

## POLYNOMIALLY DEMICOMPACT OPERATORS AND SPECTRAL THEORY FOR OPERATOR MATRICES INVOLVING DEMICOMPACTNESS CLASSES

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**ABSTRACT.** In the first part of this paper we show that, under some conditions, a polynomially demicompact operator can be demicompact. An example involving the Caputo fractional derivative of order  $\alpha$  is provided. Furthermore, we give a refinement of the left and the right Weyl essential spectra of a closed linear operator involving the class of demicompact ones. In the second part of this work we provide some sufficient conditions on the inputs of a closable block operator matrix, with domain consisting of vectors which satisfy certain conditions, to ensure the demicompactness of its closure. Moreover, we apply the obtained results to determine the essential spectra of this operator.

### 1. Introduction

Let  $X$  and  $Y$  be two Banach spaces. The set of all closed densely defined (resp. bounded) linear operators acting from  $X$  into  $Y$  is denoted by  $\mathcal{C}(X, Y)$  (resp.  $\mathcal{L}(X, Y)$ ). We denote by  $\mathcal{K}(X, Y)$  the subset of compact operators of  $\mathcal{L}(X, Y)$ . For  $T \in \mathcal{C}(X, Y)$ , we use the following notations:  $\alpha(T)$  is the dimension of the kernel  $\mathcal{N}(T)$  and  $\beta(T)$  is the codimension of the range  $\mathcal{R}(T)$  in  $Y$ . The next sets of upper semi-Fredholm, lower semi-Fredholm, Fredholm and semi-Fredholm operators from  $X$  into  $Y$  are, respectively, defined by:

$$\Phi_+(X, Y) = \{T \in \mathcal{C}(X, Y) \text{ such that } \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$

$$\Phi_-(X, Y) = \{T \in \mathcal{C}(X, Y) \text{ such that } \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$

$$\Phi(X, Y) := \Phi_-(X, Y) \cap \Phi_+(X, Y),$$

and

$$\Phi_{\pm}(X, Y) := \Phi_-(X, Y) \cup \Phi_+(X, Y).$$

For  $T \in \Phi_{\pm}(X, Y)$ , the index is defined as  $i(T) := \alpha(T) - \beta(T)$ . A complex number  $\lambda$  is in  $\Phi_{+T}$ ,  $\Phi_{-T}$ ,  $\Phi_{\pm T}$  or  $\Phi_T$  if  $\lambda - T$  is in  $\Phi_+(X, Y)$ ,  $\Phi_-(X, Y)$ ,

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$\Phi_{\pm}(X, Y)$  or  $\Phi(X, Y)$ , respectively. If  $X = Y$ , then  $\mathcal{L}(X, Y)$ ,  $\mathcal{C}(X, Y)$ ,  $\mathcal{K}(X, Y)$ ,  $\Phi(X, Y)$ ,  $\Phi_+(X, Y)$ ,  $\Phi_-(X, Y)$  and  $\Phi_{\pm}(X, Y)$  are replaced by  $\mathcal{L}(X)$ ,  $\mathcal{C}(X)$ ,  $\mathcal{K}(X)$ ,  $\Phi(X)$ ,  $\Phi_+(X)$ ,  $\Phi_-(X)$  and  $\Phi_{\pm}(X)$ , respectively. If  $T \in \mathcal{C}(X)$ , we denote by  $\rho(T)$  the resolvent set of  $T$  and by  $\sigma(T)$  the spectrum of  $T$ . Let  $T \in \mathcal{C}(X)$ . For  $x \in \mathcal{D}(T)$ , the graph norm  $\|\cdot\|_T$  of  $x$  is defined by  $\|x\|_T = \|x\| + \|Tx\|$ . It follows from the closedness of  $T$  that  $X_T := (\mathcal{D}(T), \|\cdot\|_T)$  is a Banach space. Clearly, for every  $x \in \mathcal{D}(T)$  we have  $\|Tx\| \leq \|x\|_T$ , so that  $T \in \mathcal{L}(X_T, X)$ . A linear operator  $B$  is said to be  $T$ -defined if  $\mathcal{D}(T) \subseteq \mathcal{D}(B)$ . If  $\widehat{B}$  the restriction of  $B$  to  $\mathcal{D}(T)$  is bounded from  $X_T$  into  $X$ , we say that  $B$  is  $T$ -bounded.

*Remark 1.1.* Notice that if  $T \in \mathcal{C}(X)$  and  $B$  is  $T$ -bounded, then we get the obvious relations:

$$\begin{cases} \alpha(\widehat{T}) = \alpha(T), \beta(\widehat{T}) = \beta(T), \mathcal{R}(\widehat{T}) = \mathcal{R}(T), \\ \alpha(\widehat{T} + \widehat{B}) = \alpha(T + B), \beta(\widehat{T} + \widehat{B}) = \beta(T + B), \text{ and } \mathcal{R}(\widehat{T} + \widehat{B}) = \mathcal{R}(T + B). \end{cases}$$

Hence,  $T \in \Phi(X)$ , (resp.  $\Phi_+(X)$ ,  $\Phi_-(X)$ ) if, and only if,  $\widehat{T} \in \Phi(X_T, X)$ , (resp.  $\Phi_+(X_T, X)$ ,  $\Phi_-(X_T, X)$ ).

**Definition 1.1.** Let  $T \in \mathcal{L}(X, Y)$ , where  $X$  and  $Y$  are two Banach spaces.

(i)  $T$  is said to have a left Fredholm inverse if there exist  $T_l \in \mathcal{L}(Y, X)$  and  $K \in \mathcal{K}(X)$  such that  $T_l T = I_X - K$ . The operator  $T_l$  is called left Fredholm inverse of  $T$ .

(ii)  $T$  is said to have a right Fredholm inverse if there exists  $T_r \in \mathcal{L}(Y, X)$  such that  $I_Y - T T_r \in \mathcal{K}(Y)$ . The operator  $T_r$  is called right Fredholm inverse of  $T$ .

(iii)  $T$  is said to have a Fredholm inverse if there exists a map which is both a left and a right Fredholm inverse of  $T$ .

**Definition 1.2.** Let  $T \in \mathcal{C}(X)$ , where  $X$  is a Banach space.  $T$  is said to have a left Fredholm inverse (resp. right Fredholm inverse, Fredholm inverse) if  $\widehat{T}$  has a left Fredholm inverse (resp. right Fredholm inverse, Fredholm inverse).

The sets of left and right Fredholm inverses are respectively defined by:

$$\begin{aligned} \Phi_l(X) &:= \{T \in \mathcal{C}(X) \text{ such that } T \text{ has a left Fredholm inverse}\}, \text{ and} \\ \Phi_r(X) &:= \{T \in \mathcal{C}(X) \text{ such that } T \text{ has a right Fredholm inverse}\}. \end{aligned}$$

A complex number  $\lambda$  is in  $\Phi_{lT}(X)$ ,  $\Phi_{rT}(X)$  or  $\Phi_T(X)$  if  $\lambda - T$  is in  $\Phi_l(X)$ ,  $\Phi_r(X)$  or  $\Phi(X)$ , respectively.

**Definition 1.3.** Let  $X$  and  $Y$  be two Banach spaces and let  $F \in \mathcal{L}(X, Y)$ . The operator  $F$  is called:

- Fredholm perturbation if  $T + F \in \Phi(X, Y)$  whenever  $T \in \Phi(X, Y)$ .
- Upper semi-Fredholm perturbation if  $T + F \in \Phi_+(X, Y)$  whenever  $T \in \Phi_+(X, Y)$ .
- Lower semi-Fredholm perturbation if  $T + F \in \Phi_-(X, Y)$  whenever  $T \in \Phi_-(X, Y)$ .

The set of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by  $\mathcal{F}(X, Y)$ ,  $\mathcal{F}_+(X, Y)$  and  $\mathcal{F}_-(X, Y)$ , respectively.

The concept of demicompactness appeared in the literature since 1966 in order to discuss fixed points. It was introduced by W. V. Petryshyn [16] as follows:

**Definition 1.4.** An operator  $T : \mathcal{D}(T) \subseteq X \rightarrow X$  is said to be demicompact if for every bounded sequence  $(x_n)_n$  in  $\mathcal{D}(T)$  such that  $x_n - Tx_n$  converges in  $X$ , there exists a convergent subsequence of  $(x_n)_n$ .

The family of demicompact operators on  $X$  is denoted by  $\mathcal{DC}(X)$ . It is clear that the sum, the product of demicompact operators and the product of a complex number by a demicompact operator are not necessarily demicompact. W. V. Petryshyn [16] and W. Y. Akashi [1] used the class of demicompact operators to obtain some results on Fredholm perturbation. In 2014, B. Krichen [11], gave a generalization of this notion by introducing the class of relative demicompact linear operator with respect to a given linear operator. Recently, W. Chaker, A. Jeribi and B. Krichen [5, 6] continued this study to investigate the essential spectra of densely defined linear operators. Moreover, they established the relationship between demicompact operators and upper semi-Fredholm and Fredholm ones.

