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DETERMINANTAL EXPRESSION OF THE GENERAL SOLUTION TO A RESTRICTED SYSTEM OF QUATERNION MATRIX EQUATIONS WITH APPLICATIONS

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Abstract. In this paper, we mainly consider the determinantal representations of the unique solution and the general solution to the restricted system of quaternion matrix equations

$$\begin{cases} A_1X = C_1 \\ XB_2 = C_2, \end{cases} \mathcal{R}_r(X) \subseteq T_1,\ \mathcal{N}_r(X) \supseteq S_1,$$

respectively. As an application, we show the determinantal representations of the general solution to the restricted quaternion matrix equation

$$AX + YB = E,\ \mathcal{R}_r(X) \subseteq T_1,\ \mathcal{N}_r(X) \supseteq S_1,\ \mathcal{R}_l(Y) \subseteq T_2,\ \mathcal{N}_l(Y) \supseteq S_2.$$

The findings of this paper extend some known results in the literature.

1. Introduction

Throughout, we denote the real number field by $\mathbb{R}$, the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = ijk = -1,\ a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m\times n}$, the identity matrix with the appropriate size by I. For $A \in \mathbb{H}^{m\times n}$, the symbols A^* stands for the conjugate transpose of A. The Moore-Penrose inverse of A, denoted by $A^\dagger$, is the unique matrix $X \in \mathbb{H}^{n\times m}$ satisfying the Penrose equations

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA.$$

Further, $P_A = A^\dagger A$, $Q_A = AA^\dagger$, $R_A = I_m - AA^\dagger$ and $L_A = I_n - A^\dagger A$ stand for some orthogonal projectors induced from A.

The quaternions were first explored by the Irish mathematician Sir William Rowan Hamilton in [15]. Quaternions have massive applications in diverse

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areas of mathematics like computation, geometry and algebra (see, e.g. [10, 31, 36]). Nowadays quaternion matrices play a remarkable role in control theory, mechanics, altitude control, quantum physics and signal processing (see, e.g. [1, 16–18, 28]). As a crucial technology for color image copyright protection, watermarking technology has been extensively researched and used. For the color image watermarking technology, quaternions forming the Cayley-Dickson algebra of order 4 have a structure suitable to apply in color image. Sangwine et al. [34, 35] interpreted the imaginary part of a quaternion in terms of three components of a color image: R (red), G (green) and B (blue) which means that all color components of the image are treated together, as opposed to processing each of the three components independently. That is why quaternions have found numerous applications in the field of color image processing. Moreover, when consider some engineering problems, we need to solve many different kinds of equations or linear systems (see, e.g. [3, 25–27, 29]). Constant coefficient quaternion differential equations [14] which can be transformed into linear quaternion matrix equations, play an important role in developing attitude propagation algorithms for inertial navigation or attitude estimation onboard spacecraft. Thus it is interesting and important to study the solution of linear quaternion matrix equations. (see, e.g. [32, 47]).

In 1970, Steve Robinson [33] gave an elegant proof of Cramer's rule over the complex number field. After that, using Cramer's rules to represent the generalized inverses and different solutions of some restricted equations have been studied by many authors (see, e.g. [4, 5, 8, 41–44, 46]). In Chapter 3 of [44], Wang, Wei and Qiao surveyed the results on the Cramer's rules over complex field. Known from their work, Cramer's rule is only used as a basic method to express the unique solution to some consistent matrix equation or the best approximate solution to some inconsistent matrix equation. To our best knowledge, there has been little research on expressing the general solution of the restricted system of matrix equations

$$\left\{\begin{array}{l} A_1X = C_1 \\ XB_2 = C_2, \end{array}\right. \mathcal{R}_r(X) \subseteq T_1,\ \mathcal{N}_r(X) \supseteq S_1 \tag{1}$$

and the restricted matrix equation

$$AX + YB = E,\ \mathcal{R}_r(X) \subseteq T_1,\ \mathcal{N}_r(X) \supseteq S_1,\ \mathcal{R}_l(Y) \subseteq T_2,\ \mathcal{N}_l(Y) \supseteq S_2 \tag{2}$$

by Cramer's rules.

Unlike multiplication of real or complex numbers, multiplication of quaternions is not commutative. Many authors (see, e.g. [2, 6, 7, 9, 11–13]) had tried to give the definitions of the determinant of a quaternion matrix. Unfortunately, by their definitions it is impossible for us to give a determinantal representation of an inverse of matrix. In 2008, Kyrchei [19] defined the row and column determinants of a square matrix over the quaternion skew field, and derived the Cramer's rule for some quaternionic system of linear equations. Some other results relate to the row and column determinant of quaternion matrix with applications can be founded in [20–24, 37–40].

Motivated by the work mentioned above, and keep the interesting of the row and column determinant theory of quaternion matrix, we in this paper aim to consider a series of determinantal expressions for the general solutions to the restricted system (1) and matrix equation (2), respectively. The paper is organized as follows. In Section 2, when (1) is consistent, we derive some determinantal representations for its unique solution and general solution, respectively. In Section 3, we derive the determinantal representation for the general solution of (2). To conclude this paper, in Section 4 we propose some further research topics.

2. Determinantal expressions for the unique solution and the general solution to (1)

In this section, we will consider the determinantal expressions for the unique solution and the general solution to the restricted system of matrix equations (1), respectively. We begin this section with the following results. Suppose S_n is the symmetric group on the set $I_n = \{1, \ldots, n\}$.

