

THE RANGE OF r -MAXIMUM INDEX OF GRAPHS

JEONG-OK CHOI

ABSTRACT. For a connected graph G , an r -maximum edge-coloring is an edge-coloring f defined on $E(G)$ such that at every vertex v with $d_G(v) \geq r$ exactly r incident edges to v receive the maximum color. The r -maximum index $\chi'_r(G)$ is the least number of required colors to have an r -maximum edge coloring of G . In this paper, we show how the r -maximum index is affected by adding an edge or a vertex. As a main result, we show that for each $r \geq 3$ the r -maximum index function over the graphs admitting an r -maximum edge-coloring is unbounded and the range is the set of natural numbers. In other words, for each $r \geq 3$ and $k \geq 1$ there is a family of graphs $G(r, k)$ with $\chi'_r(G(r, k)) = k$. Also, we construct a family of graphs not admitting an r -maximum edge-coloring with arbitrary maximum degrees: for any fixed $r \geq 3$, there is an infinite family of graphs $\mathcal{F}_r = \{G_k : k \geq r + 1\}$, where for each $k \geq r + 1$ there is no r -maximum edge-coloring of G_k and $\Delta(G_k) = k$.

1. Introduction

In this paper we consider only simple graphs. The *edge-connectivity* of a connected graph G is the least integer k such that there exist k edges whose deletion increases the number of components of the remaining graph. We denote the edge-connectivity of G by $\kappa'(G)$. In particular, we call an edge a *cut-edge* of a nontrivial connected graph G if $G - e$ is not connected. A k -*factor* of a graph G is a k -regular spanning subgraph of G . Given an edge-coloring f of G , not necessarily proper, a *maximum color* of a vertex $v \in V(G)$ is $\max_{e \in E_v(G)} f(e)$, where $E_v(G)$ is the set of incident edges to v . A color c *appears l times at v under f* if exactly l incident edges to v receive the color c .

For hypergraphs a vertex coloring is called a *unique-maximum coloring* if the maximum color in each edge appears exactly once on the vertices of the edge. There are results on unique-maximum coloring of hypergraphs in [1]. For a graph this coloring is just a proper coloring of the vertices. For a planar graph G , we can define an associated hypergraph with $V(G)$ as the vertex set and the

Received September 21, 2017; Revised March 29, 2018; Accepted May 2, 2018.

2010 *Mathematics Subject Classification.* 05C15.

Key words and phrases. r -maximum edge-coloring, r -maximum index of graphs.

This work was supported by the National Research Foundation of Korea, Grant # NRF2013R1A1A1076120.

faces as hyperedges. This unique-maximum coloring is called *unique-maximum coloring of a planar graph with respect to faces*. In other words, it is a vertex coloring such that for every face the maximum color occurs exactly once on the vertices of the face. The *unique-maximum index* of a planar graph G is the minimum number of colors for G to have a unique-maximum coloring. The parameter is studied in [2, 3].

Without a restriction on planar graphs, a similar concept on edges can be applied: a *unique-maximum edge-coloring* of a graph is an edge-coloring such that at every vertex the maximum color appears exactly once [4]. In [4] S. Jendrol' and Vrbljarová generalized the concept by fixing the number of maximum colors appearing at each vertex. For any natural number r , we say that a graph G has an r -maximum edge-coloring if there is an edge-coloring f defined on $E(G)$ such that at every vertex v with $d_G(v) \geq r$ the maximum color appears r times at v under f . If $\Delta(G)$ is smaller than r , then r -maximum coloring problems become trivial by using one color for all the edges. In fact, every edge-coloring is an r -maximum edge-coloring. On the other hand, if $\Delta(G)$ is larger than r , properties on r -maximum coloring reflect various aspects of structure of graphs. For example, the existence of an r -factor of a graph implies the existence of an r -maximum coloring of the graph. Note that the converse is not true. (See also [6], [7].)