The study of spectral theory of block operator matrices has been around for many years. From the most important works on this subject, we quote [9, 20]. Let us consider the following operator matrix  $L_0$  acting on the Banach space product  $X \times Y$

$$L_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

In general, the operators occurring in  $L_0$  are unbounded and  $L_0$  need not to be a closed nor a closable operator, even if its entries are closed. However, under some conditions  $L_0$  is closable and its closure  $L$  can be determined.

In the theory of unbounded block operator matrices, the Frobenius-Schur factorization is of vital importance in the study of the spectrum and the various spectral theory. In [3], the authors showed that under some hypothesis, the operator  $L_0$  is closable and they described its closure. In [14], N. Moalla, M. Damak and A. Jeribi have described the essential spectra of the closure of  $L_0$ . But to determine the essential spectra of  $L$ , they must absolutely know the essential spectrum of the entry  $A$ . In [4], the authors have generalized these results and have described the essential spectrum of the closure of  $L_0$ , which is supposed satisfying some hypothesis. In fact, they have assumed that  $\mathcal{D}(A) \subset \mathcal{D}(C)$  and the intersection of the domains of the operators  $B$  and  $D$  is sufficiently large. Moreover, they have assumed that the domain of  $L_0$  is defined by an additional relation of the form  $\Gamma_X x = \Gamma_Y y$  between the two components of its elements. In the same work, they have investigated the essential spectra of  $L$  by only knowing the essential spectrum of  $A_1 := A|_{\mathcal{D}(A) \cap \mathcal{N}(\Gamma_X)}$ .

In this paper, we will study some demicompactness properties of the closure of the operator matrix  $L_0$  under some hypotheses introduced in [4] and then we will investigate its essential spectrum. More precisely, we are concerned with the following essential spectra:

$$\begin{aligned}\sigma_{e1}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_+(X)\} := \mathbb{C} \setminus \Phi_{+T}, \\ \sigma_{e2}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_-(X)\} := \mathbb{C} \setminus \Phi_{-T}, \\ \sigma_{e4}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi(X)\} := \mathbb{C} \setminus \Phi_T, \\ \sigma_{e5}(T) &:= \bigcap_{K \in \mathcal{K}(X)} \sigma(T + K), \\ \sigma_{e7}(T) &:= \bigcap_{K \in \mathcal{K}(X)} \sigma_{ap}(T + K), \\ \sigma_{e8}(T) &:= \bigcap_{K \in \mathcal{K}(X)} \sigma_\delta(T + K), \\ \sigma_{ewl}(T) &:= \bigcap_{K \in \mathcal{K}(X)} \sigma_l(T + K), \\ \sigma_{ewr}(T) &:= \bigcap_{K \in \mathcal{K}(X)} \sigma_r(T + K),\end{aligned}$$

where,

$$\begin{aligned}\sigma_{ap}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \inf_{x \in D(T); \|x\|=1} \|(\lambda - T)x\| = 0\}, \\ \sigma_\delta(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \text{ is not surjective}\}, \\ \sigma_l(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_l(X)\} := \mathbb{C} \setminus \Phi_{lT}, \\ \sigma_r(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_r(X)\} := \mathbb{C} \setminus \Phi_{rT}.\end{aligned}$$

This paper is organized in the following way. In Section 2, we recall some definitions and results needed in the rest of the paper. In Section 3, we show that under some conditions, a polynomially demicompact operator is demicompact and we give an example involving the Caputo fractional derivative of order  $\alpha$ . In Section 4, we give a fine description of the left and the right Weyl essential spectra. In Section 5, we prove that under some conditions,  $\mu L$  is demicompact and we give a necessary condition for which  $I - L$  is an upper semi-Fredholm operator on a Banach space. In Section 6, we investigate the essential spectra of the matrix operator  $L$ .

## 2. Preliminary results

We start this section by recalling some Fredholm results related with demicompact operators.

**Theorem 2.1** ([5, 6]). *Let  $X$  be a Banach space and let  $T \in \mathcal{C}(X)$ . Then,  $T \in \mathcal{DC}(X)$  if, and only if,  $I - T \in \Phi_+(X)$ .*

**Theorem 2.2** ([5]). *Let  $X$  be a Banach space and let  $T \in \mathcal{C}(X)$ . If  $\mu T$  is demicompact for each  $\mu \in [0, 1]$ , then  $I - T$  is a Fredholm operator of index zero.*

**Theorem 2.3** ([5]). *Let  $X$  be a Banach space and let  $T : D(T) \subseteq X \rightarrow X$  be a closed linear operator. If  $T$  is a 1-set-contraction, then  $\mu T$  is demicompact for each  $\mu \in [0, 1]$ .*

Now let us recall the following results:

**Theorem 2.4** ([15, 18]). *Let  $X, Y$  and  $Z$  be Banach spaces,  $A \in \mathcal{L}(Y, Z)$  and  $B \in \mathcal{L}(X, Y)$ .*

- (i) *If  $AB \in \Phi_+(X, Z)$ , then  $B \in \Phi_+(X, Y)$ .*
- (ii) *If  $AB \in \Phi_-(X, Z)$ , then  $A \in \Phi_-(Y, Z)$ .*
- (iii) *If  $X = Y = Z$ ,  $AB \in \Phi(X)$  and  $BA \in \Phi(X)$ , then  $A \in \Phi(X)$  and  $B \in \Phi(X)$ .*
- (iv) *If  $A \in \Phi_+(Y, Z)$  and  $B \in \Phi_+(X, Y)$ , then  $AB \in \Phi_+(X, Z)$ .*
- (v) *If  $A \in \Phi(Y, Z)$  and  $B \in \Phi(X, Y)$ , then  $AB \in \Phi(X, Z)$  and  $i(A + B) = i(A) + i(B)$ .*

**Proposition 2.1** ([18]). *Let  $X$  be a Banach space and let  $T \in \mathcal{C}(X)$ . Then,  $\lambda \notin \sigma_{e5}(T)$  if, and only if,  $(\lambda - T) \in \Phi(X)$  and  $i(\lambda - T) = 0$ .*

**Theorem 2.5** ([9]). *Let  $X$  be a Banach space and let  $T \in \mathcal{C}(X)$ . Then,*

- (i)  *$\lambda \notin \sigma_{e7}(T)$  if, and only if,  $\lambda - T \in \Phi_+(X)$  and  $i(\lambda - T) \leq 0$ .*
- (ii)  *$\lambda \notin \sigma_{e8}(T)$  if, and only if,  $\lambda - T \in \Phi_-(X)$  and  $i(\lambda - T) \geq 0$ .*

**Theorem 2.6** ([2, 10]). *Let  $X$  be a Banach space and let  $T \in \mathcal{C}(X)$ . Then,*

- (i)  *$\lambda \notin \sigma_{ewl}(T)$  if, and only if,  $\lambda - T \in \Phi_l(X)$  and  $i(\lambda - T) \leq 0$ .*
- (ii)  *$\lambda \notin \sigma_{ewr}(T)$  if, and only if,  $\lambda - T \in \Phi_r(X)$  and  $i(\lambda - T) \geq 0$ .*
- (iii)  *$\sigma_{e5}(T) = \sigma_{ewl}(T) \cup \sigma_{ewr}(T)$ .*

In the rest of this section, we will give some results which guarantee the demicompactness of the sum and the product of two operators.

**Proposition 2.2.** *Let  $X$  be a Banach space and let  $A \in \mathcal{C}(X) \cap \mathcal{DC}(X)$  and  $B \in \mathcal{L}(X)$ . If the operator  $I - A$  has a left (resp. right) Fredholm inverse  $A_l$  (resp.  $A_r$ ) such that  $BA_l$  (resp.  $A_r B$ )  $\in \mathcal{DC}(X)$ , then  $A + B \in \mathcal{DC}(X)$ .*

*Proof.* Since  $A_l$  is a left Fredholm inverse of  $I - A$ , then

$$A_l(I - A) = I - K, \text{ where } K \in \mathcal{K}(X).$$

Then, the operator  $I - A - B$  can be written as follows

$$(2.1) \quad I - A - B = (I - BA_l)(I - A) - BK.$$

Now, let  $(x_n)_n$  be a bounded sequence in  $X$  satisfying  $(I - A - B)x_n$  converges to an element of  $X$ . It follows from Eq. (2.1) together with the compactness of  $BK$  and the demicompactness of  $BA_l$  that  $(I - A)x_n$  has a convergent subsequence. Using the demicompactness of  $A$ , we infer that  $(x_n)_n$  admits a convergent subsequence and this shows the demicompactness of  $A + B$ .  $\square$

**Theorem 2.7.** *Let  $X$  be a Banach space and let  $A$  and  $B$  be two bounded operators. If  $A$  has a left Fredholm inverse  $A_l$  such that  $A_l + B \in \mathcal{F}_+(X)$ , then  $AB \in \mathcal{DC}(X)$ .*

*Proof.* Let  $A_l$  be a left Fredholm inverse of  $A$ , then

$$A_l A = I - K,$$

where  $K \in \mathcal{K}(X)$ . It follows that

$$A_l(I - AB) = A_l - (I - K)B = A_l - B - KB.$$

Using the fact that  $KB \in \Phi_+(X)$  together with the fact that  $A_l - B \in \mathcal{F}_+(X)$ , we infer that  $A_l(I - AB) \in \Phi_+(X)$ . Now, the result follows from both Theorems 2.4(i) and 2.1.  $\square$

