 \geq j_0$ such that for $j \geq j_1$

$$(2.4)\qquad 0 \leq meas(S_\infty) - meas(S_j) = meas(\Omega^c_j) \leq \varepsilon^{mt'}(2c_0 A_0 d_1^p)^{-t'},$$

where $t' = \frac{t}{t-1} > 1$ and

$$(2.5)\qquad \begin{aligned} \Omega_j &= \Big\{x \in \Omega \cap B^c_j : \frac{V_1(x)}{a(x)} > d\Big\},\\ \Omega^c_j &= B^c_j \setminus \Omega_j = \Big\{x \in \Omega \cap B^c_j : \frac{V_1(x)}{a(x)} \leq d\Big\}. \end{aligned}$$

Then, by the choice of d, one sees from (2.3) that

$$(2.6)\qquad \begin{aligned} \int_{\Omega_j} a(x)|u_n|^p dx &\leq \frac{1}{d}\int_{\Omega_j} V_1(x)|u_n|^p dx\\ &\leq \frac{1}{d}\int_\Omega V_1|u_n|^p dx \leq \frac{1}{d}\|u_n\|^p_X \leq \frac{\varepsilon^m}{2}. \end{aligned}$$

Moreover, the Hölder inequality and (2.3) imply

$$(2.7)\qquad \int_{\Omega^c_j} a(x)|u_n|^p dx \leq \Big(\int_{\Omega^c_j} |u_n|^{pt} dx\Big)^{1/t}\Big(\int_{\Omega^c_j} |a(x)|^{t'} dx\Big)^{1/t'}$$

$$\le c_0\|u_n\|_X^p\|a\|_{L^{t'}(\Omega_j^c)}.$$

Noticing that $x \in \Omega_j^c$, we have $0 \le a(x) \le A_0$ and so it follows from (2.4) and (2.7) that

$$\begin{aligned}\int_{\Omega_j^c} a(x)|u_n|^p dx \ &\le c_0\|u_n\|_X^p\|a\|_{L^{t'}(\Omega_j^c)} \\ &\le c_0A_0\|u_n\|_X^p[meas(\Omega_j^c)]^{1/t'} \le \frac{\varepsilon^m}{2}.\end{aligned} \tag{2.8}$$

Therefore, we obtain

$$\int_{\Omega\cap B_j^c} a(x)|u_n|^p dx = \int_{\Omega_j} a(x)|u_n|^p dx + \int_{\Omega_j^c} a(x)|u_n|^p dx \le \varepsilon^m \tag{2.9}$$

and

$$\int_{\Omega\cap B_j^c} a(x)|u|^p dx \le \liminf_{n\to\infty} \int_{\Omega\cap B_j^c} a(x)|u_n|^p dx \le \varepsilon^m. \tag{2.10}$$

Then (2.2) yields from (2.9).

On the other hand, we have from (2.1) that $u_n \to u$ strongly in $L^p(\Omega\cap B_j, a)$. Thus, we obtain $u_n \to u$ strongly in $L^p(\Omega, a)$ from (2.9) and (2.10).

For $p < r < p_s^*$, we choose $\tau \in (0,1)$ such that $r = \tau p + (1-\tau)p_s^*$. Then, from (2.9), we have

$$\begin{aligned}\int_{\Omega\cap B_j^c} a(x)|u_n|^r dx &\le \int_{\Omega\cap B_j^c} a(x)^\tau|u_n|^{\tau p}a(x)^{1-\tau}|u_n|^{(1-\tau)p^*}dx \\ &\le A_0^{1-\tau}\|u_n\|^{\tau p}_{L^p(\Omega\cap B_j^c,a)}\|u_n\|^{(1-\tau)p^*}_{L^{p^*}(\Omega)} \\ &\le A_0^{1-\tau}S_0^{p^*}\|u_n\|_X^{(1-\tau)p^*}\|u_n\|^{\tau p}_{L^p(\Omega\cap B_j^c,a)} \\ &\le A_0^{1-\tau}S_0^{p^*}d_1^{p^*}\|u_n\|^{\tau p}_{L^p(\Omega\cap B_j^c,a)} \\ &\le A_0^{1-\tau}S_0^{p^*}d_1^{p^*}\varepsilon \equiv C_1\varepsilon,\end{aligned} \tag{2.11}$$

where S_0 is the embedding constant in (1.20) and the fact $m = \tau^{-1} > 1$ has been used. Moreover, by Fatou's lemma, we get

$$\int_{B_j^c} a(x)|u|^r dx \le \liminf_{n\to\infty} \int_{B_j^c} a(x)|u_n|^r dx \le C_1\varepsilon. \tag{2.12}$$

For such $\Omega\cap B_j$, it follows from (2.1) that $u_n(x) \to u(x)$ a.e. in $\Omega\cap B_j$. Thus, $a(x)|u_n(x)-u(x)|^r \to 0$ a.e. in $\Omega\cap B_j$. For each bounded and measurable subset $\mathcal{D} \subset (\Omega\cap B_j)$, we know that

$$\int_{\mathcal{D}} a(x)|u_n-u|^r dx \le \|a\|_{L^{\nu_1}(\mathcal{D})}\|u_n-u\|^r_{L^{p^*}(\Omega)} \le 2^rS_0^rd_1^r\|a\|_{L^{\nu_1}(\mathcal{D})} \tag{2.13}$$

with $\nu_1 = \frac{p^*}{p^*-r}$. Since $a(x) \in L^{\mu_0}(\Omega \cap B_j)$, we obtain $\{a(x)|u_n(x) - u(x)|^r\}$ is uniformly integrable and $\{\int_{\Omega\cap B_j} a(x)|u_n(x) - u(x)|^r dx\}$ is bounded. Then the Vitali convergence theorem implies

$$\lim_{n\to\infty} \int_{\Omega\cap B_j} a(x)|u_n(x) - u(x)|^r dx = 0. \tag{2.14}$$

Moreover, an application of (2.11), (2.12) and (2.14) gives that

$$\lim_{n\to\infty} \int_{\Omega} a(x)|u_n(x) - u(x)|^r dx = 0. \tag{2.15}$$

This completes the proof of Lemma 2.1. □

Similarly, we have:

Lemma 2.2. *Let (H_2) and (H_3) be satisfied and $q \le r < q_s^*$. Assume that $\{v_n\}$ is a bounded sequence in Y. Then there exists $v \in Y \cap L^r(\Omega, a)$ such that up to a subsequence, $v_n \to v$ strongly in $L^r(\Omega, a)$.*

To obtain the existence of solutions to system (1.1), we need to prove that the functional J defined by (1.24) satisfies $(PS)_c$ condition. Let $c \in \mathbb{R}$. The sequence $\{(u_n, v_n)\}$ in E is called $(PS)_c$ sequence of J if

$$J(u_n, v_n) \to c, \quad J'(u_n, v_n) \to 0 \ \text{ in } E^* \quad \text{as } n \to \infty. \tag{2.16}$$

The functional J satisfies $(PS)_c$ condition in E if any $(PS)_c$ sequence possesses a convergent subsequence in E

Lemma 2.3. *Under the assumptions of Theorems 1.4 and 1.5, if $\{(u_n, v_n)\} \subset E$ is a $(PS)_c$ sequence, then $\{(u_n, v_n)\}$ is bounded in E.*

Proof. Let $\{(u_n, v_n)\}$ be a $(PS)_c$ sequence in E. First, we assume the conditions in Theorem 1.4 hold. We choose θ such that $p_0 = \max\{p(1+\alpha), q(1+\beta)\} < \theta < m \le k$. Then, for large n, there is a constant $c > 0$ such that

$$\begin{aligned}
&c + 1 + \|(u_n, v_n)\|_E \\
&\ge J(u_n, v_n) - \frac{1}{\theta} J'(u_n, v_n)(u_n, v_n) \\
&= \big(\frac{1}{p}\mu_1([u_n]_{s,p}^p) - \frac{1}{\theta} M_1([u_n]_{s,p}^p)\big)[u]_{s,p}^p + (\frac{1}{p} - \frac{1}{\theta}) \int_\Omega V_1|u_n|^p dx \\
&\quad + \big(\frac{1}{q}\mu_2([v_n]_{s,q}^q) - \frac{1}{\theta} M_2([v_n]_{s,q}^q)\big)[v_n]_{s,q}^q + (\frac{1}{q} - \frac{1}{\theta}) \int_\Omega V_2|v_n|^q dx \\
&\quad - \ell(\frac{1}{k} - \frac{1}{\theta}) \int_\Omega F(x, u_n, v_n) dx - (\frac{1}{m} - \frac{1}{\theta}) \int_\Omega a(x)(\lambda|u_n|^m + \mu|v_n|^m) dx \\
&\ge (\frac{1}{p(1+\alpha)} - \frac{1}{\theta}) M_1([u_n]_{s,p}^p)[u_n]_{s,p}^p + (\frac{1}{q(1+\beta)} - \frac{1}{\theta}) M_2([v_n]_{s,q}^q)[v_n]_{s,q}^q \\
&\quad + (\frac{1}{q} - \frac{1}{\theta}) \int_\Omega (V_1|u_n|^p + V_2|v_n|^q) dx.
\end{aligned} \tag{2.17}$$

Then, if $\ell > 0$, it follows from (2.17) that the sequence $\{(u_n, v_n)\}$ is bounded in E. For assumption (H_5) and $\ell \in (-\ell_0, 0)$, we get from Young's inequality with any $\varepsilon > 0$ that

$$
\begin{aligned}
F(x,u,v) &= a(x)|u|^\sigma |v|^\tau \\
&\le a(x)m^{-1}(\sigma\varepsilon|u|^m + \tau\varepsilon^{-\frac{\sigma}{\tau}}|v|^m), \quad \forall (x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2,
\end{aligned}
\tag{2.18}
$$

where $\sigma + \tau = m = k$. Let $\varepsilon = (\frac{\lambda\tau}{\mu\sigma})^{\tau/m}$. Since $\ell \in (-\ell_0, 0)$, we have

$$
\sigma_1 \equiv \frac{-\ell\sigma}{m}\varepsilon - \lambda < 0, \quad \sigma_2 \equiv \frac{-\ell\tau}{m}\varepsilon^{-\frac{\sigma}{\tau}} - \mu < 0. \tag{2.19}
$$

Thus,

$$
\begin{aligned}
&- \ell F(x,u,v) - a(x)(\lambda|u|^m + \mu|v|^m) \\
&\le a(x)(\sigma_1|u|^m + \sigma_2|v|^m) \le 0, \quad \forall (x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2.
\end{aligned}
\tag{2.20}
$$

Then, it follows from (2.17) that

$$
\begin{aligned}
&c + 1 + \|(u_n, v_n)\|_E \\
\ge\ & J(u_n, v_n) - \frac{1}{\theta}J'(u_n, v_n)(u_n, v_n) \\
=\ & \big(\frac{1}{p}\mu_1([u_n]^p_{s,p}) - \frac{1}{\theta}M_1([u_n]^p_{s,p})\big)[u_n]^p_{s,p} + (\frac{1}{p} - \frac{1}{\theta})\int_\Omega V_1|u_n|^p dx \\
& + \big(\frac{1}{q}\mu_2([v_n]^q_{s,q}) - \frac{1}{\theta}M_2([v_n]^q_{s,q})\big)[v_n]^q_{s,q} + (\frac{1}{q} - \frac{1}{\theta})\int_\Omega V_2|v_n|^q dx \\
& + (\frac{1}{m} - \frac{1}{\theta})\int_\Omega [\ell F(x,u,v) + a(x)(\lambda|u_n|^m + \mu|v_n|^m)]dx \\
\ge\ & (\frac{1}{p(1+\alpha)} - \frac{1}{\theta})M_1([u_n]^p_{s,p})[u_n]^p_{s,p} \\
& + (\frac{1}{q(1+\beta)} - \frac{1}{\theta})M_2([v_n]^q_{s,q})[v_n]^q_{s,q} \\
& + (\frac{1}{q} - \frac{1}{\theta})\int_\Omega (V_1|u_n|^p + V_2|v_n|^q)dx,
\end{aligned}
\tag{2.21}
$$

which implies that the sequences $\{(u_n, v_n)\}$ is bounded in E.