The r -maximum index of a graph G , denoted by $\chi'_r(G)$, is the smallest number of required colors to have an r -maximum edge coloring of G . A natural question on r -maximum index is what the possible numbers for $\chi'_r(G)$ will be. When $r \leq 2$, every graph admits an r -maximum edge coloring. Moreover, $\chi'_r(G) \leq 3$ for $r = 1, 2$.

Theorem 1.1 ([4]). *For $r = 1, 2$, $\chi'_r(G) \leq 3$ for all connected nontrivial graph G .*

In addition, they characterized graphs according to the value of $\chi'_r(G) = i$, $i = 1, 2, 3$. On the other hand, not every graph has an r -maximum edge-coloring when $r \geq 3$. For $r \geq 3$ they expected $\chi'_r(G) \leq 3$ for every graph G admitting an r -maximum edge-coloring.

As answering this question, we show that for each $r \geq 3$ the r -maximum index function over the graphs admitting an r -maximum edge-coloring is unbounded and the range is the set of natural numbers. In other words, for each $r \geq 3$ and $k \geq 1$ there is a family of graphs $G(r, k)$ with $\chi'_r(G(r, k)) = k$. Also, we construct a family of graphs not admitting an r -maximum edge-coloring with arbitrary maximum degrees: for any fixed $r \geq 3$, there is an infinite family of graphs $\mathcal{F}_r = \{G_k : k \geq r + 1\}$, where for each $k \geq r + 1$ there is no r -maximum edge-coloring of G_k and $\Delta(G_k) = k$.

We make a remark that there is a similar name for the parameter called a *maximum edge coloring* or known as the Maximum Edge Coloring (MEC) problem. This maximum edge coloring is a generalization of a proper edge coloring. (For this topic, see [5].) The r -maximum edge-coloring is not related

to a proper edge-coloring, and it is completely different from the maximum edge coloring. In this paper, we only discuss on r -maximum edge-coloring.

In the next section, we first explore effects of an r -maximum edge-coloring and the r -maximum index as adding an edge in many different ways.

2. Preliminaries

The maximum edge-coloring of a graph is related to r -regular subgraphs. It is trivial to see the following results.

Proposition 2.1. *If G has an r -factor, then $\chi'_r(G) \leq 2$.*

Proposition 2.2. *If G has an r -maximum edge-coloring f , then all the incident edges to a vertex of degree r must get the same color under f .*

Proposition 2.3. *Suppose that $\delta(G) \geq r$ and G has an r -maximum edge-coloring f . Let c be the maximum value of f . Then the induced subgraph by the edges receiving c is r -regular.*

The r -maximum index of a graph is not inherited from its supergraph. All four combinations are possible. For example, $\chi'_r(K_r) = 1$ but $\chi'_r(K_{2r}) = 2$. But there is a $2r$ -vertex graph whose r -maximum index does not exist [4]. We provide sufficient conditions to make an r -maximum index as an invariant under some operations on a graph.

Proposition 2.4. *Given a graph H admitting an r -maximum edge-coloring with $\chi'_r(H) > 1$, let H' be a graph obtained by adding a new leaf to a vertex v with $d_H(v) > r$. Then $\chi'_r(H') \leq \chi'_r(H)$.*

Proof. For any given r -maximum edge-coloring of H , the number of colors that all incident edges to v received is at least two because $d_H(v) \geq r$. If we assign a smaller color than the maximum color at v to the newly added edge, then it is still an r -maximum edge-coloring. Therefore, $\chi'_r(H') \leq \chi'_r(H)$. \square

By arguing similarly we obtain the following fact.