Definition 2.1 (Definitions 2.4-2.5 [19]). (1) The ith row determinant of $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ is defined by

$$\operatorname{rdet}_i A = \sum_{\sigma\in S_n} (-1)^{n-r} a_{ii_{k_1}} a_{i_{k_1} i_{k_1+1}} \cdots a_{i_{k_1+l_1} i} \cdots a_{i_{k_r} i_{k_r+1}} \cdots a_{i_{k_r+l_r} i_{k_r}}$$

for all $i = 1, \ldots, n$. The elements of the permutation σ are indices of each monomial. The left-ordered cycle notation of the permutation σ is written as follows:

$$\sigma = (i i_{k_1} i_{k_1+1} \cdots i_{k_1+l_1}) (i_{k_2} i_{k_2+1} \cdots i_{k_2+l_2}) \cdots (i_{k_r} i_{k_r+1} \cdots i_{k_r+l_r}).$$

The index i opens the first cycle from the left and other cycles satisfy the following conditions, $i_{k_2} < i_{k_3} < \cdots < i_{k_r}$ and $i_{k_t} < i_{k_t+s}$ for all $t = 2, \ldots, r$ and $s = 1, \ldots, l_t$.

(2) The jth column determinant of $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ is defined by

$$\operatorname{cdet}_j A = \sum_{\tau\in S_n} (-1)^{n-r} a_{j_{k_r} j_{k_r+l_r}} \cdots a_{j_{k_r+1} j_{k_r}} \cdots a_{j j_{k_1+l_1}} \cdots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j}$$

for all $j = 1, \ldots, n$. The elements of the permutation τ are indices of each monomial. The right-ordered cycle notation of the permutation τ is written as follows:

$$\tau = (j_{k_r+l_r} \cdots j_{k_r+1} j_{k_r}) (j_{k_2+l_2} \cdots j_{k_2+1} j_{k_2}) \cdots (j_{k_1+l_1} \cdots j_{k_1+1} j_{k_1} j).$$

The index j opens the first cycle from the right and other cycles satisfy the following conditions, $j_{k_2} < j_{k_3} < \cdots < j_{k_r}$ and $j_{k_t} < j_{k_t+s}$ for all $t = 2, \ldots, r$ and $s = 1, \ldots, l_t$.

Suppose that $A_{.j}(b)$ denotes the matrix obtained from A by replacing its jth column with the column b, and $A_{i.}(b)$ denotes the matrix obtained form A by replacing its ith row with the row b.

Lemma 2.1 ([20]). *Suppose that $A, B, C \in \mathbb{H}^{n\times n}$ are given, and $X \in \mathbb{H}^{n\times n}$ is unknown. If $\det(A^*A) \neq 0$ and $\det(BB^*) \neq 0$, then $AXB = C$ has a unique solution, which can be written as*

$$x_{ij} = \frac{\operatorname{rdet}_j (BB^*)_{j.}\left(c_{i.}^A\right)}{\det(A^*A)\det(BB^*)} \text{ or } x_{ij} = \frac{\operatorname{cdet}_j (A^*A)_{.i}\left(c_{.j}^B\right)}{\det(A^*A)\det(BB^*)},$$

where

$$c_{i.}^A := \begin{bmatrix} \operatorname{cdet}_i (A^*A)_{.i}(d_{.1}) & ,\ldots, & \operatorname{cdet}_i (A^*A)_{.i}(d_{.n}) \end{bmatrix}$$
$$c_{.j}^B := \begin{bmatrix} \operatorname{rdet}_j (BB^*)_{j.}(d_{1.}) & ,\ldots, & \operatorname{rdet}_j (BB^*)_{j.}(d_{n.}) \end{bmatrix}^T$$

with $d_{i.}$, $d_{.j}$ are the ith row vector and jth column vector of A^*CB^*, respectively, for all $i, j = 1, \ldots, n$.

The following lemma is given by Mitra [30], which can be generalized into the quaternion skew filed.

Lemma 2.2. (1) *Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{p\times q}$, $C \in \mathbb{H}^{m\times q}$ be known and $X \in \mathbb{H}^{n\times p}$ be unknown. Then the matrix equation $AXB = C$ is consistent if and only if $AA^\dagger CB^\dagger B = C$. In this case, its general solution can be expressed as*

$$X = A^\dagger CB^\dagger + L_A U + V R_B = A^\dagger CB^\dagger + Z - A^\dagger AZBB^\dagger,$$

where U, V and Z are arbitrary matrices over $\mathbb{H}$ with appropriate dimensions.

(2) *Let $A_i \in \mathbb{H}^{m_i\times n}$, $B_i \in \mathbb{H}^{p\times q_i}$, $C_i \in \mathbb{H}^{m_i\times q_i}$, $i = 1, 2$ be known and $X \in \mathbb{H}^{n\times p}$ be unknown. Denote $A_1^*A_1 + A_2^*A_2 = T$, $B_1B_1^* + B_2B_2^* = S$, then a necessary and sufficient condition for the consistent equations $A_1XB_1 = C_1, A_2XB_2 = C_2$ to have a common solution is*

$$A_1^*A_1T^\dagger A_2^*C_2B_2^*S^\dagger B_1B_1^* = A_2^*A_2T^\dagger A_1^*C_1B_1^*S^\dagger B_2B_2^*.$$

Then we can show the main results of this section.