Next, we consider the conditions in Theorem 1.5. Since $1 < m < p_0 = p(1+\alpha)$, it follows from Young's inequality with $\epsilon > 0$ and assumption (H_7) that

$$
\begin{aligned}
&\int_\Omega a(x)|u_n|^m \le \|u_n\|^m_{p_s^*}\|a\|_\delta \le S_0^m[u_n]^m_{s,p}\|a\|_\delta \le \epsilon[u_n]^{p_0}_{s,p} + C_\epsilon\|a\|^\tau_\delta, \\
&\int_\Omega a(x)|v_n|^m \le \|v_n\|^m_{p_s^*}\|a\|_\delta \le S_0^m[u_n]^m_{s,p}\|a\|_\delta \le \epsilon[v_n]^{p_0}_{s,p} + C_\epsilon\|a\|^\tau_\delta,
\end{aligned}
\tag{2.22}
$$

with $\tau=\frac{p_0}{p_0-1}$. We take $\theta\in(p_0,k)$ and then (2.17) becomes

$$\begin{aligned}
&c+1+\|(u_n,v_n)\|_E\\
&\geq J(u_n,v_n)-\frac{1}{\theta}J'(u_n,v_n)(u_n,v_n)\\
&=(\frac{1}{p}-\frac{1}{\theta})a_0\big([u_n]_{s,p}^p+[v_n]_{s,p}^p\big)\\
&\quad+(\frac{1}{p_0}-\frac{1}{\theta})a_1\big([u_n]_{s,p}^{p_0}+[v_n]_{s,p}^{p_0}\big)+(\frac{1}{p}-\frac{1}{\theta})\int_\Omega(V_1|u_n|^p+V_2|v_n|^p)dx \qquad (2.23)\\
&\quad-\ell(\frac{1}{k}-\frac{1}{\theta})\int_\Omega F(x,u_n,v_n)dx-(\frac{1}{m}-\frac{1}{\theta})\int_\Omega a(x)(\lambda|u_n|^m+\mu|v_n|^m)dx\\
&\geq(\frac{1}{p_0}-\frac{1}{\theta})a_1\big([u_n]_{s,p}^{p_0}+[v_n]_{s,p}^{p_0}\big)-(\frac{1}{m}-\frac{1}{\theta})\epsilon([u_n]^{p_0}+[v_n]_{s,p}^{p_0})-C_\epsilon\|a\|_\delta^\tau\\
&\quad+(\frac{1}{p}-\frac{1}{\theta})\int_\Omega(V_1|u_n|^p+V_2|v_n|^p)dx.
\end{aligned}$$

This implies that the sequence $\{(u_n,v_n)\}$ is bounded in E if we choose small $\epsilon>0$. Then we complete the proof of Lemma 2.3. □

Since the sequence $\{(u_n,v_n)\}$ given by Lemma 2.3 is bounded in E, there exist a constant $d_2>0$ and $(u,v)\in E$, and a subsequence of $\{(u_n,v_n)\}$, still denoted by $\{(u_n,v_n)\}$, such that $\|(u_n,v_n)\|_E+\|(u,v)\|_E\leq d_2$ for any $n\geq 1$ and

$$\begin{aligned}
&(u_n,v_n)\rightharpoonup(u,v)\text{ weakly in }E,\\
&(u_n(x),v_n(x))\to(u(x),v(x))\ \text{ a.e. in }\Omega, \qquad (2.24)\\
&(u_n,v_n)\to(u,v)\text{ strongly in }L^r_{loc}(\Omega)\times L^t_{loc}(\Omega),\forall r\in[1,p_s^*),\forall t\in[1,q_s^*).
\end{aligned}$$

Lemma 2.4. *Let the assumptions in Theorems 1.4 and 1.5 be satisfied. If the sequence* $\{(u_n,v_n)\}$ *satisfies* (2.24)*, then*

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega(F_u(x,u_n,v_n)-F_u(x,u,v))(u_n-u)dx=0,\\
&\lim_{n\to\infty}\int_\Omega(F_v(x,u_n),v_n)-F_v(x,u,v)(v_n-v)dx=0, \qquad (2.25)\\
&\lim_{n\to\infty}\int_\Omega F(x,u_n,v_n)dx=\int_\Omega F(x,u,v)dx.
\end{aligned}$$

Proof. Since the sequence $\{(u_n,v_n)\}$ satisfies (2.24), we obtain from Lemma 2.1 that $\|u_n-u\|_{L^k(\Omega,a)}\to 0$ as $n\to\infty$. By assumption (1.11), it follows

$$|F_u(x,u_n,v_n)|\leq c_1a(x)(|u_n)|^{k-1}+|v_n|^{k-1}). \qquad (2.26)$$

Furthermore, by Hölder inequality, we have

$$\begin{aligned}
&\int_\Omega |F_u(x,u_n,v_n)-F_u(x,u,v)||u_n-u|dx \\
&\leq c_1\int_\Omega a(x)(|u_n|^{k-1}+|v_n|^{k-1}+|u|^{k-1}+|v|^{k-1})|u_n-u|dx \\
&\leq c_1\|u_n-u\|_{L^k(\Omega,a)}\Big(\|u_n\|^{k-1}_{L^k(\Omega,a)}+\|u\|^{k-1}_{L^k(\Omega,a)}+\|v_n\|^{k-1}_{L^k(\Omega,a)}+\|v\|^{k-1}_{L^k(\Omega,a)}\Big)
\end{aligned} \tag{2.27}$$

and we derive the first limit in (2.25). The other two limits in (2.25) can be proved similarly. This completes the proof of Lemma 2.4. □

Lemma 2.5. *Let the assumptions in Theorems 1.4 and 1.5 be satisfied. Assume that $\{(u_n,v_n)\}\}$ is a bounded $(PS)_c$ sequence. Then J satisfies $(PS)_c$ condition.*

Proof. By Lemma 2.3, we can assume that the sequence $\{(u_n,v_n)\}$ satisfies (2.24). Then, by Lemma 2.1, the sequence $\{(u_n,v_n)\}$ is bounded in E and there exists a subsequence, still denoted by $\{(u_n,v_n)\}$ such that (2.24) holds.