Proposition 2.5. *Given a graph H admitting an r -maximum edge-coloring with $\chi'_r(H) > 1$, let H' be a graph obtained by adding a new edge connecting two vertices u and v with $d_H(u), d_H(v) > r$. Then $\chi'_r(H') \leq \chi'_r(H)$.*

Proof. Since both of $d_H(u)$ and $d_H(v)$ are strictly greater than r , for each of u and v there is an incident edge that has a non-maximum color among the incident edges to the vertex. Let c_u and c_v be a non-maximum color at u and v , respectively. Without loss of generality we assume that $c_u \leq c_v$. We assign c_u for the color of the newly added edge. The edge-coloring of H' is now an r -maximum edge-coloring. Therefore, $\chi'_r(H') \leq \chi'_r(H)$. \square

For example, note that the star $K_{m,1}$ with $m \geq r + 1$ has an r -maximum edge-coloring. For a graph G admitting an r -maximum edge-coloring assume that G has a vertex v of degree at least $r + 1$. Let u be the vertex of degree

m in $K_{m,1}$. We denote $G_1 + G_2$ the disjoint union of two graphs G_1 and G_2 . If $G' = (G + K_{m,1}) \cup \{uv\}$, then $\chi'_r(G') \leq \max\{\chi'_r(G), 2\}$ by Proposition 2.5. We generalize this observation as follows.

Corollary 2.6. *For G_i with $i = 1, 2$ suppose that each graph G_i admits an r -maximum edge-coloring. Also suppose that $d_{G_1}(u) > r$ and $d_{G_2}(v) > r$ for some vertices $u \in V(G_1)$ and $v \in V(G_2)$. Let $G' = (G_1 + G_2) \cup \{uv\}$. Then $\chi'_r(G') \leq \max\{\chi'_r(G_1), \chi'_r(G_2)\}$.*

Theorem 2.7. *If a connected n -vertex graph G has only one vertex of degree larger than r which is not a cut-vertex and all the rest of the vertices have degree r , then G does not have an r -maximum edge-coloring.*

Proof. Suppose to the contrary that G has an r -maximum edge-coloring, say f . Let u be the vertex of maximum degree. Then $\chi'_r(G) > 1$. Otherwise, the unique color appears more than r times at u . Therefore, there are at least two edges e_1 and e_2 incident to u such that $f(e_1) \neq f(e_2)$. Let x be the end vertex of e_1 different from u and y be the end vertex of e_2 different from u . Since u is not a cut-vertex, $G - u$ is connected. Therefore, there is an x, y -path in $G - u$, say this path $xv_1v_2 \cdots v_t y$ for some $t \geq 0$. Then by applying Proposition 2.2 to f on G , the edge xv_1 must get the same color as $f(e_1)$, and we repeat this argument until we reach the edge $v_t y$. If $t = 0$, xy must get the same color as $f(e_1)$ and $f(e_2)$. Now we have a contradiction: $f(e_1) = f(v_t y) = f(e_2)$. \square

There is a relation between the edge-connectivity and r on the maximum edge-coloring. If $r \leq \kappa'(G)$, then G admits an r -maximum edge-coloring if and only if G has an r -factor since $r \leq \kappa'(G) \leq \delta(G)$.

3. Main results

In this section we show that for each $r \geq 3$, $\chi'_r(G)$ is arbitrary.

It is easy to see that we can characterize all graphs admitting an r -maximum edge-coloring according to the values 1, 2, and at least 3 for r -maximum index in the following way.

Theorem 3.1. *Let G be a connected graph. Then*

- (1) $\chi'_r(G) = 1$ if and only if $\Delta(G) \leq r$.
- (2) $\chi'_r(G) = 2$ if and only if $\Delta(G) > r$ and G has a spanning subgraph H such that $d_H(v) = r$ if $d_G(v) \geq r$ and $d_H(u) = t$ for some $t < r$ if $d_G(v) < r$.
- (3) For G admitting an r -maximum edge-coloring, $\chi'_r(G) \geq 3$ if and only if G does not satisfies any of the conditions above.

Note that in particular, if $\delta(G) \geq r$, a spanning subgraph H satisfying the condition in (2) in Theorem 3.1 is an r -factor of G .