### 3. Polynomially demicompact operators

In this section, we will generalize the following result proved in [12] for the class of polynomially demicompact operators on  $X$ . In fact, the authors proved that a polynomially compact operator  $T$ , element of  $\mathcal{P}(X) := \{T \in \mathcal{L}(X) \text{ such that there exists a nonzero complex polynomial } P(z) = \sum_{r=0}^p a_r z^r \text{ satisfying } P(1) \neq 0, P(1) - a_0 \neq 0, \text{ and } P(T) \in \mathcal{K}(X)\}$ , is demicompact. In order to state our results, we need to introduce the following set, denoted by  $\mathcal{PDC}(X)$ , which is defined by:

$$\mathcal{PDC}(X) = \bigcup_{P \in \mathbb{C}[z] \setminus \{0\}, P(1) \neq 0} \mathcal{H}_P,$$

where

$$\mathcal{H}_P := \left\{ T \in \mathcal{L}(X) \text{ such that } \frac{1}{P(1)} P(T) \in \mathcal{DC}(X) \right\}.$$

We note that  $\mathcal{PDC}(X)$  contains the set  $\mathcal{P}(X)$ .

**Theorem 3.1.** *Let  $X$  be a Banach space. Then,  $T \in \mathcal{PDC}(X)$  if, and only if,  $T$  is demicompact.*

*Proof.* We first establish the following relation that we are using in the proof. Since  $I - T$  commutes with  $I$ , Newton's binomial formula allows us to write the following relation

$$T^j = I + \sum_{i=1}^j (-1)^i C_j^i (I - T)^i.$$

By making some simple calculations, we may write

$$P(T) = P(1)I + \sum_{j=1}^p a_j \left( \sum_{i=1}^j (-1)^i C_j^i (I - T)^i \right).$$

Since  $P(1) \neq 0$ , we have

$$(3.1) \quad I - \frac{1}{P(1)}P(T) = \frac{1}{P(1)} \sum_{j=1}^p a_j \left( \sum_{i=1}^j (-1)^i C_j^i (I - T)^i \right).$$

Now, let  $(x_n)_n$  be a bounded sequence in  $X$  satisfying  $(I - T)x_n \rightarrow x_0$ . Using the continuity of  $T$  together with the relation (3.1), we infer that there exists  $x$  such that

$$\left( I - \frac{1}{P(1)}P(T) \right) x_n \rightarrow x.$$

By demicompactness of  $\frac{1}{P(1)}P(T)$ , we conclude that  $(x_n)_n$  admits a convergent subsequence. The converse can be checked by taking  $P(z) = z$ .  $\square$

**Theorem 3.2.** *Let  $X$  be a Banach space and let  $T \in \mathcal{L}(X)$ . Suppose that there exists a complex polynomial  $P$  such that  $P(0) = 0$ . Then,  $I - P(T)$  is demicompact if, and only if,  $I - T$  is demicompact.*

*Proof.* Assume that  $I - P(T)$  is a demicompact operator, it follows from Theorem 2.1 that  $P(T) \in \Phi_+(X)$ . Now, take  $x \in \mathcal{N}(T)$ , then  $Tx = 0$  which implies that for all  $j \geq 1$ ,  $T^j = 0$ . Hence,

$$P(T)x = \sum_{j=0}^m a_j T^j x = P(0)x + \sum_{j=1}^m a_j T^j x = 0,$$

where  $P(z) = \sum_{j=0}^m a_j z^j$ . Then,  $\mathcal{N}(T) \subset \mathcal{N}(P(T))$  is obvious and this shows that  $\alpha(T) < \infty$ . Next, since  $\mathcal{R}(P(T))$  is closed, we deduce from Theorem 3.12 in [18] that there exists  $k > 0$  such that  $\forall y \in X$ ,  $\|y\| \leq k\|P(T)y\|$ . In particular,

$$\begin{aligned} \|x\| &\leq k\|P(T)x\| \\ &\leq k \sum_{j=1}^m |a_j| \|T\|^{j-1} \|Tx\|. \end{aligned}$$

The use of Theorem 3.12 in [18] shows that  $\mathcal{R}(T)$  is closed. Therefore,  $T \in \Phi_+(X)$  and we conclude by Theorem 2.1 in [6] that  $I - T$  is demicompact. Conversely, the result can be obtained by taking  $P(z) = z$ .  $\square$

*Remark 3.1.* Using Theorem 3.1 in [5] and Theorem 2.1 in [6], we deduce that if  $P$  is a complex polynomial such that  $P(0) = 0$  and  $T \in \mathcal{L}(X)$ , then  $P(T) \in \Phi_+(X)$  if, and only if,  $T \in \Phi_+(X)$ .

**Theorem 3.3** ([13]). *If  $x(t) \in C^1[0, T]$  for  $T > 0$ , then*

$${}_C D_{0,t}^{(\alpha_2)} {}_C D_{0,t}^{(\alpha_1)} x(t) = {}_C D_{0,t}^{(\alpha_1)} {}_C D_{0,t}^{(\alpha_2)} x(t) = {}_C D_{0,t}^{(\alpha_1 + \alpha_2)} x(t), \quad t \in [0, T],$$

where  $\alpha_1$  and  $\alpha_2 \in \mathbb{R}_+$  and  $\alpha_1 + \alpha_2 \leq 1$ .

**Theorem 3.4** ([13]). *If  $x(t) \in C^m[0, T]$ ,  $m \in \mathbb{N}$  for  $T > 0$ , then*

$${}_C D_{0,t}^{(\alpha)} x(t) = {}_C D_{0,t}^{(\alpha_n)} \cdots {}_C D_{0,t}^{(\alpha_2)} {}_C D_{0,t}^{(\alpha_1)} x(t); \quad t \in [0, T],$$

where  $\alpha = \sum_{i=1}^n \alpha_i$ ;  $\alpha_i \in (0, 1]$ ,  $m - 1 \leq \alpha < m$  and there exists  $i_k < n$ , such that  $\sum_{j=1}^{i_k} \alpha_j = k$ , and  $k = 1, 2, \dots, m - 1$ .

**Example 3.1.** Let  $C_\omega$  be the space of continuous  $\omega$ -periodic functions  $x : \mathbb{R} \rightarrow \mathbb{R}$  and  $C'_\omega$  the space of continuously differentiable  $\omega$ -periodic functions  $x : \mathbb{R} \rightarrow \mathbb{R}$ .  $C_\omega$  equipped with the maximum norm  $\|\cdot\|_\infty$  and  $C'_\omega$  with the norm given by  $\|u\|_\infty^1 = \max\{\|u\|_\infty, \|u'\|_\infty\}$  for  $u \in C'$  are Banach spaces. Let us consider the following differential equation:

$$x'(t) = a(t)x'(t - h_1) + b(t)x(t - h_2) + f(t).$$

Here,  $a$  and  $b$  are continuous  $\omega$ -periodic functions such that  $|a(t)| < k$ , ( $k < \infty$ ), where  $k < \frac{1}{\omega}$  if  $\omega > 2$  or  $k < \frac{1}{2}$  if  $\omega \leq 2$ ;  $f \in C_\omega$  is a given function and  $x \in C'_\omega$  is an unknown function. This equation can be rewritten in the operator form:

$$Gx - Ax = f,$$

where  $G : C'_\omega \rightarrow C_\omega$  is given by the formula:

$$(Gx)(t) = x'(t),$$

and the operator  $A : C'_\omega \rightarrow C_\omega$  by the formula:

$$(Ax)(t) = a(t)x'(t - h_1) + b(t)x(t - h_2).$$

Let us consider the polynomial  $P(z) = \frac{1}{\mu^n} z^n$  and the operator  $T = \mu {}_C D^{(\frac{1}{n})}$ ;  $n \in \mathbb{N} \setminus \{0\}$ , where  ${}_C D^{(\frac{1}{n})}$  is the Caputo derivative of fractional order  $\frac{1}{n}$  and  $\mu \in \mathbb{C} \setminus \{0\}$ . Applying Theorem 3.4, we get

$$P(T) = \frac{1}{\mu^n} T^n(x) = \frac{1}{\mu^n} [\mu {}_C D^{(\frac{1}{n})}]^n x(t) = x'(t).$$

Clearly,  $P(T)$  is a bounded linear operator with  $\|P(T)\| = 1$  and therefore,  $P(T)$  is 1-set-contractive. It follows from the use of Theorem 2.3 that  $Q(T) = \alpha^n P(T)$  is demicompact, for all  $\alpha \in ]0, 1[$ . Remark that  $Q(1) \neq 0$ , we get from Theorem 3.1

$$\mu {}_C D^{(\frac{1}{n})} \in \mathcal{DC}(X) \quad \forall \mu \in \mathbb{C} \setminus \{0\}.$$