We now prove $(u_n,v_n)\to(u,v)$ in E. Let $\varphi\in X$ and $\psi\in Y$ be fixed and denote by T_φ and S_ψ the linear functionals on X and Y defined by (1.23) respectively. Then it follows from the Hölder inequality that

$$|T_\varphi(u)|\leq\|\varphi\|_X^{p-1}\|u\|_X,\ \ \forall u\in X;\quad |S_\psi(v)|\leq\|\psi\|_Y^{q-1}\|v\|_Y,\ \ \forall v\in Y. \tag{2.28}$$

On the other hand, it follows from (1.23) and (2.24) that

$$\lim_{n\to\infty}T_u(u_n-u)=0,\ \lim_{n\to\infty}S_v(v_n-v)=0,\ \lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\beta_n=0, \tag{2.29}$$

where

$$\begin{aligned}
\alpha_n&=(M_1([u_n]^p_{s,p})-M_1([u]^p_{s,p}))T_u(u_n-u),\\
\beta_n&=M_2([v_n]^q_{s,q})-M_2([v]^q_{s,q}))S_v(v_n-v).
\end{aligned} \tag{2.30}$$

Furthermore, as $n\to\infty$, we have

$$\begin{aligned}
o_n(1)=(J'(u_n,v_n)-J'(u,v))(u_n-u,v_n-v)&=P_n+\alpha_n+Q_n+\beta_n\\
&-\frac{\ell}{k}\int_\Omega[(F_u(x,u_n,v_n)-F_u(x,u,v))(u_n-u)\\
&\qquad+(F_v(x,u_n,v_n)-F_v(x,u,v))(v_n-v)]dx\\
&+\int_\Omega V_1(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)dx\\
&+\int_\Omega V_2(x)(|v_n|^{q-2}v_n-|v|^{q-2}v)(v_n-v)dx\\
&-\lambda\int_\Omega a(x)(|u_n|^{m-2}u_n-|u|^{m-2}u)(u_n-u)dx\\
&-\mu\int_\Omega a(x)(|v_n|^{m-2}v_n-|v|^{m-2}v)(v_n-v)dx,
\end{aligned} \tag{2.31}$$

where

$$P_n = M_1([u_n]_{s,p}^p)(T_{u_n}(u_n-u) - T_u(u_n-u)),$$
$$Q_n = M_2([v_n]_{s,q}^q)(S_{v_n}(v_n-v) - S_v(v_n-v)). \tag{2.32}$$

Furthermore, By Lemmas 2.1 and 2.4, we have from (2.31) that

$$\begin{aligned} o_n(1) = P_n + Q_n &+ \int_\Omega V_1(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)dx \\ &+ \int_\Omega V_2(|v_n|^{q-2}v_n - |v|^{q-2}v)(v_n-v)dx. \end{aligned} \tag{2.33}$$

Let us now recall the well-known vector inequalities:

$$\begin{aligned} &|\xi-\eta|^t \le c_p(|\xi|^{t-2}\xi - |\eta|^{t-2}\eta)(\xi-\eta) \quad \text{for } t\ge 2 \\ &|\xi-\eta|^t \le C_p\Big[(|\xi|^{t-2}\xi - |\eta|^{t-2}\eta)(\xi-\eta)\Big]^{t/2}(|\xi|^t+|\eta|^t)^{(2-t)/2} \\ &\qquad\qquad \text{for } 1<t<2, \end{aligned} \tag{2.34}$$

for all $\xi,\eta\in\mathbb{R}^N$, where c_p and C_p are positive constants depending only on p.

Noticing the facts $P_n, Q_n \ge 0$ and

$$\begin{aligned} &V_1(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u) \ge 0, \\ &V_2(x)(|v_n|^{q-2}v_n - |v|^{q-2}v)(v_n-v) \ge 0, \quad \forall x\in\Omega, \end{aligned} \tag{2.35}$$

we have from (2.33) that

$$\begin{aligned} \lim_{n\to\infty} P_n = \lim_{n\to\infty} Q_n & \\ &= \lim_{n\to\infty}\int_\Omega V_1(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)dx \\ &= \lim_{n\to\infty}\int_\Omega V_2(|v_n|^{q-2}v_n - |v|^{q-2}v)(v_n-v)dx = 0. \end{aligned} \tag{2.36}$$

Moreover, the assumption $M_1(t), M_2(t) \ge m_0 = \min\{a_0,b_0\} > 0$ in $\mathbb{R}^+$ implies that

$$\lim_{n\to\infty}\big(T_{u_n}(u_n-u) - T_u(u_n-u)\big) = \lim_{n\to\infty}\big(S_{v_n}(v_n-v) - S_v(v_n-v)\big) = 0. \tag{2.37}$$

We now prove $u_n\to u$ in X as $n\to\infty$. Assume first that $p\ge 2$. From (2.34), one sees

$$\begin{aligned} &[u_n-u]_{s,p}^p \\ &= \iint_{\mathbb{R}^{2N}} \big|u_n(x)-u_n(y)-u(x)+u(y)\big|^p|x-y|^{-(N+sp)}dxdy \\ &\le c_p \iint_{\mathbb{R}^{2N}} \Big[|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y)) - |u(x)-u(y)|^{p-2} \\ &\quad \times (u(x)-u(y))\Big](u_n(x)-u_n(y)-u(x)+u(y))|x-y|^{-(N+sp)}dxdy \end{aligned} \tag{2.38}$$

$$= c_p[T_{u_n}(u_n - u) - T_u(u_n - u)] \to 0 \quad \text{as } n \to \infty,$$

and

$$\begin{aligned} &\|u_n - u\|^p_{L^p(\Omega,V_1)} \\ &\le c_p \int_\Omega V_1(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.39}$$

Therefore, for $p \ge 2$ we have $u_n \to u$ in X as $n \to \infty$.