It is trivial to see that $\chi'_r(G) = 1$ if $r \geq \Delta(G)$. However, $\chi'_r(G)$ may not exist if $r < \Delta(G)$. In fact, it is known that for every $r \geq 3$ there is a graph

$G(r)$ with $\Delta = 2r - 2$ that has no r -maximum edge-coloring [4]. We show that there is no correlation between the maximum degree of a graph and existence of r -maximum edge-coloring. For any fixed $r \geq 3$ there are graphs with arbitrary maximum degree at least $r + 1$ that do not admit an r -maximum edge-coloring.

Theorem 3.2. *For any fixed $r \geq 3$, there is an infinite family of graphs $\mathcal{F}_r = \{G_k: k \geq r + 1\}$, where for each $k \geq r + 1$ there is no r -maximum edge-coloring of G_k and $\Delta(G_k) = k$.*

Proof. Let H be an r -regular graph such that $\alpha'(H) \geq 2$ and $\kappa'(H) \geq 3$, where $\alpha'(H)$ is the size of a maximum matching in H . For any $r \geq 3$, such H exists, for example $H = K_{r+1}$. We pick two independent edges $e = xy$ and $e' = uv$. Let H_i be an i -th copy of H . In this case we rename the vertices and the edges by putting subscript i . For example, the corresponding vertex to x in H_i is denoted by x_i .

We construct G_k as follows:

- (1) We take k copies of H , and take disjoint union of them: we start with $H_1 + H_2 + \dots + H_k$.
- (2) We delete the edges: $\{e_1, e_2, \dots, e_k, e'_1, \dots, e'_k\}$ from (1).
- (3) Add a new vertex z to (2).
- (4) Add new edges: $\{y_1x_2, y_2x_3, \dots, y_{k-1}x_k, y_kx_1\}$ to (2).
- (5) Add new edges: $\{u_iz, v_iz: i = 1, 2, \dots, k\}$ to (2).

(Case 1) r is odd.

Let $G = G_{\frac{r+1}{2}}$. Then $\Delta(G) = r + 1 = d_G(z)$. All the rest of the vertices have degree r .

Claim 1. G does not admit an r -maximum edge-coloring.

Proof. Suppose that there is an r -maximum edge-coloring of G , say f . Let $c = f(u_1z)$. Note that all the other incident edges to u_1 appear in H_1 and must receive the color c since the degree of u_1 in G is r . Moreover, by Proposition 2.2, all the incident edges to each vertex except for z must receive only one (maximum) color. For every $i = 1, 2, \dots, \frac{r+1}{2}$, the graph $H_i - \{e_i, e'_i\}$ is connected since $\kappa'(H_i) \geq 3$. Therefore, for any vertex $w \in V(H_1) - \{u_1\}$, there is a w, u_1 -path in $H_1 - \{e_1, e'_1\}$. Applying the same argument as in the proof of Theorem 2.7, we conclude that in G the edges from $H_1 - \{e_1, e'_1\}$ must receive the color c under f . In fact, the incident edges of a and the incident edges of b must get the same color under f whenever there is an a, b -path in G for vertices a and b with $d_G(a) = d_G(b) = r$. Hence, we get $f(u_iz) = f(v_jz)$ and $f(u_iz) = f(u_jz)$ for all i, j . Then c appears $r + 1$ times at z under f , which is contradiction. \square

If we add arbitrarily many leaves making z as their neighbor, then we can make the degree of z become an arbitrary size at least $r + 2$. In this case, by repeating the proof of Claim 1, the color of the previously existing edges must get the same color anyway. This forces that all $r + 1$ incident edges to

z must get the maximum color. Therefore this new graph does not have an r -maximum edge-coloring.

(Case 2) r is even.

We let $G = G_{\frac{r}{2}+1}$. Then $\Delta(G) = r + 2$. For larger maximum degree we add leaves to z as in r odd case. In this case we can make $\Delta(G) \geq r + 2$. Then the proof is exactly the same as above.