Then,

$$\mu {}_C D^{(\frac{1}{n})} \in \mathcal{DC}(X) \quad \forall \mu \in [0, 1].$$

Using Theorem 2.2, we infer that

$$I - {}_C D^{(\frac{1}{n})} \in \Phi(X) \quad \text{and} \quad i(I - {}_C D^{(\frac{1}{n})}) = 0.$$



#### 4. Characterization of left and right Weyl essential spectra

The aim of this section is to give a refinement of the left and the right Weyl essential spectra. For this, let  $X$  be a Banach space and  $T \in \mathcal{C}(X)$ . Let us consider the following sets  $\Lambda_X$ ,  $\Upsilon_T(X)$ , and  $\Psi_T(X)$ , respectively, defined by:

$$\begin{aligned}\Lambda_X &= \{J \in \mathcal{L}(X) \text{ such that } \mu J \text{ is demicompact for all } \mu \in [0, 1]\}, \\ \Upsilon_T(X) &= \{K \in \mathcal{L}(X) \text{ such that for all } \lambda \in \rho(T + K), \\ &\quad -(\lambda - T - K)^{-1}K \in \Lambda_X\}, \\ \Psi_T(X) &= \{K \text{ is } T\text{-bounded such that for all } \lambda \in \rho(T + K), \\ &\quad -K(\lambda - T - K)^{-1} \in \Lambda_X\}.\end{aligned}$$

We also denote for  $T \in \mathcal{C}(X)$ , the following sets:

$$\sigma_e^l(T) = \bigcap_{K \in \Upsilon_T(X)} \sigma_l(T + K) \text{ and } \sigma_e^r(T) = \bigcap_{K \in \Psi_T(X)} \sigma_r(T + K).$$

**Theorem 4.1.** *Let  $X$  be a Banach space and let  $T \in \mathcal{C}(X)$ . Then,*

$$\sigma_{ewl}(T) = \sigma_e^l(T) \text{ and } \sigma_{ewr}(T) = \sigma_e^r(T).$$

*Proof.* Let us notice that for  $T \in \mathcal{C}(X)$ ,  $K$  is a  $T$ -bounded operator such that  $\lambda \in \rho(T + K)$ , then, according to closed graph theorem (Lemma 2.1 in [17]),  $K(\lambda - T - K)^{-1}$  is a closed linear operator defined on  $X$  and then bounded. We first prove that  $\sigma_{ewl}(T) \subset \sigma_e^l(T)$  (resp.  $\sigma_{ewr}(T) \subset \sigma_e^r(T)$ ). Indeed, for  $\lambda \notin \sigma_e^l(T)$  (resp.  $\lambda \notin \sigma_e^r(T)$ ), there exists  $K \in \Upsilon_T(X)$  (resp.  $K \in \Psi_T(X)$ ) such that  $\lambda \notin \sigma_l(T + K)$  (resp.  $\lambda \notin \sigma_r(T + K)$ ). Hence,

$$\begin{aligned}\lambda - T - K &\in \Phi_l(X) \text{ and } i(\lambda - T - K) = 0, \\ (\text{resp. } \lambda - T - K &\in \Phi_r(X) \text{ and } i(\lambda - T - K) = 0).\end{aligned}$$

Which implies that

$$\begin{aligned}\lambda - T - K &\in \Phi_l(X) \text{ and } i(\lambda - T - K) \leq 0, \\ (\text{resp. } \lambda - T - K &\in \Phi_r(X) \text{ and } i(\lambda - T - K) \geq 0).\end{aligned}$$

Next, since  $K \in \Upsilon_T(X)$ , (resp.  $K \in \Psi_T(X)$ ) then  $-(\lambda - T - K)^{-1}K \in \Lambda_X$ , (resp.  $-K(\lambda - T - K)^{-1} \in \Lambda_X$ ), whenever  $\lambda \in \rho(T + K)$ . Thus, applying Theorem 2.2, one has

$$I + (\lambda - T - K)^{-1}K \in \Phi(X) \text{ and } i[I + (\lambda - T - K)^{-1}K] = 0,$$

which implies that

$$\begin{aligned}I + (\lambda - T - K)^{-1}K &\in \Phi_l(X) \text{ and } i[I + (\lambda - T - K)^{-1}K] \leq 0 \\ (\text{resp. } I + (\lambda - T - K)^{-1}K &\in \Phi_r(X) \text{ and } i[I + (\lambda - T - K)^{-1}K] \geq 0).\end{aligned}$$

Using the equality

$$\lambda - T = (\lambda - T - K)[I + (\lambda - T - K)^{-1}K],$$

(resp.

$$\lambda - T = [I + K(\lambda - T - K)^{-1}](\lambda - T - K),)$$

we deduce from Theorem 2.5 in [8] that

$$\lambda - T \in \Phi_l(X) \text{ and } i(\lambda - T) \leq 0.$$

(resp.

$$\lambda - T \in \Phi_r(X) \text{ and } i(\lambda - T) \geq 0.)$$

We conclude from Theorem 2.6 that  $\lambda \notin \sigma_{ewl}(T)$  (resp.  $\lambda \notin \sigma_{ewr}(T)$ ). The inverse inclusion follows from the fact that  $\mathcal{K}(X) \subset \Upsilon_T(X)$ , (resp.  $\mathcal{K}(X) \subset \Psi_T(X)$ ).  $\square$

**Corollary 4.1.** *Let  $X$  be a Banach space,  $T \in \mathcal{C}(X)$  and let  $\Gamma(X)$  be a subset of  $X$  containing  $\mathcal{K}(X)$ .*

$$(i) \text{ If } \Gamma(X) \subset \Upsilon_T(X), \text{ then } \sigma_{ewl}(T) = \bigcap_{K \in \Gamma(X)} \sigma_l(T + K).$$

$$(ii) \text{ If } \Gamma(X) \subset \Psi(X), \text{ then } \sigma_{ewr}(T) = \bigcap_{K \in \Gamma(X)} \sigma_r(T + K).$$

*Proof.* Since  $\mathcal{K}(X) \subset \Gamma(X) \subset \Upsilon_T(X)$  (resp.  $\mathcal{K}(X) \subset \Gamma(X) \subset \Psi(X)$ ), we obtain

$$\bigcap_{K \in \Upsilon_T(X)} \sigma_l(T + K) \subset \bigcap_{K \in \Gamma(X)} \sigma_l(T + K) \subset \bigcap_{K \in \mathcal{K}(X)} \sigma_l(T + K) := \sigma_{ewl}(T).$$

(resp.

$$\bigcap_{K \in \Psi_T(X)} \sigma_r(T + K) \subset \bigcap_{K \in \Gamma(X)} \sigma_r(T + K) \subset \bigcap_{K \in \mathcal{K}(X)} \sigma_r(T + K) := \sigma_{ewr}(T).)$$

The use of Theorem 4.1 allows us to conclude that

$$\sigma_{ewl}(T) = \bigcap_{K \in \Gamma(X)} \sigma_l(T + K),$$

and

$$\sigma_{ewr}(T) = \bigcap_{K \in \Gamma(X)} \sigma_r(T + K).$$

Hence, we get the desired result.  $\square$

## 5. Demicompactness results for operator matrices

Let  $X$ ,  $Y$  and  $Z$  be three Banach spaces. In this paper, we consider the linear operators  $\Gamma_X$  from  $X$  into  $Z$  and  $\Gamma_Y$  from  $Y$  into  $Z$ . In this section, we are concerned with some new results which can be used to determinate the essential spectra of the matrix operator  $L$ , the closure of  $L_0$ , on the space  $X \times Y$ . Let us consider an operator which is formally defined by a matrix

$$(5.1) \quad L_0 := \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the operator  $A$  acts on  $X$  and has domain  $\mathcal{D}(A)$ ,  $D$  is defined on  $\mathcal{D}(D)$  and acts on the Banach space  $Y$  and the intertwining operators  $B$  and  $C$  are defined on the domains  $\mathcal{D}(B)$  and  $\mathcal{D}(C)$ , respectively, and act between these spaces. Then the operator  $L_0$  is defined on the domain  $[\mathcal{D}(A) \cap \mathcal{D}(C)] \times [\mathcal{D}(D) \cap \mathcal{D}(B)]$  and

$$\mathcal{D}(L_0) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ such that } x \in \mathcal{D}(A), y \in \mathcal{D}(D) \cap \mathcal{D}(B) \text{ and } \Gamma_X x = \Gamma_Y y \right\}.$$

In the following, it is always assumed that the entries of this matrix satisfy the following conditions, introduced in [4].

- (H1) The operator  $A$  is densely defined and closable.  
It follows that  $\mathcal{D}(\bar{A})$ , the domain of closure  $\bar{A}$  of  $A$ , coincides with the Banach space  $X_A$  which is contained in  $X$ .
- (H2)  $\mathcal{D}(A) \subset \mathcal{D}(\Gamma_X) \subset X_A$  and  $\Gamma_X$  is bounded as a mapping from  $X_A$  into  $Z$ .  
The extension of  $\Gamma_X$  by continuity to  $X_A = \mathcal{D}(A)$  is denoted by  $\bar{\Gamma}_X$ , which is a bounded operator from  $X_A$  into  $Z$ .
- (H3) The lineal  $\mathcal{D}(A) \cap \mathcal{N}(\Gamma_X)$  is dense in  $X$  and the resolvent set of restriction  $A_1 := A|_{\mathcal{D}(A) \cap \mathcal{N}(\Gamma_X)}$  is not empty, i.e.,  $\rho(A_1) \neq \emptyset$ .
- (H4)  $\mathcal{D}(A) \subset \mathcal{D}(C) \subset X_A$  and  $C$  is closable as an operator from  $X_A$  into  $Y$ .