Theorem 2.3. *Suppose that $A_1 \in \mathbb{H}^{m\times n}$, $B_2 \in \mathbb{H}^{p\times q}$, $C_1 \in \mathbb{H}^{m\times p}$, $C_2 \in \mathbb{H}^{n\times q}$, $T_1 \subset \mathbb{H}^n$ and $S_1 \subset \mathbb{H}^p$ are known. Then we can get the following results.*

(a) (1) *is consistent if and only if*

(3) $\mathcal{R}_r(C_1) \subseteq A_1T_1,\ \mathcal{N}_r(C_1) \supseteq S_1,\ \mathcal{R}_r(C_2) \subseteq T_1,\ \mathcal{N}_r(C_2) \supseteq S_1B_2$ *and* $A_1C_2 = C_1B_2$.

In this case, the general solution of (1) *can be written as*

$$X = X_0 + P_{T_1\cap\mathcal{N}_r(A_1)}W_1P_{S_1^\perp\cap\mathcal{N}_r(B_2^*)}, \tag{4}$$

where

$$X_0 = (A_1P_{T_1})^\dagger C_1P_{S_1^\perp} + P_{T_1\cap\mathcal{N}_r(A_1)}\left(C_2 - (A_1P_{T_1})^\dagger C_1P_{S_1^\perp}B_2\right)\left(P_{S_1^\perp}B_2\right)^\dagger,$$

and W_1 is an arbitrary matrix with proper size.

(b) *If the equalities in* (3) *are all satisfied and* $T_1 \cap \mathcal{N}_r(A_1) = 0$ *or* $S_1^{\perp} \cap \mathcal{N}_r(B_2^*) = 0$, *then the solution of* (1) *is unique. Let* E_1^*, F_1 *be two full column rank matrices such that* $T_1 = \mathcal{N}_r(E_1), S_1 = \mathcal{R}_r(F_1)$. *Then the unique solution of* (1) *can be expressed as* $X = (x_{ij}) = (A_1 P_{T_1})^{\dagger} C_1 P_{S_1^{\perp}}$, *which possess the determinantal representations*:

$$x_{ij} = \frac{\operatorname{rdet}_j (B_2B_2^* + F_1F_1^*)_{j.} \left(c_{i.}^A\right)}{\det (A_1^*A_1 + E_1^*E_1) \det (B_2B_2^* + F_1F_1^*)} \quad or \tag{5}$$
$$x_{ij} = \frac{\operatorname{cdet}_j (A_1^*A_1 + E_1^*E_1)_{.i} \left(c_{.j}^B\right)}{\det (A_1^*A_1 + E_1^*E_1) \det (B_2B_2^* + F_1F_1^*)},$$

where

$$c_{i.}^A := \left[\ \operatorname{cdet}_i (A_1^*A_1 + E_1^*E_1)_{.i} (d_{.1}) \quad , \ldots, \quad \operatorname{cdet}_i (A_1^*A_1 + E_1^*E_1)_{.i} (d_{.n})\ \right]$$

$$c_{.j}^B := \left[\ \operatorname{rdet}_j (B_2B_2^* + F_1F_1^*)_{j.} (d_{1.}) \quad , \ldots, \quad \operatorname{rdet}_j (B_2B_2^* + F_1F_1^*)_{j.} (d_{n.})\ \right]^T$$

with $d_{i.}$, $d_{.j}$ *are the* i*th row vector and* j*th column vector of* $A_1^*C_1B_2B_2^*$, *respectively, for all* $i = 1, \ldots, n$, $j = 1, \ldots, p$.

(c) *If the equalities in* (3) *are all satisfied,* $T_1 \cap \mathcal{N}_r(A_1) \neq 0$ *and* $S_1^{\perp} \cap \mathcal{N}_r(B_2^*) \neq 0$, *then the solution of* (1) *is not unique. Suppose that* E_2^*, K_2^*, F_2, L_2 *are full column rank matrices such that*

$$T_1 = \mathcal{N}_r(E_2),\ T_1 \cap \mathcal{N}_r(A_1) = \mathcal{R}_r(K_2^*),$$
$$S_1^{\perp} = \mathcal{N}_r(F_2^*),\ S_1^{\perp} \cap \mathcal{N}_r(B_2^*) = \mathcal{R}_r(L_2).$$

In this case, $X = (x_{ij}) \in \mathbb{H}^{n \times p}$ *possess the determinantal representations,*

$$x_{ij} = \frac{\operatorname{rdet}_j (B_2B_2^* + F_2F_2^* + L_2L_2^*)_{j.} \left(c_{i.}^A\right)}{\det (A_1^*A_1 + E_2^*E_2 + K_2^*K_2) \det (B_2B_2^* + F_2F_2^* + L_2L_2^*)}, \tag{6}$$

or

$$x_{ij} = \frac{\operatorname{cdet}_j (A_1^*A_1 + E_2^*E_2 + K_2^*K_2)_{.i} \left(c_{.j}^B\right)}{\det (A_1^*A_1 + E_2^*E_2 + K_2^*K_2) \det (B_2B_2^* + F_2F_2^* + L_2L_2^*)}, \tag{7}$$

where

$$c_{i.}^A := \left[\underset{i}{\operatorname{cdet}} (A_1^*A_1 + E_2^*E_2 + K_2^*K_2)_{.i} (d_{.1}), \ldots, \underset{i}{\operatorname{cdet}} (A_1^*A_1 + E_2^*E_2 + K_2^*K_2)_{.i} (d_{.n})\right]$$