For $\Delta(G) = r + 1$ when r is even, we make the following construction: from the above graph G with $\Delta(G) = r + 2$, let G' be the graph adding a new vertex w and a new edge v_1w after deleting the edge v_1z . Then $\Delta(G') = r + 1$. Applying the same argument from above a color assigned on u_1z forces the rest of the edges of G' to have the same color. Therefore, it is not possible to admit an r -maximum edge-coloring. \square

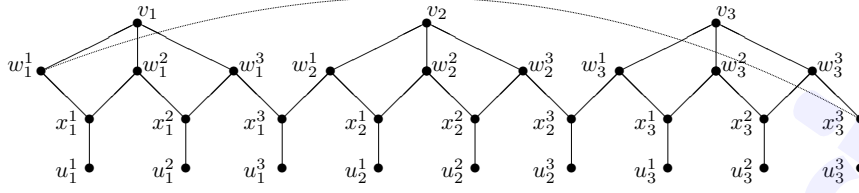
In a graph admitting an r -maximum edge-coloring, a structure that forces a certain color on certain edges may require many colors to be used. In [4] the authors proposed an open problem asking whether $\chi'_r(G) \leq 3$ for every $r \geq 3$ and for every graph G admitting an r -maximum edge-coloring. But unlike the case $r = 2$, from $r \geq 3$ we show that some graphs require many colors to allow an r -maximum coloring. Our construction requires $r \geq 3$ and with $r = 2$ the construction does not contain a structure forcing many colors anymore. We will construct a family of graphs using H_n^m , which is described below. (Figure 1 presents H_3^3 .)

Construction of H_n^m :

- Let $V = \{v_1, v_2, \dots, v_n\}$.
- Let $W = \{w_1^1, \dots, w_1^m, w_2^1, \dots, w_2^m, \dots, w_n^1, \dots, w_n^m\}$.
- Let $X = \{x_1^1, \dots, x_1^m, x_2^1, \dots, x_2^m, \dots, x_n^1, \dots, x_n^m\}$.
- Let $U = \{u_1^1, \dots, u_1^m, u_2^1, \dots, u_2^m, \dots, u_n^1, \dots, u_n^m\}$.
- $V(H_n^m) = V \cup W \cup X \cup U$
- There are three types of edges.
 - **Edges between V and W :** Each v_i has exactly m neighbors in W : w_i^1, \dots, w_i^m .
 - **Edges between W and X :** Each x_i^j has exactly $m - 1$ neighbors in W : starting from w_i^j , take the first consecutive $m - 1$ vertices in order in W as neighbors of x_i^j reading subscripts and superscripts modulo m .
 - **Edges between X and U :** Each x_i^j in X has only one neighbor u_i^j in U .

We see that every vertex in U is a leaf. All other vertices have degree m .

Lemma 3.3. *For any $m \geq 3$ and $n \geq 1$, $\chi'_m(H_n^m) = 1$. In any m -maximum edge-coloring of H_n^m , any color c assigned on the edge incident to a leaf forces c as the color for the rest of the edges.*

Figure 1: H_3^3

Two vertices w_1^1 and x_1^1 are adjacent in H_3^3 .
There are 9 leaves and 21 vertices of degree 3.

Proof. Assigning the same color to all edges satisfies the condition for an m -maximum edge-coloring. Now we will show that edge-coloring with one color is the only possible way.