*Remark 5.1.* It follows from (H3) that  $A_1$  is a closed operator, whence  $\mathcal{D}(A_1)$  is a closed subspace of  $X_A$ . The closed graph theorem and (H4) imply that for  $\lambda \in \rho(A_1)$  the operator  $C_\lambda := C(A_1 - \lambda I)^{-1}$  from  $X$  into  $Y$  is bounded.

*Remark 5.2.* Under the assumptions (H1)-(H3), we infer that for each  $\lambda \in \rho(A_1)$ ,

- (i) the following decomposition holds:

$$\mathcal{D}(A) = \mathcal{D}(A_1) \oplus \mathcal{N}(A - \lambda I).$$

- (ii)  $\Gamma_\lambda := \Gamma_X|_{\mathcal{N}(A - \lambda I)}$  is injective and  $\mathcal{R}(\Gamma_\lambda) = \Gamma_X(\mathcal{N}(A - \lambda I)) = \Gamma_X(\mathcal{D}(A)) := Z_1$  does not depend on  $\lambda$ .
- (iii) the inverse  $K_\lambda$  of the operator  $\Gamma_\lambda$  is

$$K_\lambda := (\Gamma_X|_{\mathcal{N}(A - \lambda I)})^{-1} : Z_1 \rightarrow \mathcal{N}(A - \lambda I) \subset X.$$

Concerning the operators  $K_\lambda$ ,  $D$ ,  $\Gamma_Y$  and  $B$  we impose the following conditions:

- (H5) For some (hence for all)  $\lambda \in \rho(A_1)$ , the operator  $K_\lambda$  is bounded as a mapping from  $Z$  into  $X$ .
- (H6) The operator  $D$  is densely defined and closed with  $\rho(D) \neq \emptyset$ .
- (H7)  $\mathcal{D}(\Gamma_Y) \supset \mathcal{D}(D) \cap \mathcal{D}(B)$ , the set

$$Y_1 = \{y \text{ such that } y \in \mathcal{D}(D) \cap \mathcal{D}(B) \text{ and } \Gamma_Y y \in Z_1\}$$

is dense in  $Y$  and the restriction of  $\Gamma_Y$  to this set is bounded as an operator from  $Y$  into  $Z$ .

(H8) For some (hence for all)  $\lambda \in \rho(A_1)$ , the operator  $(A_1 - \lambda I)^{-1}B$  is closable and its closure  $\overline{(A_1 - \lambda I)^{-1}B}$  is bounded.

We recall the following results which describes the operator  $L_0$ .

*Remark 5.3.* We will denote by

(i)  $\overline{\Gamma}_X$  the extension of  $\Gamma_X$  by continuity to  $\mathcal{D}(\overline{A}) := X_A$ . It is a bounded operator from  $X_A$  into  $Z$ .

(ii)  $\overline{\Gamma}_Y^\circ$  the extension of  $\Gamma_Y|_{Y_1}$ .

(iii)  $\overline{K}_\lambda$  the extension of  $K_\lambda$  to the closure  $\overline{Z}_1$  of  $Z_1$  with respect to the norm of  $Z$ . Without loss of generality we assume that  $\overline{Z}_1 = Z$ . Clearly, the operator  $\overline{K}_\lambda$  is also bounded as a mapping from  $\overline{Z}_1$  to  $X_A$ .

We consider in the space  $Y$ , for  $\lambda \in \rho(A_1)$ , the operator

$$M_\lambda := D + CK_\lambda\Gamma_Y - C_\lambda B.$$

The operator  $M_\lambda$  is defined on the set  $Y_1$ , which is dense in  $Y$  according to (H7). Here we observe that  $\Gamma_Y$  is bounded on this domain by assumption (H7), that  $K_\lambda$  is bounded by assumption (H8). From (H4) and (H8), we infer that if  $M_\lambda$  is closable as an operator in  $Y$  for some  $\lambda \in \rho(A_1)$ , then it is closable for all  $\lambda \in \rho(A_1)$ . We emphasize also that the domain of  $M_\lambda$  does not depend on  $\lambda$ .

**Theorem 5.1** ([4]). *Let conditions (H1)-(H8) be satisfied. Then, the operator  $L_0$  is closable in  $X \times Y$  if, and only if, the operator  $M_\lambda := D + CK_\lambda\Gamma_Y - C_\lambda B$  is closable for some  $\lambda \in \rho(A_1)$ , and equivalently, for all  $\lambda \in \rho(A_1)$ . Moreover, the closure  $L$  of  $L_0$  is given by:*

$$(5.2) \quad L = \lambda I + \begin{pmatrix} I & 0 \\ C_\lambda & I \end{pmatrix} \begin{pmatrix} A_1 - \lambda I & 0 \\ 0 & \overline{M}_\lambda - \lambda I \end{pmatrix} \begin{pmatrix} I & G_\lambda \\ 0 & I \end{pmatrix},$$

where  $G_\lambda := -\overline{K}_\lambda\overline{\Gamma}_Y^\circ + \overline{(A_1 - \lambda I)^{-1}B}$ .

*Remark 5.4.* For each  $\lambda \in \mathbb{C}$ , we have:

$$(5.3) \quad \begin{aligned} \lambda - L &= \begin{pmatrix} I & 0 \\ C_\mu & I \end{pmatrix} \begin{pmatrix} \lambda - A_1 & 0 \\ 0 & \lambda - \overline{M}_\mu \end{pmatrix} \begin{pmatrix} I & G_\mu \\ 0 & I \end{pmatrix} - (\lambda - \mu)N_\mu \\ &:= UV(\lambda)W - (\lambda - \mu)N_\mu, \end{aligned}$$

where

$$N_\mu = \begin{pmatrix} 0 & G_\mu \\ C_\mu & C_\mu G_\mu \end{pmatrix}.$$

**Proposition 5.1.** *Let  $L_0$  assumptions (H1)-(H8) hold and suppose that there is  $\mu \neq 0$  such that  $\frac{1}{\mu} \in \rho(A_1)$ . If the operator  $\mu\overline{M}_{\frac{1}{\mu}}$  is demicompact, then  $\mu L$  is a demicompact operator.*

*Proof.* Let  $(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix})_n \in \mathcal{D}(L)$  be a bounded sequence such that

$$\begin{pmatrix} x'_n \\ y'_n \end{pmatrix} := (I - \mu L) \begin{pmatrix} x_n \\ y_n \end{pmatrix} \rightarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Recalling the factorization (5.2) for  $\frac{1}{\mu} \in \rho(A_1)$ , one has

$$L = \frac{1}{\mu}I - \begin{pmatrix} I & 0 \\ C_{\frac{1}{\mu}} & I \end{pmatrix} \begin{pmatrix} \frac{1}{\mu} - A_1 & 0 \\ 0 & \frac{1}{\mu} - \overline{M}_{\frac{1}{\mu}} \end{pmatrix} \begin{pmatrix} I & G_{\frac{1}{\mu}} \\ 0 & I \end{pmatrix}.$$

Then,

$$\begin{pmatrix} x'_n \\ y'_n \end{pmatrix} = \begin{pmatrix} I & 0 \\ C_{\frac{1}{\mu}} & I \end{pmatrix} \begin{pmatrix} I - \mu A_1 & 0 \\ 0 & I - \mu \overline{M}_{\frac{1}{\mu}} \end{pmatrix} \begin{pmatrix} I & G_{\frac{1}{\mu}} \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

means exactly,

$$\begin{pmatrix} I & 0 \\ -C_{\frac{1}{\mu}} & I \end{pmatrix} \begin{pmatrix} x'_n \\ y'_n \end{pmatrix} = \begin{pmatrix} I - \mu A_1 & 0 \\ 0 & I - \mu \overline{M}_{\frac{1}{\mu}} \end{pmatrix} \begin{pmatrix} I & G_{\frac{1}{\mu}} \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

Hence, we get the following system:

$$(5.4) \quad \begin{cases} (I - \mu A_1)^{-1} x'_n = x_n + G_{\frac{1}{\mu}} y_n, \\ -C_{\frac{1}{\mu}} x'_n + y'_n = (I - \mu \overline{M}_{\frac{1}{\mu}}) y_n. \end{cases}$$

Since the operator  $C_{\frac{1}{\mu}}$  is bounded, then we infer from the second equation of the system (5.4) that  $(I - \mu \overline{M}_{\frac{1}{\mu}}) y_n$  is convergent. Combining this result together with the demicompactness of  $\mu \overline{M}_{\frac{1}{\mu}}$ , we show that  $(y_n)_n$  has a convergent subsequence. Hence, the first equation of the system (5.4) allows us to conclude that  $(x_n)_n$  has a convergent subsequence, which proves the demicompactness of  $\mu L$  and this shows our claim.  $\square$

For more generalization, we give the following result.