$$c_{.j}^B := \left[\underset{j}{\operatorname{rdet}} (B_2B_2^* + F_2F_2^* + L_2L_2^*)_{j.} (d_{1.}), \ldots, \underset{j}{\operatorname{rdet}} (B_2B_2^* + F_2F_2^* + L_2L_2^*)_{j.} (d_{n.})\right]^T$$

with $d_{i.}$, $d_{.j}$ *are the* i*th row vector and* j*th column vector of*

$$A_1^*C_1B_2B_2^* + A_1^*C_1L_2L_2^* + K_2^*K_2C_2B_2^* + K_2^*K_2X_0L_2L_2^* + K_2^*K_2L_{A_1P_{T_1}}W_2R_{P_{S_1^{\perp}}B_2}L_2L_2^*,$$

respectively, for all $i = 1, \ldots, n$, $j = 1, \ldots, p$ *and* W_2 *is arbitrary.*

Proof. (a) It is easy to prove that if the restricted system (1) is consistent then the equalities in (3) are all satisfied. For the other direction, note that

$$\mathcal{R}_r(X) \subseteq T_1,\ \mathcal{N}_r(X) \supseteq S_1 \Leftrightarrow X = P_{T_1} W P_{S_1^\perp},$$

where W is an arbitrary matrix with proper size. Then the restricted system (1) is consistent if and only if the following system of matrix equations

$$\left\{\begin{array}{l} A_1 P_{T_1} W P_{S_1^\perp} = C_1 \\ P_{T_1} W P_{S_1^\perp} B_2 = C_2, \end{array}\right. \tag{8}$$

is consistent relate to W. By $\mathcal{R}_r(C_1) \subseteq A_1 T_1$, $\mathcal{N}_r(C_1) \supseteq S_1$, $\mathcal{R}_r(C_2) \subseteq T_1$ and $\mathcal{N}_r(C_2) \supseteq S_1 B_2$, we can get the two equations in (8) are consistent, respectively. Moreover, by Lemma 2.2 (2) and note that $A_1 C_2 = C_1 B_2$, then

$$P_{T_1} A_1^* A_1 P_{T_1} (P_{T_1}(A_1^* A_1 + I) P_{T_1})^\dagger P_{T_1} C_2 B_2^* P_{S_1^\perp} \left(P_{S_1^\perp}(I + B_2 B_2^*) P_{S_1^\perp}\right)^\dagger P_{S_1^\perp}$$

$$= P_{T_1} (P_{T_1}(A_1^* A_1 + I) P_{T_1})^\dagger P_{T_1} A_1^* A_1 P_{T_1} C_2 B_2^* P_{S_1^\perp} \left(P_{S_1^\perp}(I + B_2 B_2^*) P_{S_1^\perp}\right)^\dagger P_{S_1^\perp}$$

$$= P_{T_1} (P_{T_1}(A_1^* A_1 + I) P_{T_1})^\dagger P_{T_1} A_1^* C_1 P_{S_1^\perp} B_1 B_1^* P_{S_1^\perp} \left(P_{S_1^\perp}(I + B_2 B_2^*) P_{S_1^\perp}\right)^\dagger P_{S_1^\perp}$$

$$= P_{T_1} (P_{T_1}(A_1^* A_1 + I) P_{T_1})^\dagger P_{T_1} A_1^* C_1 P_{S_1^\perp} \left(P_{S_1^\perp}(I + B_2 B_2^*) P_{S_1^\perp}\right)^\dagger P_{S_1^\perp} B_1 B_1^* P_{S_1^\perp},$$

which is saying that the system (8) is consistent. By Lemma 2.2(1), the general solution of the first equation in (8) can be expressed

$$W = (A_1 P_{T_1})^\dagger C_1 P_{S_1^\perp} + L_{A_1 P_{T_1}} V_1 + V_2 R_{P_{S_1^\perp}},$$

where V_1 and V_2 are arbitrary matrices with proper sizes. After taking it into the second equation in (8), we can get

$$P_{T_1} L_{A_1 P_{T_1}} V_1 P_{S_1^\perp} B_2 = C_2 - P_{T_1} (A_1 P_{T_1})^\dagger C_1 P_{S_1^\perp} B_2.$$

Moreover, V_1 can be expressed as

$$\begin{aligned} V_1 = {} & \left(P_{T_1} L_{A_1 P_{T_1}}\right)^\dagger \left(C_2 - P_{T_1} (A_1 P_{T_1})^\dagger C_1 P_{S_1^\perp} B_2\right) \left(P_{S_1^\perp} B_2\right)^\dagger \\ & + L_{P_{T_1} L_{A_1 P_{T_1}}} W_1 + W_2 R_{P_{S_1^\perp} B_2}, \end{aligned}$$

where W_1 and W_2 are arbitrary. In this case, the general solution of (8) can be expressed as

$$W = W_0 + L_{A_1 P_{T_1}} W_1 R_{P_{S_1^\perp} B_2} + L_{A_1 P_{T_1}} L_{P_{T_1} L_{A_1 P_{T_1}}} W_2 + V_2 R_{P_{S_1^\perp}},$$

with

$$\begin{aligned} W_0 = {} & (A_1 P_{T_1})^\dagger C_1 P_{S_1^\perp} \\ & + L_{A_1 P_{T_1}} \left(P_{T_1} L_{A_1 P_{T_1}}\right)^\dagger \left(C_2 - (A_1 P_{T_1})^\dagger C_1 P_{S_1^\perp} B_2\right) \left(P_{S_1^\perp} B_2\right)^\dagger. \end{aligned}$$