In the follow, we consider the case $1 < p < 2$. From (2.24), there exists $d_2 > 0$ such that $\|u_n\|_X \le d_2$ for all $n \ge 1$. Now, from (2.34), it follows

$$\begin{aligned} &[u_n - u]^p_{s,p} \\ &\le C_p[T_{u_n}(u_n - u) - T_u(u_n - u)]^{p/2}([u_n]^p_{s,p} + [u]^p_{s,p})^{(2-p)/2} \\ &\le C_p[T_{u_n}(u_n - u) - T_u(u_n - u)]^{p/2}([u_n]^{p(2-p)/2}_{s,p} + [u]^{p(2-p)/2}_{s,p}) \\ &\le 2C_p d_2^{p(2-p)/2}[T_{u_n}(u_n - u) - T_u(u_n - u)]^{p/2}, \end{aligned} \tag{2.40}$$

and

$$\begin{aligned} &\|u_n - u\|^p_{L^p(\Omega,V_1)} \\ &\le 2C_p d_2^{p(2-p)/2}\Big(\int_\Omega V_1(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx\Big)^{p/2}, \end{aligned} \tag{2.41}$$

where we have used the following inequality

$$(a+b)^{(2-p)/2} \le a^{(2-p)/2} + b^{(2-p)/2} \quad \text{for all } \ a, b \ge 0 \ \text{ and } \ 1 < p < 2. \tag{2.42}$$

Then, from (2.36)–(2.37) and (2.40)–(2.41), it derives that $u_n \to u$ in X as $n \to \infty$. Arguing as the above, we have $v_n \to v$ in Y as $n \to \infty$. Thus, J satisfies the $(PS)_c$ condition and the proof of Lemma 2.5 is finished. □

3. Proof of Theorems 1.4 and 1.5

In this Section, we prove the existence of infinitely many solutions of (1.1). For this paper, we will make use of the following lemma.

Lemma 3.1 ([22]). *Let E be an infinite dimensional real Banach space, the functional $J \in C^1(E, \mathbb{R})$ be even and satisfy the $(PS)_c$ condition for all $c > 0$ and $J(0) = 0$. In addition, assume $E = Y \oplus Z$, in which Y is finite dimensional, and J satisfies*

(J_1) there exist constants $\rho, \alpha_0 > 0$ such that $J(z) \ge \alpha_0$ on $\partial B_\rho \cap Z$;

(J_2) for each finite dimensional subspace $E_0 \subset E$, there is an $R = R(E_0)$ such that $J(z) \le 0$ on $E_0 \setminus \overline{B}_R$, where $B_R = \{z \in E : \|z\|_E < R\}, \partial B_R = \{z \in E : \|z\|_E = R\}$.

Then, J possesses an unbounded sequence of critical values, i.e., there exists a sequence $\{z_n\} \subset E$ such that $J'(z_n) = 0$ and $J(z_n) \to \infty$ as $n \to \infty$.

Proof of Theorem 1.4. We verify the conditions in Lemma 3.1 under the assumptions of Theorem 1.4. By the definition of J, it is easy to see that J is an even functional in E and $J(0,0)=0$. Furthermore, by Lemma 2.5, $J(u,v)$ satisfies $(PS)_c$ condition. Next, we prove that J satisfies (J_1) and (J_2) in Lemma 3.1.

By (1.11), for any small $\varepsilon>0$, there exists $C_2>0$ such that

$$\begin{aligned} &0\le F(x,u,v)\\ &\quad\le \varepsilon a(x)(|u|^p+|v|^q)+C_2a(x)(|u|^{p_s^*}+|v|^{q_s^*}),\ \ \forall (x,u,v)\in\Omega\times\mathbb{R}^2. \end{aligned}\tag{3.1}$$

On the other hand, it follows from (H_3) that there exist $j_0>1$ and $A_0>0$ such that $0\le a(x)\le A_0$ for $x\in\Omega\cap B_{j_0}^c$. Moreover, the assumption $a(x)\in L^{\mu_0}_{loc}(\Omega)$ implies $a(x)\in L^{\nu}_{loc}(\Omega)$ with $\mu_0=\frac{N}{qs}$ and $\nu=\frac{N}{ps}$. Then, we have

$$\begin{aligned} \int_\Omega a(x)|u|^p dx &= \int_{\Omega\cap B_{j_0}} a(x)|u|^p dx+\int_{\Omega\cap B_{j_0}^c} a(x)|u|^p dx\\ &\le S_0^p\|a\|_{L^\nu(\Omega\cap B_{j_0})}\|u\|_X^p+A_0S_p^p\|u\|_X^p\\ &\le \beta_1\|u\|_X^p \end{aligned}\tag{3.2}$$

with $\beta_1=A_0S_p^p+S_0^p\|a\|_{L^\nu(\Omega\cap B_{j_0})}$, and $S_0=S_{p_s^*}$ is the embedding constant in (1.18). Similarly, we have

$$\begin{aligned} &\int_\Omega a(x)|v|^q dx\le \beta_2\|v\|_Y^q,\\ &\int_\Omega a(x)|u|^{p_s^*} dx\le \beta_2\|u\|_X^{p_s^*},\\ &\int_\Omega a(x)|v|^{q_s^*} dx\le \beta_2\|v\|_Y^{q_s^*} \end{aligned}\tag{3.3}$$

with some constant $\beta_2>0$. An application of (3.1)–(3.3) shows that there exists a constant $\beta_3>0$, which is independent of ε, such that

$$\int_\Omega F(x,u,v)dx\le \varepsilon\beta_3(\|u\|_X^p+\|v\|_Y^q)+\beta_3(\|u\|_X^{p_s^*}+\|v\|_Y^{q_s^*}).\tag{3.4}$$