**Theorem 5.2.** *Let assumptions (H1)-(H8) be satisfied and suppose that for every  $\mu \in \rho(A_1)$ , there is  $\lambda \in \mathbb{C} \setminus \{0\}$  such that  $\lambda A_1$  is demicompact. Then, if  $C_{\mu}$  is compact and  $\lambda \overline{M}_{\mu}$  is demicompact, then  $\lambda L \in \mathcal{DC}(X \times Y)$ .*

*Proof.* Let  $\mu \in \rho(A_1)$ . By assumption, there is a complex nonzero number  $\lambda$  verifying  $\lambda A_1 \in \mathcal{DC}(X)$ . Take a bounded sequence  $(\frac{x_n}{y_n})_n \in \mathcal{D}(L)$  such that

$$\begin{pmatrix} x'_n \\ y'_n \end{pmatrix} := (I - \lambda L) \begin{pmatrix} x_n \\ y_n \end{pmatrix} \rightarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Let  $\mu \in \rho(A_1)$  be such that there is a complex nonzero number  $\lambda$  verifying  $\lambda A_1 \in \mathcal{DC}(X)$ . Thanks to Remark 5.4, one has

$$\frac{1}{\lambda} - L = \begin{pmatrix} I & 0 \\ C_{\mu} & I \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda} - A_1 & 0 \\ 0 & \frac{1}{\lambda} - \overline{M}_{\mu} \end{pmatrix} \begin{pmatrix} I & G_{\mu} \\ 0 & I \end{pmatrix} - (\frac{1}{\lambda} - \mu) N_{\mu}.$$

Thus,

$$I - \lambda L = \begin{pmatrix} I & 0 \\ C_{\mu} & I \end{pmatrix} \begin{pmatrix} I - \lambda A_1 & 0 \\ 0 & I - \lambda \overline{M}_{\mu} \end{pmatrix} \begin{pmatrix} I & G_{\mu} \\ 0 & I \end{pmatrix} - (1 - \lambda \mu) N_{\mu}.$$

Therefore,

$$(5.5) \quad \begin{pmatrix} x'_n \\ y'_n \end{pmatrix} = \begin{pmatrix} I & 0 \\ C_\mu & I \end{pmatrix} \begin{pmatrix} I - \lambda A_1 & 0 \\ 0 & I - \lambda \overline{M}_\mu \end{pmatrix} \begin{pmatrix} I & G_\mu \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} - (1 - \lambda\mu)N_\mu \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

Observe that (5.5) is equivalent to

$$\begin{aligned} & \begin{pmatrix} I & 0 \\ -C_\mu & I \end{pmatrix} \begin{pmatrix} x'_n \\ y'_n \end{pmatrix} + (1 - \lambda\mu) \begin{pmatrix} I & 0 \\ -C_\mu & I \end{pmatrix} N_\mu \begin{pmatrix} x_n \\ y_n \end{pmatrix} \\ &= \begin{pmatrix} I - \lambda A_1 & 0 \\ 0 & I - \lambda \overline{M}_\mu \end{pmatrix} \begin{pmatrix} I & G_\mu \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}. \end{aligned}$$

Moreover, by making some simple calculations, we may show that

$$\begin{pmatrix} x'_n \\ -C_\mu x'_n + y'_n \end{pmatrix} + \begin{pmatrix} (I - \lambda\mu)G_\mu y_n \\ (I - \lambda\mu)C_\mu x_n \end{pmatrix} = \begin{pmatrix} (I - \lambda A_1)x_n + (I - \lambda A_1)G_\mu y_n \\ (I - \lambda \overline{M}_\mu)y_n \end{pmatrix},$$

in equivalent way,

$$(5.6) \quad \begin{cases} x'_n - \lambda(\mu - A_1)G_\mu y_n = (I - \lambda A_1)x_n, \\ -C_\mu x'_n + y'_n + (I - \lambda\mu)C_\mu x_n = (I - \lambda \overline{M}_\mu)y_n. \end{cases}$$

We deduce from the fact that  $G_\mu$  is compact and  $(y_n)_n$  is bounded, that  $\lambda(\mu - A_1)G_\mu y_n$  has a convergent subsequence. Hence, from the first equation of system (5.6), we infer that  $(I - \lambda A_1)x_n$  has a convergent subsequence. Using the demicompactness of  $\lambda A_1$ , we deduce that there exists a convergent subsequence of  $(x_n)_n$ . Now, since  $C_\mu$  is bounded, we conclude from the second equation of system (5.6) that  $(I - \lambda \overline{M}_\mu)y_n$  has a convergent subsequence. This together with the fact that  $\lambda \overline{M}_\mu$  is demicompact allows us to conclude that  $(y_n)_n$  has a convergent subsequence. Therefore, there exists a subsequence of  $(x_n)_n$  which converges on  $\mathcal{D}(L)$ . Thus,  $\lambda L$  is demicompact.  $\square$

**Theorem 5.3.** *Let assumptions (H1)-(H8) be satisfied and let  $\mu \in \rho(A_1)$  such that  $(1 - \mu)N_\mu \in \mathcal{F}_+(X \times Y)$ . If the operators  $A_1$  and  $\overline{M}_\mu$  are demicompact, then  $I - L$  is an upper semi-Fredholm operator.*

*Proof.* According to the factorization (5.3), we have:

$$\begin{aligned} I - L &= \begin{pmatrix} I & 0 \\ C_\mu & I \end{pmatrix} \begin{pmatrix} I - A_1 & 0 \\ 0 & I - \overline{M}_\mu \end{pmatrix} \begin{pmatrix} I & G_\mu \\ 0 & I \end{pmatrix} - (1 - \mu)N_\mu \\ &:= UV(1)W - (1 - \mu)N_\mu. \end{aligned}$$

Since  $A_1$  and  $\overline{M}_\mu$  are demicompact, it follows from Theorem 2.1, the operators  $I - A_1$  and  $I - \overline{M}_\mu$  are upper semi-Fredholm, hence  $V(1) \in \Phi_+(X \times Y)$ . Taking into account that the operators  $U$  and  $W$  are invertible, we deduce that  $UV(1)W$  is an upper semi-Fredholm operator. Now, owing to the fact that  $(1 - \mu)N_\mu \in \mathcal{F}_+(X \times Y)$ , we conclude that  $I - L$  is an upper semi-Fredholm operator.  $\square$

Before giving the next results, we may state the following notations: Assume that  $L_0$  satisfies (H1)-(H8) and suppose that  $[1, +\infty[ \subset \rho(A_1(\lambda))$ . According to the Frobenius-Schur factorization, one has

$$\lambda L = I - \begin{pmatrix} I & 0 \\ C_1(\lambda) & I \end{pmatrix} \begin{pmatrix} I - A_1(\lambda) & 0 \\ 0 & I - \overline{M}_1(\lambda) \end{pmatrix} \begin{pmatrix} I & G_1(\lambda) \\ 0 & I \end{pmatrix},$$

where,

$$\begin{aligned} A_1(\lambda) &= \lambda A|_{\mathcal{D}(A) \cap \mathcal{N}(\Gamma_X)}, \\ C_1(\lambda) &= \lambda C(A_1(\lambda) - I)^{-1}, \\ K_1(\lambda) &= (\Gamma_X|_{\mathcal{N}(\lambda A - I)})^{-1}, \\ \Gamma_Y(\lambda) &= \Gamma_X|_{\mathcal{N}(\lambda A - I)}, \\ \overline{\Gamma}_Y^\circ(\lambda) &\text{ the extension of } \Gamma_Y(\lambda)|_{Y_1}, \\ M_1(\lambda) &= \lambda(D + CK_1(\lambda)\Gamma_Y(\lambda) - C_1(\lambda)B), \text{ and} \\ G_1(\lambda) &= -\overline{K}_1(\lambda)\overline{\Gamma}_Y^\circ(\lambda) + \overline{\lambda(A_1(\lambda) - I)^{-1}B}. \end{aligned}$$

**Theorem 5.4.** *Let assumptions (H1)-(H8) hold and let  $\lambda \in \mathbb{C}$  such that  $CK_1(\lambda)\Gamma_Y(\lambda) - C_1(\lambda)B$  is bounded on  $Y$  and  $[1, +\infty[ \subset \rho(A_1(\lambda))$ . If  $\lambda D$  is demicompact and  $I - \lambda D$  has a left (resp. right) Fredholm inverse  $D_l$  (resp.  $D_r$ ) such that  $\lambda(CK_1(\lambda)\Gamma_Y(\lambda) - C_1(\lambda)B)D_l$  (resp.  $\lambda D_r(CK_1(\lambda)\Gamma_Y(\lambda) - C_1(\lambda)B)$ ) is demicompact, then  $\lambda L \in \mathcal{DC}(X \times Y)$ .*