Note that

$$P_{T_1} L_{A_1 P_{T_1}} = P_{T_1} - P_{T_1} (A_1 P_{T_1})^\dagger A_1 P_{T_1} = P_{T_1 \cap \mathcal{N}_r(A_1)},$$

$$R_{P_{S_1^\perp}B_2}P_{S_1^\perp}=P_{S_1^\perp}-P_{S_1^\perp}B_2\left(P_{S_1^\perp}B_2\right)^\dagger P_{S_1^\perp}=P_{S_1^\perp\cap\mathcal{N}_r\left(B_2^*\right)},$$

then the general solution of (1) can be expressed as

$$\begin{aligned}X&=P_{T_1}WP_{S_1^\perp}\\&=P_{T_1}\left(W_0+L_{A_1P_{T_1}}L_{P_{T_1}L_{A_1P_{T_1}}}W_2+L_{A_1P_{T_1}}W_1R_{P_{S_1^\perp}B_2}+V_2R_{P_{S_1^\perp}}\right)P_{S_1^\perp}\\&=P_{T_1}W_0P_{S_1^\perp}+P_{T_1\cap\mathcal{N}_r(A_1)}W_1P_{S_1^\perp\cap\mathcal{N}_r\left(B_2^*\right)},\end{aligned}$$

where W_1 is an arbitrary matrix with proper size.

(b) If $T_1\cap\mathcal{N}_r\left(A_1\right)=0$ or $S_1^\perp\cap\mathcal{N}_r\left(B_2^*\right)=0$, then $P_{T_1\cap\mathcal{N}_r(A_1)}=0$ or $P_{S_1^\perp\cap\mathcal{N}_r\left(B_2^*\right)}=0$. It follows that the solution of (1) is unique. In order to prove the determinantal expression of the unique solution of (1), we need to show that: (1) has the same solutions with the following restricted equation

$$A_1^*A_1X=A_1^*C_1,\ XB_2B_2^*=C_2B_2^*,\mathcal{R}_r\left(X\right)\subseteq T_1,\ \mathcal{N}_r\left(X\right)\supseteq S_1. \tag{9}$$

Firstly, it is easy to show that all the solutions of (1) satisfy (9). For the other direction, suppose that X_0 is an arbitrary solution of (9), then

$$A_1^*A_1X_0=A_1^*C_1,\ X_0B_2B_2^*=C_2B_2^*.$$

On account of

$$\mathcal{R}_r\left(C_1\right)\subseteq\mathcal{R}_r\left(A_1P_{T_1}\right),\ \mathcal{N}_r\left(C_2\right)\supseteq\mathcal{N}_r\left(P_{S_1^\perp}B_2\right),$$

then there exist two matrices W_1 and W_2 such that

$$A_1^*A_1X_0=A_1^*A_1P_{T_1}W_1,\ X_0B_2B_2^*=W_2C_{P_{S_1^\perp}B_2}B_2^*.$$

By the reducing rules, we have

$$A_1X_0=A_1P_{T_1}W_1=C_1,\ C_2=X_0B_2=W_2C_{P_{S_1^\perp}B_2},$$

which is equivalent that X_0 satisfied (1). Next, we will show the determinantal expression of the unique solution of (1). Denote $T_1=\mathcal{N}_r\left(E_1\right),\ S_1=\mathcal{R}_r\left(F_1\right),$ then

$$\mathcal{R}_r\left(X\right)\subseteq T_1\Leftrightarrow E_1X=0,\quad \mathcal{N}_r\left(X\right)\supseteq S_1\Leftrightarrow XF_1=0.$$

In this case, (9) can be rewritten as

$$\begin{bmatrix}A_1^*A_1 & E_1^*\\ E_1 & 0\end{bmatrix}\begin{bmatrix}X & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}B_2B_2^* & F_1\\ F_1^* & 0\end{bmatrix}=\begin{bmatrix}A_1^*C_1B_2B_2^* & 0\\ 0 & 0\end{bmatrix}.$$

Multiply $\begin{bmatrix}I & E_1^*\end{bmatrix}$ and $\begin{bmatrix}I\\ F_1^*\end{bmatrix}$ from the two sides gives

$$\left(A_1^*A_1+E_1^*E_1\right)X\left(B_2B_2^*+F_1F_1^*\right)=A_1^*C_1B_2B_2^*.$$

Note that $A_1^*A_1+E_1^*E_1$ and $B_2B_2^*+F_1F_1^*$ are nonsingular, then by Lemma 2.1 the determinantal expressions of the unique solution of (1) can be expressed as (5).

(c) If $T_1 \cap \mathcal{N}_r(A_1) \neq 0$ and $S_1^{\perp} \cap \mathcal{N}_r(B_2^*) \neq 0$, the solution of (1) is not unique which can be expressed as (4). Next we will show the determinantal expression of the general solution to (1). Suppose that E_2^*, K_2^*, F_2, L_2 are full column rank matrices such that

$$T_1 = \mathcal{N}_r(E_2),\ T_1 \cap \mathcal{N}_r(A_1) = \mathcal{R}_r(K_2^*),$$
$$S_1^{\perp} = \mathcal{N}_r(F_2^*),\ S_1^{\perp} \cap \mathcal{N}_r(B_2^*) = \mathcal{R}_r(L_2).$$