Arguing as the above, we have some $\beta_4>0$, which is independent of ε, such that

$$\int_\Omega a(x)(\lambda|u|^m+\mu|v|^m)dx\le \varepsilon\beta_4(\|u\|_X^p+\|v\|_Y^q)+\beta_4(\|u\|_X^{p_s^*}+\|v\|_Y^{q_s^*}).\tag{3.5}$$

with small $\varepsilon>0$. Furthermore, let $\|(u,v)\|_E\le 1$. Note that for $t>1$,

$$a^t+b^t\ge 2^{1-t}(a+b)^t,\quad \forall a,b\ge 0.\tag{3.6}$$

Using the assumption (H_1) and (3.6), we have

$$\begin{aligned}\mu_1([u]_{s,p}^p)+\|u\|_{p,V_1}^p &\geq a_1(1+\alpha)^{-1}[u]_{s,p}^{p(1+\alpha)}+\|u\|_{p,V_1}^p\\ &\geq a_1'(1+\alpha)^{-1}([u]_{s,p}^{p(1+\alpha)}+\|u\|_{p,V_1}^{p(1+\alpha)})\\ &\geq a_1'(1+\alpha)^{-1}2^{1-p(1+\alpha)}\|u\|_X^{p(1+\alpha)}\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\mu_2([v]_{s,q}^q)+\|v\|_{q,V_2}^p &\geq b_1(1+\alpha)^{-1}[v]_{s,q}^{q(1+\beta)}+\|v\|_{q,V_2}^q\\ &\geq b_1'(1+\beta)^{-1}([v]_{s,q}^{q(1+\beta)}+\|v\|_{q,V_2}^{q(1+\beta)})\\ &\geq b_1'(1+\beta)^{-1}2^{1-q(1+\beta)}\|v\|_Y^{q(1+\beta)},\end{aligned} \tag{3.8}$$

where $a_1'=\min\{a_1,1\}, b_1'=\{b_1,1\}$. Then, it follows from (3.4)–(3.5) and (3.7)–(3.8) that

$$\begin{aligned}J(u,v)&=\frac{1}{p}(\mu_1([u]_{s,p}^p)+\|u\|_{p,V_1}^p)+\frac{1}{q}(\mu_2([v]_{s,q}^q)+\|v\|_{q,V_2}^q)\\ &\quad-\frac{\ell}{k}\int_\Omega F(x,u,v)dx-\frac{1}{m}(\lambda\|u\|_{m,a}^m+\mu\|v\|_{m,a}^m)\\ &\geq \alpha_0(\|u\|_X+\|v\|_Y)^{p_0}-\varepsilon\beta_0(\|u\|_X+\|v\|_Y)^p\\ &\quad-\beta_0(\|u\|_X+\|v||_Y)^{p_s^*}\\ &\equiv \alpha_0\rho^{p_0}-\varepsilon\beta_0\rho^p-\beta_0\rho^{p_s^*},\end{aligned} \tag{3.9}$$

where $\rho=\|(u,v)\|_E$ and

$$\alpha_0=\frac{2}{2^{2p_0}p_0}\min\{a_1',b_1'\},\ \beta_0=\beta_4+\frac{\ell}{k}\beta_3,\ \ p_0=\max\{p(1+\alpha),q(1+\beta)\}. \tag{3.10}$$

We now choose $0<\rho\leq\min\{1,(\frac{\alpha_0}{2\beta_0})^{1/(p_s^*-p_0)}\}$ and $0<\varepsilon\leq\frac{\alpha_0}{4\beta_0}\rho^{p_0-p}$ and obtain

$$J(u,v)\geq\alpha_0\rho^{p_0}-\varepsilon\beta_0\rho^p-\beta_0\rho^{p_s^*}\geq\frac{\alpha_0}{2}\rho^{p_0}-\varepsilon\beta_0\rho^p\geq\frac{\alpha_0}{4}\rho^{p_0}>0. \tag{3.11}$$

Thus, condition (J_1) is satisfied.

We now verify (J_2). For any finite dimensional subspace $E_0\subset E$, we assert that there exists a constant $R_0>\rho$ such that $J<0$ on $E_0\setminus\bar{B}_{R_0}$. Otherwise, there is a sequence $(u_n,v_n)\subset E_0$ such that $\|(u_n,v_n)\|_E=\|u_n\|_X+\|v_n\|_Y\to\infty$ as $n\to\infty$ and $J(u_n,v_n)\geq 0$ for all $n\geq 1$. Without loss of generality, we assume $\|u_n\|_X\to\infty$ and $\|v_n\|_Y\to\infty$ as $n\to\infty$. Clearly, the fact $J(u_n,v_n)\geq 0$ implies that

$$\begin{aligned}&\frac{\ell}{k}\int_\Omega F(x,u_n,v_n)dx+\frac{1}{m}\int_\Omega a(x)(\lambda|u_n|^m+\mu|v_n|^m)dx\\ &\leq\frac{1}{p}\Big(\mu_1([u_n]_{s,p}^p)+\|u_n\|_{L^p(\Omega,V_1)}^p\Big)+\frac{1}{q}\Big(\mu_2([v_n]_{s,q}^q)+\|v_n\|_{L^q(\Omega,V_2)}^q\Big)\\ &\leq C_3\Big(\|u_n\|_X^{p(1+\alpha)}+\|v_n\|_Y^{q(1+\beta)}\Big)\leq C_3\Big(\|u_n\|_X+\|v_n\|_Y\Big)^{p_0}\end{aligned} \tag{3.12}$$

with some constant $C_3 > 0$, which is independent of n.

Let

$$U_n(x) = \frac{u_n(x)}{\|u_n\|_X + \|v_n\|_Y} \text{ and } V_n(x) = \frac{v_n(x)}{\|u_n\|_X + \|v_n\|_Y},\ x \in \Omega.$$

Then, $\|(U_n, V_n)\|_E = 1$ for all $n \geq 1$. Using Lemmas 2.1-2.5, up to a subsequence, we can assume that $(U_n, V_n) \rightharpoonup (u_0, v_0)$ in E, $(U_n, V_n) \to (u_0, v_0)$ in $L^r(\Omega) \times L^t(\Omega)$ with $r \in [p, p_s^*), t \in [q, q_s^*)$, and $(U_n(x), V_n(x)) \to (u_0(x), v_0(x))$ a.e. in Ω. Set $\mathcal{D} = \{x \in \Omega : u_0^2(x) + v_0^2(x) > 0\}$, $\mathcal{D}^c = \Omega \setminus \bar{\mathcal{D}}$. Clearly, $|(u_n(x), v_n(x))| \to \infty$ in $\mathcal{D}$ as $n \to \infty$. If $meas(\mathcal{D}) > 0$, then, it follows from (1.11) that

$$\lim_{n\to\infty} \frac{F(x, u_n(x), v_n(x))}{|(u_n(x), v_n(x))|^{p_0}} = +\infty, \quad x \in \mathcal{D}. \tag{3.13}$$