Consider an m -maximum edge-coloring of H_n^m . Let c be the color that was assigned on an edge $e = x_1^1 u_1^1$. Since the degree of x_1^1 is m , all incident edges to this vertex must receive the same color. In other words, if a color appears at a vertex of degree m , then that color is the only color appearing at the vertex. Note that for any two vertices y, y' with degree m , if there is an y, y' -path via the vertices of degree m , then y and y' must get the same color for their incident edges. Note that H_n^m is connected. In fact, we can see that for any x_i^j and $w_{i'}^{j'}$, there is an $x_i^j, w_{i'}^{j'}$ -path using vertices only in $W \cup X$. To see this it is sufficient to show that there is an $x_1^j, w_1^{j'}$ -path using vertices only in $W \cup X$. It is because that x_1^i and w_1^i are always adjacent for every i . Once the vertices in $W \cup X$ allow only one color c , then the vertices in $V \cup U$ are forced to allow color c only. \square

Given an integer $L \geq 2$, let $H_n^m(k)$ be the k -th copy of H_n^m for $k = 1, 2, \dots, L$. Let $G(m, L)$ be the graph constructed in the following way:

- Concatenate $H_{m^k}^m(k)$ and $H_{m^{k+1}}^m(k+1)$ for $k = 1, 2, \dots, L-1$.
 - Identify u_i^t in $V(H_{m^k}^m(k))$ and $v_{m(i-1)+t}$ in $V(H_{m^{k+1}}^m(k+1))$ for $1 \leq i \leq m^k$ and $1 \leq t \leq m$.

Note that there are only three values for the degrees: 1, m , and $m+1$. The leaves of $G(m, L)$ are the leaves of $H_{m^L}^m(L)$.

Theorem 3.4. $\chi_r'(G(r, L)) = L$ for $r \geq 3$. In other words, for any fixed integer $r \geq 3$, the r -maximum index function (over the graphs admitting an r -maximum edge-coloring) is unbounded and the range is the set of natural numbers.

Proof. Let $r \geq 3$ and consider $G(r, L)$. First of all, if we assign color k for the edges of $E(H_{r^k}^r(k))$ in $G(r, L)$, then this edge-coloring becomes an r -maximum edge-coloring. Therefore, $\chi_r'(G(r, L)) \leq L$.

Let f be an optimal r -maximum edge-coloring of $G(r, L)$. Let $c = f(e)$, where e is the edge incident to a leaf in $H_{r^L}^r(L)$. By the proof of Lemma 3.3 all the edges of $H_{r^L}^r(L)$ must receive color c . Now in $G(r, L)$ the color c already appeared r times at every vertex of degree $m + 1$. Therefore, the uncolored incident edge at every such vertex must receive a smaller color, say c_1 . Applying the same argument, all the edges of $H_{r^{L-1}}^r(L - 1)$ will get color c_1 . We inductively apply this argument to conclude that all the edges of $H_{r^{L-i}}^r(L - i)$ will get color c_i , where $c > c_1 > \dots > c_{i-1} > c_i > \dots > c_{L-1}$ and $i = 0, 1, \dots, L - 1$. Therefore, f requires at least L distinct values, and $\chi_r'(G(r, L)) \geq L$. \square

In [4] the family of trees and the family of complete graphs were characterized in terms of an r -maximum index. They are bounded above by 3 for every r . Note that $G(r, L)$ is bipartite for $r \geq 3$. Therefore, the r -maximum index function is not bounded over the family of bipartite graphs. It would be an interesting question whether there is any well-known family of graphs that has a bounded range for the r -maximum index function when $r \geq 3$.

Remark 3.5. Note that the above construction with $G(2, L)$ does not give the same conclusion as in Theorem 3.4. The graph $H_{2^k}^2(k)$ is disconnected for each k , and $G(2, L)$ is a tree. Therefore, in $G(2, L)$, at any level k a color assigned on an edge incident to a leaf does not necessarily force the same color for the rest of the edges in the copy $H_{2^k}^2(k)$.

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JEONG-OK CHOI
 DIVISION OF LIBERAL ARTS AND SCIENCES
 GWANGJU INSTITUTE OF SCIENCE AND TECHNOLOGY
 GWANGJU 61005, KOREA
 Email address: jchoi351@gist.ac.kr