Denote

$$T_{11} = \mathcal{N}_r\begin{pmatrix} E_2 \\ K_2 \end{pmatrix} \text{ and } S_{11} = \mathcal{N}_r\begin{pmatrix} F_2^* \\ L_2^* \end{pmatrix},$$

then it is easy to prove

$$(E_2^* E_2 + K_2^* K_2) P_{T_{11}} = 0,\ P_{S_{11}^{\perp}}(F_2 F_2^* + L_2 L_2^*) = 0,$$
$$K_2 P_{T_1} = K_2,\ P_{S_1^{\perp}} L_2 = L_2,\ E_2 P_{T_1} = 0,\ P_{S_1^{\perp}} F_2 = 0.$$

By the results in (a), the general solution of (1) can be expressed as (4). It can be verified that

$$\begin{aligned}
&(A_1^* A_1 + E_2^* E_2 + K_2^* K_2)\left(P_{T_1} W_0 P_{S_1^{\perp}} + P_{T_1} L_{A_1 P_{T_1}} W_1 R_{P_{S_1^{\perp}} B_2} P_{S_1^{\perp}}\right)\\
&(B_2 B_2^* + F_2 F_2^* + L_2 L_2^*)\\
=\ & A_1^* A_1 P_{T_1} W_0 P_{S_1^{\perp}} B_2 B_2^* + A_1^* A_1 P_{T_1} W_0 P_{S_1^{\perp}} L_2 L_2^* + K_2^* K_2 P_{T_1} W_0 P_{S_1^{\perp}} B_2 B_2^*\\
&+ K_2^* K_2 P_{T_1} W_0 P_{S_1^{\perp}} L_2 L_2^* + K_2^* K_2 P_{T_1} L_{A_1 P_{T_1}} W_1 R_{P_{S_1^{\perp}} B_2} P_{S_1^{\perp}} L_2 L_2^*\\
=\ & A_1^* C_1 B_2 B_2^* + A_1^* C_1 L_2 L_2^* + K_2^* K_2 C_2 B_2^* + K_2^* K_2 P_{T_1} W_0 P_{S_1^{\perp}} L_2 L_2^*\\
&+ K_2^* K_2 L_{A_1 P_{T_1}} W_1 R_{P_{S_1^{\perp}} B_2} L_2 L_2^*\\
=\ & A_1^* C_1 B_2 B_2^* + A_1^* C_1 L_2 L_2^* + K_2^* K_2 C_2 B_2^* + W,
\end{aligned}$$

where

$$W = K_2^* K_2 P_{T_1} W_0 P_{S_1^{\perp}} L_2 L_2^* + K_2^* K_2 L_{A_1 P_{T_1}} W_1 R_{P_{S_1^{\perp}} B_2} L_2 L_2^*.$$

Note that

$$T_{11} \cap \mathcal{N}_r(A_1) = 0 \text{ and } S_{11} \cap \mathcal{N}_r(B_2^*) = 0,$$

thus $A_1^* A_1 + E_2^* E_2 + K_2^* K_2$ and $B_2 B_2^* + F_2 F_2^* + L_2 L_2^*$ are nonsingular, and X can be written as

$$\begin{aligned}
X =\ & (A_1^* A_1 + E_2^* E_2 + K_2^* K_2)^{-1}(A_1^* C_1 B_2 B_2^* + A_1^* C_1 L_2 L_2^* + K_2^* K_2 C_2 B_2^* + W)\\
& (B_2 B_2^* + F_2 F_2^* + L_2 L_2^*)^{-1}.
\end{aligned}$$

By Lemma 2.1 the general solution of (1) can be expressed as (6)-(7). □

As applications, we can get the following results.

Corollary 2.4. *Suppose that $A_1 \in \mathbb{H}^{m\times n}$, $B_2 \in \mathbb{H}^{p\times q}$, $C_1 \in \mathbb{H}^{m\times p}$ and $C_2 \in \mathbb{H}^{n\times q}$ are given such that the system of matrix equations*

$$A_1 X = C_1,\ X B_2 = C_2 \tag{10}$$

is consistent. Let E^* *and* F *be two full column rank matrices such that* $\mathcal{N}_r(A_1) = \mathcal{R}_r(E^*)$ *and* $\mathcal{N}_r(B_2^*) = \mathcal{R}_r(F)$. *In this case, the general solution of* (10) *possess the following determinantal representations*:

$$x_{ij} = \frac{\operatorname{rdet}_j \left(B_2B_2^* + FF^*\right)_{j.} \left(c_{i.}^A\right)}{\det\left(A_1^*A_1 + E^*E\right)\det\left(B_2B_2^* + FF^*\right)} \text{ or } \tag{11}$$
$$x_{ij} = \frac{\operatorname{cdet}_j \left(A_1^*A_1 + E^*E\right)_{.i} \left(c_{.j}^B\right)}{\det\left(A_1^*A_1 + E^*E\right)\det\left(B_2B_2^* + FF^*\right)},$$

where

$$c_{i.}^A := \left[\begin{array}{ccc} \operatorname{cdet}_i\left(A_1^*A_1 + E^*E\right)_{.i}(d_{.1}) & ,\ldots, & \operatorname{cdet}_i\left(A_1^*A_1 + E^*E\right)_{.i}(d_{.n}) \end{array}\right],$$

$$c_{.j}^B := \left[\begin{array}{ccc} \operatorname{rdet}_j\left(B_2B_2^* + FF^*\right)_{j.}(d_{1.}) & ,\ldots, & \operatorname{rdet}_j\left(B_2B_2^* + FF^*\right)_{j.}(d_{n.}) \end{array}\right]^T,$$

with $d_{i.}$, $d_{.j}$ *are the* i*th row vector and* j*th column vector of*

$$A_1^*C_1\left(B_2B_2^* + FF^*\right) + E^*EC_2B_2^* + E^*EL_{A_1}VR_{B_2}FF^*,$$

respectively, for all $i = 1, \ldots, n$, $j = 1, \ldots, p$, *with an arbitrary matrix* $V \in \mathbb{H}^{n\times p}$.