Noticing that there exists $C_4 > 0$ such that

$$\begin{aligned} C_4(|U_n(x)|^{p_0} + |V_n(x)|^{p_0}) \leq \frac{|(u_n(x), v_n(x))|^{p_0}}{\|(u_n, v_n)\|_E^{p_0}} \\ \leq 2^{p_0-1}(|U_n(x)|^{p_0} + |V_n(x)|^{p_0}), \quad x \in \mathcal{D}, \end{aligned} \tag{3.14}$$

and

$$\lim_{n\to\infty} \int_{\mathcal{D}} (|U_n(x)|^{p_1} + |V_n(x)|^{p_0}) dx = \int_{\mathcal{D}} (|u_0(x)|^{p_1} + |v_0(x)|^{p_0}) dx > 0, \tag{3.15}$$

we have from (3.13)–(3.15) that

$$\lim_{n\to\infty} \int_{\mathcal{D}} \frac{F(x, u_n, v_n)}{\|(u_n, v_n)\|_E^{p_0}} dx = \lim_{n\to\infty} \int_{\mathcal{D}} \frac{F(x, u_n, v_n)}{|(u_n, v_n)|^{p_0}} \frac{|(u_n, v_n)|^{p_0}}{\|(u_n, v_n)\|_E^{p_0}} dx = +\infty. \tag{3.16}$$

On the other hand, from the assumptions $(H_2), (H_3)$ and (H_5), one sees

$$\liminf_{n\to\infty} \int_{\mathcal{D}^c} \frac{F(x, u_n, v_n) dx}{\|(u_n, v_n)\|_E^{p_0}} \geq 0, \ \ \liminf_{n\to\infty} \int_{\mathcal{D}^c} \frac{a(x)(\lambda |u_n|^m + \mu |v_n|^m) dx}{\|(u_n, v_n)\|_E^{p_0}} \geq 0. \tag{3.17}$$

Therefore, for $\ell > 0$, it follows from (3.16) and (3.17) that

$$\lim_{n\to\infty} \|(u_n, v_n)\|_E^{-p_0} \Big[\frac{\ell}{k} \int_\Omega F(x, u_n, v_n) dx + \frac{1}{m} \int_\Omega a(x)(\lambda |u_n|^m + \mu |v_n|^m) dx\Big] = +\infty. \tag{3.18}$$

Then the application of (3.12) and (3.16)–(3.18) yields $C_3 \geq +\infty$. This is impossible. So, $meas(\mathcal{D}) = 0$ and $(u_0(x), v_0(x)) = (0, 0)$ a.e. in Ω.

By the equivalence of all norms in E_0, one sees that there exists a constant $\gamma > 0$ such that

$$\|(u, v)\|_q \geq \gamma \|(u, v)\|_{E_0} = \gamma \|(u, v)\|_E, \quad \forall (u, v) \in E_0. \tag{3.19}$$

Hence,

$$0 = \|(u_0, v_0)\|_q = \lim_{n\to\infty} \|(U_n, V_n)\|_q \geq \gamma \lim_{n\to\infty} \|(U_n, V_n)\|_E = \gamma. \tag{3.20}$$

It is a contradiction. This shows that there exists $R_0 > \rho$ such that for any $R > R_0$, $J(u,v) < 0$ when $\|(u,v)\|_E \geq R$. Therefore, condition (J_2) is verified.

We now verify (J_1) and (J_2) under the assumptions (H_1)–(H_4) and (H_6). By Young's inequality, it derive

$$F(x,u,v) = a(x)|u|^\sigma |v|^\tau \leq a(x)(|u|^k + |v|^k) \tag{3.21}$$

with $k = m = \sigma + \tau$. Similarly, for any $\varepsilon > 0$, we have some $\beta_5 > 0$ such that

$$\begin{aligned} &\frac{1}{m}\int_\Omega |-\ell F(x,u,v) - a(x)(\lambda |u|^m + \mu |v|^m)|dx \\ &\leq \varepsilon \beta_5 (\|u\|_X^p + \|v\|_Y^q) + \beta_5 (\|u\|_X^{p_s^*} + \|v\|_Y^{q_s^*}). \end{aligned} \tag{3.22}$$

Arguing as the proof of (3.9) and (3.11), condition (J_1) is satisfied. As the proof of the above, condition (J_2) is satisfied if $\ell \geq 0$. In the following, let $\ell \in (-\ell_0, 0)$. From (2.20) and (3.12), one sees that

$$\begin{aligned} \sigma_0 \int_\Omega a(x)(|u_n|^m + |v_n|^m)dx &\leq -\frac{1}{m}\int_\Omega a(x)(\sigma_1 |u_n|^m + \sigma_2 |v_n|^m)dx \\ &\leq C_5 (\|u_n\|_X + \|v_n\|_Y)^{p_0} \end{aligned} \tag{3.23}$$

with some $C_5 > 0$ and $\sigma_0 = \frac{1}{m}\min\{-\sigma_1, \sigma_2\} > 0$, in which σ_1 and σ_2 are given in (2.19).