*Proof.* Let  $(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix})_n$  be a bounded sequence in  $\mathcal{D}(L)$  which verifies

$$\begin{pmatrix} x'_n \\ y'_n \end{pmatrix} := (I - \lambda L) \begin{pmatrix} x_n \\ y_n \end{pmatrix} \rightarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

It follows from Eq. (5.2) that

$$\begin{pmatrix} x'_n \\ y'_n \end{pmatrix} = \begin{pmatrix} I & 0 \\ C_1(\lambda) & I \end{pmatrix} \begin{pmatrix} I - A_1(\lambda) & 0 \\ 0 & I - \overline{M}_1(\lambda) \end{pmatrix} \begin{pmatrix} I & G_1(\lambda) \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

thus,

$$\begin{pmatrix} I & 0 \\ -C_1(\lambda) & I \end{pmatrix} \begin{pmatrix} x'_n \\ y'_n \end{pmatrix} = \begin{pmatrix} I - A_1(\lambda) & 0 \\ 0 & I - \overline{M}_1(\lambda) \end{pmatrix} \begin{pmatrix} I & G_1(\lambda) \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

which allows us to get the following system

$$(5.7) \quad \begin{cases} x'_n = (I - A_1(\lambda))x_n + (I - A_1(\lambda))G_1(\lambda)y_n, \\ -C_1(\lambda)x'_n + y'_n = (I - \overline{M}_1(\lambda))y_n. \end{cases}$$

Since both  $\lambda D$  and  $\lambda(CK_1(\lambda)\Gamma_Y(\lambda) - C_1(\lambda)B)D_l$  (resp.  $\frac{\lambda}{\alpha}D_r(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha}) - C_1(\frac{\lambda}{\alpha})B)$ ) are demicompact, we infer from Proposition 2.2 that the operator  $\lambda D + \lambda(CK_1(\lambda)\Gamma_Y(\lambda) - C_1(\lambda)B)$  is such too. Now, it is easy to show that if a closable operator is demicompact, then its closure is also demicompact. Consequently, we get the demicompactness of  $\overline{M}_1(\lambda)$ . Moreover, it should be observed that the second equation of the system (5.7) implies the convergence

of  $((I - \overline{M}_1(\lambda))y_n)_n$ , hence  $(y_n)_n$  has a convergent subsequence. Next, the first equation of the system (5.7) implies that  $(x_n)_n$  has a convergent subsequence. Therefore, there exists a convergent subsequence of  $(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix})_n$  which converges in  $\mathcal{D}(L)$ . Hence, the demicompactness of  $\lambda L$  is proved.  $\square$

The following corollary gives a sufficient condition to guarantee the demicompactness of  $L$ , the closure of the closable matrix operator  $L_0$ .

**Corollary 5.1.** *Let assumptions (H1)-(H8) hold. Suppose that  $CK_1\Gamma_Y - C_1B$  is bounded on  $Y$  and  $[1, +\infty[ \subset \rho(A_1)$ . If  $D$  is demicompact and  $I - D$  has a left (resp. right) Fredholm inverse  $D_l$  (resp.  $D_r$ ) such that  $(CK_1\Gamma_Y - C_1B)D_l$  (resp.  $D_r(CK_1\Gamma_Y - C_1B)$ ) is demicompact, then  $L \in \mathcal{DC}(X \times Y)$ .*

*Proof.* The proof is a direct application of Theorem 5.4 for  $\lambda = 1$ .  $\square$

## 6. Essential spectra of matrix operators by means of demicompactness classes

We start this section by giving some notations that we will need in the proof. Assume that  $L_0$  satisfies (H1)-(H8) and let  $\alpha \in \mathbb{C} \setminus \{0\}$ . Suppose that  $[1, +\infty[ \subset \rho(A_1(\frac{1}{\alpha}))$ . Applying Remark 5.4 on the operator  $\frac{1}{\alpha}L$  and for the case  $\lambda = 1$ , one has

$$\begin{aligned} I - \frac{1}{\alpha}L &= \begin{pmatrix} I & 0 \\ C_\mu(\frac{1}{\alpha}) & I \end{pmatrix} \begin{pmatrix} I - A_1(\frac{1}{\alpha}) & 0 \\ 0 & I - \overline{M}_\mu(\frac{1}{\alpha}) \end{pmatrix} \begin{pmatrix} I & G_\mu(\frac{1}{\alpha}) \\ 0 & I \end{pmatrix} \\ &\quad - (1 - \mu)N_\mu(\frac{1}{\alpha}) \\ (6.1) \quad &:= UV(1)W - (1 - \mu)N_\mu(\frac{1}{\alpha}), \end{aligned}$$

where

$$N_\mu(\frac{1}{\alpha}) := \begin{pmatrix} 0 & G_\mu(\frac{1}{\alpha}) \\ C_\mu(\frac{1}{\alpha}) & C_\mu(\frac{1}{\alpha})G_\mu(\frac{1}{\alpha}) \end{pmatrix}.$$

**Theorem 6.1.** *Let assumptions (H1)-(H8) hold and suppose that  $[1, +\infty[ \subset \rho(A_1(\frac{1}{\alpha}))$ , then we have:*

(i) *If for all  $\alpha \in \mathbb{C} \setminus \{0\}$ , the operator  $\frac{1}{\alpha}D$  is demicompact and*

$$\frac{1}{\alpha}(CK_1(\frac{1}{\alpha})\Gamma_Y(\frac{1}{\alpha}) - C_1(\frac{1}{\alpha})B)$$

*is bounded on  $Y$  and if  $I - \frac{1}{\alpha}D$  has a left (resp. right) Fredholm inverse  $D_l$  (resp.  $D_r$ ) such that  $\frac{1}{\alpha}(CK_1(\frac{1}{\alpha})\Gamma_Y(\frac{1}{\alpha}) - C_1(\frac{1}{\alpha})B)D_l$  (resp.  $\frac{1}{\alpha}D_r(CK_1(\frac{1}{\alpha})\Gamma_Y(\frac{1}{\alpha}) - C_1(\frac{1}{\alpha})B)$ ) is demicompact and  $(1 - \mu)N_\mu(\frac{1}{\alpha}) \in \mathcal{F}_+(X \times Y)$ , then*

$$\sigma_{e_1}(L) \setminus \{0\} = \sigma_{e_1}(A_1) \setminus \{0\} \cup \sigma_{e_1}(\alpha \overline{M}_\mu(\frac{1}{\alpha})) \setminus \{0\}.$$

(ii) *If for all  $\lambda \in [0, 1]$  and  $\alpha \in \mathbb{C} \setminus \{0\}$  the operator  $\frac{\lambda}{\alpha}D$  is demicompact and*

$$\frac{\lambda}{\alpha}(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha}) - C_1(\frac{\lambda}{\alpha})B)$$



is bounded on  $Y$  and if  $I - \frac{\lambda}{\alpha}D$  has a left (resp. right) Fredholm inverse  $D_l$  (resp.  $D_r$ ) such that  $\frac{\lambda}{\alpha}(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha}) - C_1(\frac{\lambda}{\alpha})B)D_l$  (resp.  $\frac{\lambda}{\alpha}D_r(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha}) - C_1(\frac{\lambda}{\alpha})B)$ ) is demicompact and  $(1 - \mu)N_\mu(\frac{\lambda}{\alpha}) \in \mathcal{F}(X \times Y)$ , then

$$\sigma_{e_i}(L) \setminus \{0\} = \sigma_{e_i}(A_1) \setminus \{0\} \cup \sigma_{e_i}(\alpha \overline{M}_\mu(\frac{1}{\alpha})) \setminus \{0\}, \text{ where } i \in \{4, 5\},$$

and

$$\sigma_{e_i}(L) \setminus \{0\} \subseteq \sigma_{e_i}(A_1) \setminus \{0\} \cup \sigma_{e_i}(\alpha \overline{M}_\mu(\frac{1}{\alpha})) \setminus \{0\}, \text{ where } i \in \{7, 8\}.$$

*Proof.* (i) Let  $\alpha \in \mathbb{C} \setminus \{0\}$  be such that  $\alpha \notin \sigma_{e_1}(L)$ . Then,

$$(6.2) \quad \alpha - L = \alpha(I - \frac{1}{\alpha}L) \in \Phi_+(X \times Y).$$

Clearly,  $\alpha I$  is an upper semi Fredholm operator. We get then the following equivalence

$$\alpha - L \in \Phi_+(X \times Y) \iff (I - \frac{1}{\alpha}L) \in \Phi_+(X \times Y).$$