Proof. Similarly, we can choose two full column rank matrices E^* and F such that $\mathcal{N}_r(A_1) = \mathcal{R}_r(E^*)$ and $\mathcal{N}_r(B^*) = \mathcal{R}_r(F)$. Suppose that X is an arbitrary solution to (10), then we can prove

$$\begin{aligned} &\left(A_1^*A_1 + E^*E\right)X\left(B_2B_2^* + FF^*\right) \\ =\ & A_1^*C_1B_2B_2^* + A_1^*C_1FF^* + E^*EC_2B_2^* + E^*EL_{A_1}VR_{B_2}FF^*, \end{aligned}$$

where V is an arbitrary matrix with proper size. Note that $A_1^*A_1 + E^*E$ and $B_2B_2^* + FF^*$ are nonsingular, then by Lemma 2.1, the general solution to (10) can be expressed as (11). □

Corollary 2.5. *Suppose that* $A \in \mathbb{H}^{m\times n}$ *and* $C \in \mathbb{H}^{m\times n}$ *are given such that* $AX = C$ *has a Hermitian solution. Let* E^* *be a full column rank matrix such that* $\mathcal{N}_r(A) = \mathcal{R}_r(E^*)$. *In this case, its Hermitian solution can be expressed as* $X = \frac{1}{2}\left(X_1 + X_1^*\right)$ *where* $X_1 = (x_{ij})$ *possess the following determinantal representations*

$$x_{ij} = \frac{\operatorname{rdet}_j\left(A^*A + E^*E\right)_{j.}\left(c_{i.}^A\right)}{\det\left(A^*A + E^*E\right)^2} \text{ or } x_{ij} = \frac{\operatorname{cdet}_j\left(A^*A + E^*E\right)_{.i}\left(c_{.j}^B\right)}{\det\left(A^*A + E^*E\right)^2},$$

where

$$c_{i.}^A := \left[\begin{array}{ccc} \operatorname{cdet}_i\left(A^*A + E^*E\right)_{.i}(d_{.1}) & ,\ldots, & \operatorname{cdet}_i\left(A^*A + E^*E\right)_{.i}(d_{.n}) \end{array}\right],$$

$$c_{.j}^B := \left[\begin{array}{ccc} \operatorname{rdet}_j\left(A^*A + E^*E\right)_{j.}(d_{1.}) & ,\ldots, & \operatorname{rdet}_j\left(A^*A + E^*E\right)_{j.}(d_{n.}) \end{array}\right]^T,$$

with $d_{i.}$, $d_{.j}$ *are the* i*th row vector and* j*th column vector*

$$A^*C\left(A^*A + E^*E\right) + E^*EC^*A + E^*EL_AVL_AE^*E$$

for all $i, j = 1, \ldots, n$, *with an arbitrary matrix* $V \in \mathbb{H}^{n\times n}$.

Corollary 2.6. *Suppose that* $A \in \mathbb{H}^{m\times n}$, $C \in \mathbb{H}^{m\times q}$, $T_1 \subset \mathbb{H}^n$ *and* $S_1 \subset \mathbb{H}^q$. *Denote* $T_{11} = \mathcal{R}_r(P_{T_1}A^*)$, *then the restricted quaternion matrix equation*

$$AX = C,\ \mathcal{R}_r(X) \subseteq T_1,\ \mathcal{N}_r(X) \supseteq S_1$$

is consistent if and only if $\mathcal{R}_r(C) \subseteq AT_1$ *and* $\mathcal{N}_r(C) \supseteq S_1$. *In this case, the general solution can be expressed as*

$$\begin{aligned} X &= (AP_{T_1})^{\dagger} CP_{S_1^{\perp}} + P_{T_1}L_{AP_{T_1}}U_1P_{S_1^{\perp}} + P_{T_1}V_1P_{S_1^{\perp}} \\ &= (AP_{T_1})^{\dagger} CP_{S_1^{\perp}} + P_{T_1}Z_1P_{S_1^{\perp}} - P_{T_{11}}Z_1P_{S_1^{\perp}} \\ &= (AP_{T_1})^{\dagger} CP_{S_1^{\perp}} + P_{T_1\cap N_r(A)}U_1P_{S_1^{\perp}} + P_{T_1}V_1P_{S_1^{\perp}}, \end{aligned}$$

where U_1, V_1 *and* Z_1 *are arbitrary matrices with proper sizes. Suppose that* E_1^*, K_1^* *are full column rank matrices such that*

$$T_1 = \mathcal{N}_r(E_1),\ T_1 \cap \mathcal{N}_r(A) = \mathcal{R}_r(K_1^*).$$

Then the general solution $X = (x_{ij})$ *posses the determinantal representation:*

$$x_{ij} = \frac{\operatorname{cdet}_i (A^*A + E_1^*E_1 + K_1^*K_1)_{.i}(d_{.j})}{\det (A^*A + E_1^*E_1 + K_1^*K_1)},$$

with $d_{.j}$ *is the* j*th column vector of* $A^*C + K_1^*K_1Z$ *for all* $i = 1, \ldots, n$, $j = 1, \ldots, q$, *and* Z *is an arbitrary matrix over* $\mathbb{H}$ *with appropriate dimension.*