Arguing as the above, for $p_0 < m$, we have

$$\lim_{n\to\infty} \frac{a(x)(|u_n|^m + |v_n|^m)}{|(u_n, v_n)|^{p_0}} = +\infty, \quad x \in \mathcal{D}. \tag{3.24}$$

Then we have from (3.14)–(3.15) and (3.24) that

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathcal{D}} \frac{a(x)(|u_n|^m + |v_n|^m)}{\|(u_n, v_n)\|_E^{p_0}} dx \\ =& \lim_{n\to\infty} \int_{\mathcal{D}} \frac{a(x)(|u_n|^m + |v_n|^m)}{|(u_n, v_n)|^{p_0}} \frac{|(u_n, v_n)|^{p_0}}{\|(u_n, v_n)\|_E^{p_0}} dx = +\infty. \end{aligned} \tag{3.25}$$

Clearly,

$$\liminf_{n\to\infty} \int_{\mathcal{D}^c} \frac{a(x)(|u_n|^m + |v_n|^m)}{\|(u_n, v_n)\|_E^{p_0}} dx \geq 0. \tag{3.26}$$

Consequently,

$$\lim_{n\to\infty} \sigma_0 \int_\Omega \frac{a(x)(|u_n|^m + |v_n|^m)}{\|(u_n, v_n)\|_E^{p_0}} dx = +\infty. \tag{3.27}$$

The application of (3.23) and (3.27) gives that $C_5 \geq +\infty$. This is a contradiction. So, meas($\mathcal{D}$)=0 and $U_0(x) = 0$ and $V_0(x) = 0$ a.e. in Ω. Then, from (3.19) and (3.20), we derive that condition (J_2) hold true under the assumptions (H_1)–(H_4) and (H_6).

Then the application of Lemma 3.1 shows that system (1.1) admits infinitely many solutions $(u_n, v_n) \in E$ with $J(u_n, v_n) \to \infty$ as $n \to \infty$. Hence, we complete the proof of Theorem 1.4. □

Proof of Theorem 1.5. Similar as the proof of Theorem 1.4, we verify the conditions (J_1) and (J_2) in the position of assumptions in Theorem 1.5.

When $p < m < p_0 = p(1+\alpha)$, we note that the verification of (J_1) and (J_2) is similar as the first part of proof for Theorem 1.4, and so we only consider the case $1 < m < p$. Note that

$$\begin{aligned} \lambda \int_\Omega a(x)|u|^m dx \le \lambda S_0^m [u]_{s,p}^m \|a\|_\delta \le \varepsilon [u]_{s,p}^{p_0} + S_0^{m\tau} \lambda^\tau \|a\|_\delta^\tau \varepsilon^{1-p_0/m}, \\ \mu \int_\Omega a(x)|v|^m dx \le \mu S_0^m [v]_{s,p}^m \|a\|_\delta \le \varepsilon [v]_{s,p}^{p_0} + S_0^{m\tau} \mu^\tau \|a\|_\delta^\tau \varepsilon^{1-p_0/m}, \end{aligned} \tag{3.28}$$

where $\tau = p_0/m$. Then from (1.24), (3.4)–(3.8) and (3.28), it follows that

$$\begin{aligned} J(u,v) &\ge \frac{a_1}{p_0}\Big([u]_{s,p}^{p_0} + [v]_{s,p}^{p_0}\Big) + \frac{1}{p}\Big(\|u\|_{m,V_1}^p + \|v\|_{m,V_1}^p\Big) \\ &\quad - \frac{\ell}{k}\int_\Omega F(x,u,v)dx - \frac{1}{m}(\lambda \|u\|_{m,a}^m + \mu \|v\|_{m,a}^m) \\ &\ge \Big(\frac{a_1}{p_0} - \frac{\varepsilon}{m}\Big) 2^{1-p_0}\Big([u]_{s,p}^{p_0} + [v]_{s,p}^{p_0}\Big) + \frac{1}{p}\Big(\|u\|_{m,V_1}^p + \|v\|_{m,V_1}^p\Big) \\ &\quad - \beta_5 \varepsilon (\|u\|_X + \|v\|_Y)^p - \beta_6 (\|u\|_X + \|v||_Y)^{p_s^*} \\ &\quad - \frac{1}{m} S_0^{p_0} \|a\|_\delta^\tau \varepsilon^{1-p_0/m} (\lambda^\tau + \mu^\tau) \end{aligned} \tag{3.29}$$

with $\beta_5 = \frac{\ell}{k}\beta_3 2^{p-1}, \beta_6 = \frac{\ell}{k}\beta_3 2^{p_s^*-1}$. Let $\alpha_1 = 2^{1-p_0} \min\Big\{\frac{1}{p}, \frac{a_1}{2^{p_0} p_0}\Big\}, \rho = \min\Big\{1, (\frac{\alpha_1}{2\beta_6})^{p_s^*-p_0}\Big\}$ and $0 < \varepsilon = \min\Big\{\frac{ma_1}{2p_0}, \frac{\alpha_1}{4\beta_5}\rho^{p_0-p}\Big\}$ and Furthermore, assume $\rho = \|(u,v)\|_E \le 1$. Then we have

$$\alpha_1 \rho^{p_0} - \beta_6 \rho^{p_s^*} \ge \frac{\alpha_1}{2}\rho^{p_0}, \text{ and } \frac{\alpha_1}{2}\rho^{p_0} - \beta_5 \varepsilon \rho^p \ge \frac{\alpha_1}{4}\rho^{p_0} \tag{3.30}$$

Then It follows from (3.29) and (3.30) that there exists $\lambda_0 > 0$ such that

$$\begin{aligned} J(u,v) &\ge \alpha_1 \rho^{p_0} - \beta_5 \varepsilon \rho^p - \beta_6 \rho^{p_s^*} - \frac{1}{m} S_0^{p_0} \|a\|_\delta^\tau \varepsilon^{1-p_0/m}(\lambda^\tau + \mu^\tau) \\ &\ge \frac{\alpha_1}{4}\rho^{p_0} - \frac{1}{m} S_0^{p_0} \|a\|_\delta^\tau \varepsilon^{1-p_0/m}(\lambda^\tau + \mu^\tau) \ge \frac{\alpha_1}{8}\rho^{p_0} \end{aligned} \tag{3.31}$$

provided $\lambda^\tau + \mu^\tau \le \lambda_0$. This shows condition (J_1). The verification of condition (J_2) is the same as that in the proof of Theorem 1.4 and is omitted.

Then the application of Lemma 3.1 shows that system (1.1) admits infinitely many solutions $(u_n, v_n) \in E$ with $J(u_n, v_n) \to \infty$ as $n \to \infty$. Hence, we complete the proof of Theorem 1.5. □

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