Since both  $\frac{1}{\alpha}D$  and  $\frac{1}{\alpha}(CK_1(\frac{1}{\alpha})\Gamma_Y(\frac{1}{\alpha}) - C_1(\frac{1}{\alpha})B)D_l$  (resp.  $\frac{1}{\alpha}D_r(CK_1(\frac{1}{\alpha})\Gamma_Y(\frac{1}{\alpha}) - C_1(\frac{1}{\alpha})B)$ ) are demicompact, it follow from Theorem 5.4 that the operator  $\frac{1}{\alpha}L$  is such too. Hence, thanks to Theorem 2.1, the operator  $I - \frac{1}{\alpha}L \in \Phi_+(X \times Y)$ . Using the fact that  $(1 - \mu)N_\mu(\frac{1}{\alpha}) \in \mathcal{F}_+(X \times Y)$ , we infer that  $I - \frac{1}{\alpha}L \in \Phi_+(X \times Y)$  if, and only if, the operator  $UV(1)W$  is such too. Now, observe that  $U$  and  $W$  are invertible and have bounded inverses, hence  $I - \frac{1}{\alpha}L \in \Phi_+(X \times Y)$  if, and only if,  $V(1)$  has this property, if and only if, both  $I - A_1(\frac{1}{\alpha})$  and  $I - \overline{M}_\mu(\frac{1}{\alpha})$  are demicompact operators. Equivalently, the operators  $\alpha - A_1$  and  $\alpha - \alpha \overline{M}_\mu(\frac{1}{\alpha})$  are demicompact. Thus,

$$\sigma_{e_1}(L) \setminus \{0\} = \sigma_{e_1}(A_1) \setminus \{0\} \cup \sigma_{e_1}(\alpha \overline{M}_\mu(\frac{1}{\alpha})) \setminus \{0\}.$$

(ii) We claim that

$$\sigma_{e_4}(L) \setminus \{0\} = \sigma_{e_4}(A_1) \setminus \{0\} \cup \sigma_{e_4}(\alpha \overline{M}_\mu(\frac{1}{\alpha})) \setminus \{0\}.$$

For this purpose, take  $\alpha \in \mathbb{C} \setminus \{0\}$ . Since  $\alpha I$  is a Fredholm operator, then  $\alpha - L \in \Phi(X \times Y)$  if, and only if, the operator  $(I - \frac{1}{\alpha}L) \in \Phi(X \times Y)$ . Next, using the demicompactness of both  $\frac{\lambda}{\alpha}D$  and  $\frac{\lambda}{\alpha}(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha}) - C_1(\frac{\lambda}{\alpha})B)D_l$  (resp.  $\frac{\lambda}{\alpha}D_r(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha}) - C_1(\frac{\lambda}{\alpha})B)$ ) for all  $\lambda \in [0, 1]$ , we deduce from Theorem 5.4 that the operator  $\frac{\lambda}{\alpha}L$  is demicompact. Hence, according to Theorem 2.2, we have  $I - \frac{1}{\alpha}L \in \Phi(X \times Y)$ . Using Eq. (6.1) and the fact that  $(1 - \mu)N_\mu(\frac{1}{\alpha}) \in \mathcal{F}(X \times Y)$ , we infer that  $I - \frac{1}{\alpha}L$  is a Fredholm operator if, and only if, the operator  $UV(1)W$  is such too. Now, observe that  $U$  and  $W$  are invertible and have bounded inverses, hence  $I - \frac{1}{\alpha}L \in \Phi(X \times Y)$  if, and only if,  $V(1)$  has this property if, and only if,  $\alpha - A_1$  and  $\alpha - \alpha \overline{M}_\mu(\frac{1}{\alpha})$  are Fredholm operators. Thus, the desired result follows.

Now, we prove the same equality for the Schechter's essential spectrum. To this end, we take  $\alpha \in \mathbb{C} \setminus \{0\}$ . It is easy to see that  $\alpha - L \in \Phi(X \times Y)$  and  $i(\alpha - L) = 0$  if, and only if, the operator  $(I - \frac{1}{\alpha}L) \in \Phi(X \times Y)$  and  $i(I - \frac{1}{\alpha}L) = 0$ . By demicompactness of both  $\frac{\lambda}{\alpha}D$  and  $\frac{\lambda}{\alpha}(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha}) - C_1(\frac{\lambda}{\alpha})B)D_i$  (resp.  $\frac{\lambda}{\alpha}D_r(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha}) - C_1(\frac{\lambda}{\alpha})B)$ ) for all  $\lambda \in [0, 1]$ , we get from Theorem 5.4 that the operator  $\frac{\lambda}{\alpha}L$  is demicompact. Hence, according to Theorem 2.2, the operator  $I - \frac{1}{\alpha}L \in \Phi(X \times Y)$  and  $i(I - \frac{1}{\alpha}L) = 0$ . Using Eq. (6.1) and the fact that  $(1 - \mu)N_\mu(\frac{1}{\alpha}) \in \mathcal{F}(X \times Y)$ , we infer that  $I - \frac{1}{\alpha}L$  is a Fredholm operator with index zero if, and only if, the operator  $UV(1)W$  is such too. Note that  $U$  and  $W$  are invertible and have bounded inverses, then  $I - \frac{1}{\alpha}L$  is Fredholm with index zero if, and only if,  $V(1)$  has this property, if and only if,  $I - A_1(\frac{1}{\alpha})$  and  $I - \overline{M}_\mu(\frac{1}{\alpha})$  are Fredholm operators with indexes zero. Therefore,  $\alpha - A_1$  and  $\alpha - \alpha\overline{M}_\mu(\frac{1}{\alpha})$  have also this property. Hence  $\alpha \notin \sigma_{e_5}(A_1) \setminus \{0\} \cap \sigma_{e_5}(\alpha\overline{M}_\mu(\frac{1}{\alpha})) \setminus \{0\}$ . Thus,

$$(6.3) \quad \sigma_{e_5}(A_1) \setminus \{0\} \cup \sigma_{e_5}(\alpha\overline{M}_\mu(\frac{1}{\alpha})) \setminus \{0\} \subseteq \sigma_{e_5}(L) \setminus \{0\}.$$

Conversely, let  $0 \neq \alpha \notin \sigma_{e_5}(A_1) \cap \sigma_{e_5}(\alpha\overline{M}_\mu(\frac{1}{\alpha}))$ , then  $\alpha - A_1$  and  $\alpha - \alpha\overline{M}_\mu(\frac{1}{\alpha})$  are Fredholm operators with indexes zero. Equivalently,  $I - A_1(\frac{1}{\alpha})$  and  $I - \overline{M}_\mu(\frac{1}{\alpha})$  are such too. The boundedness of the operators  $U$  and  $W$  and their inverses and the fact that  $(1 - \mu)N_\mu(\frac{1}{\alpha}) \in \mathcal{F}(X \times Y)$  imply that  $I - \frac{1}{\alpha}L \in \Phi(X \times Y)$  and  $i(I - \frac{1}{\alpha}L) = 0$ . Therefore,  $\alpha - L \in \Phi(X \times Y)$  and  $i(\alpha - L) = 0$ , hence  $\alpha \notin \sigma_{e_5}(L) \setminus \{0\}$ . This immediately shows that

$$(6.4) \quad \sigma_{e_5}(L) \setminus \{0\} \subseteq \sigma_{e_5}(A_1) \setminus \{0\} \cup \sigma_{e_5}(\alpha\overline{M}_\mu(\frac{1}{\alpha})) \setminus \{0\}.$$

Now, the use of Eqs. (6.3) and (6.4) makes us to conclude that

$$\sigma_{e_5}(L) \setminus \{0\} = \sigma_{e_5}(A_1) \setminus \{0\} \cup \sigma_{e_5}(\alpha\overline{M}_\mu(\frac{1}{\alpha})) \setminus \{0\}.$$

We give now the proof for  $i = 7$ . Note that the case  $i = 8$  can be checked similarly. Let  $\alpha \in \mathbb{C} \setminus \{0\}$ , reasoning in the same way as the case  $i = 5$ , we prove that  $I - \frac{1}{\alpha}L \in \Phi(X \times Y)$  and  $i(I - \frac{1}{\alpha}L) = 0$ . This implies that  $I - \frac{1}{\alpha}L \in \Phi_+(X \times Y)$  and  $i(I - \frac{1}{\alpha}L) \leq 0$ . If  $\alpha \notin \sigma_{e_7}(A_1) \cap \sigma_{e_7}(\alpha\overline{M}_\mu(\frac{1}{\alpha}))$ , then  $\alpha - A_1$  and  $\alpha - \alpha\overline{M}_\mu(\frac{1}{\alpha})$  are upper semi Fredholm operators with negative indexes. It remains to get the same properties for the operators  $I - A_1(\frac{1}{\alpha})$  and  $I - \overline{M}_\mu(\frac{1}{\alpha})$ . Since  $U$  and  $W$  are invertible and have bounded inverses and using the fact that  $(1 - \mu)N_\mu(\frac{1}{\alpha}) \in \mathcal{F}_+(X \times Y)$ , we infer that  $I - \frac{1}{\alpha}L \in \Phi_+(X \times Y)$  and  $i(I - \frac{1}{\alpha}L) \leq 0$ . Therefore,  $\alpha - L \in \Phi_+(X \times X)$  and  $i(\alpha - L) \leq 0$ . Now, by applying Lemma 2.5, we conclude that  $\alpha \notin \sigma_{e_7}(L) \setminus \{0\}$  and then,

$$\sigma_{e_7}(L) \setminus \{0\} \subseteq \sigma_{e_7}(A_1) \setminus \{0\} \cup \sigma_{e_7}(\alpha\overline{M}_\mu(\frac{1}{\alpha})) \setminus \{0\}.$$

Hence, the theorem is proved.  $\square$

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