Corollary 2.7. *Suppose that* $B \in \mathbb{H}^{n\times q}$, $C \in \mathbb{H}^{m\times q}, T_2 \subset \mathbb{H}^{1\times n}$ *and* $S_2 \subset \mathbb{H}^{1\times q}$. *Denote* $T_{22} = \mathcal{R}_l(B^*Q_{T_2})$, *then the restricted matrix equation*

$$XB = C,\ \mathcal{R}_l(X) \subseteq T_2,\ \mathcal{N}_l(X) \supseteq S_2$$

is consistent if and only if $\mathcal{R}_l(C) \subseteq T_2B$ *and* $\mathcal{N}_l(C) \supseteq S_2$. *In this case, the general solution can be expressed as*

$$\begin{aligned} X &= Q_{S_2^{\perp}} C (Q_{T_2}B)^{\dagger} + Q_{S_2^{\perp}}U_2Q_{T_2} + Q_{S_2^{\perp}}V_2R_{Q_{T_2}B}Q_{T_2} \\ &= Q_{S_2^{\perp}} C (Q_{T_2}B)^{\dagger} + Q_{S_2^{\perp}}Z_2Q_{T_2} - Q_{S_2^{\perp}}Z_2Q_{T_{22}} \\ &= Q_{S_2^{\perp}} C (Q_{T_2}B)^{\dagger} + Q_{S_2^{\perp}}U_2Q_{T_2} + Q_{S_2^{\perp}}V_2Q_{T_2\cap N_l(B)}, \end{aligned}$$

where U_2, V_2 *and* Z_2 *are arbitrary matrices with proper sizes. Suppose that* E_2^*, K_2^* *are full row rank matrices such that,*

$$T_2 = \mathcal{N}_l(E_2),\ T_2 \cap \mathcal{N}_l(B) = \mathcal{R}_l(K_2^*).$$

Then the general solution $X = (x_{ij})$ *posses the determinantal representation:*

$$x_{ij} = \frac{\operatorname{rdet}_j (BB^* + E_2E_2^* + K_2K_2^*)_{j.}(d_{i.})}{\det (BB^* + E_2E_2^* + K_2K_2^*)},$$

with $d_{.i}$ *are the* i*th row vector of* $CB^* + ZK_2K_2^*$ *for all* $i = 1, \ldots, m$, $j = 1, \ldots, n$, *and* Z *is an arbitrary matrix over* $\mathbb{H}$ *with appropriate dimension.*

Remark 2.1. Similarly, we can get the corresponding results relate to the following restricted system of matrix equations

$$\left\{ \begin{array}{l} A_1X = C_1 \\ XB_2 = C_2, \end{array} \right. \quad \mathcal{R}_l\left(X\right) \subseteq T_1,\ \mathcal{N}_l\left(X\right) \supseteq S_1.$$

3. Determinantal expressions for the general solution to (2)

We begin this section by the following Lemma.

Lemma 3.1 ([45])**.** *Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{s\times q}$, $C \in \mathbb{H}^{m\times q}$, $D \in \mathbb{H}^{t\times q}$ and $E \in \mathbb{H}^{m\times q}$ be given. Denote $M = R_AC$, $N = DL_B$, then the matrix equation $AXB + CYD = E$ is consistent if and only if $R_MR_AE = 0$, $R_AEL_D = 0$, $EL_BL_N = 0$, $R_CEL_D = 0$.*

Then we can show the main results of this section.

Theorem 3.2. *Suppose that $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{t\times q}$ and $E \in \mathbb{H}^{m\times q}$, $T_1 \subset \mathbb{H}^n$, $S_1 \subset \mathbb{H}^p$, $T_2 \subset \mathbb{H}^{1\times t}$ and $S_2 \subset \mathbb{H}^{1\times m}$ are known, $X \in \mathbb{H}^{n\times q}$, $Y \in \mathbb{H}^{m\times t}$ are unknown. Denote $M = R_{AP_{T_1}}Q_{S_2^{\perp}}$, $N = Q_{T_2}BL_{P_{S_1^{\perp}}}$, $A^*R_{Q_{S_2^{\perp}}}A + A^*A = T$ and $I + L_{Q_{T_2}B} = S$. Then we can get the following results.*

(a) *The restricted quaternion matrix equation* (2) *is consistent if and only if*

$$R_MR_{AP_{T_1}}E = 0, R_{AP_{T_1}}EL_{Q_{T_2}B} = 0, EL_{P_{S_1^{\perp}}}L_N = 0, R_{Q_{S_2^{\perp}}}EL_{Q_{T_2}B} = 0. \tag{12}$$

In this case, the general solution of (2) *can be expressed as*

$$\begin{aligned} X &= \left(TP_{T_1}\right)^{\dagger} CP_{S_1^{\perp}} + P_{T_1}L_{AP_{T_1}}U_1P_{S_1^{\perp}} + P_{T_1}V_1P_{S_1^{\perp}} \\ &= \left(TP_{T_1}\right)^{\dagger} CP_{S_1^{\perp}} + P_{T_1}Z_1P_{S_1^{\perp}} - P_{T_{11}}Z_1P_{S_1^{\perp}} \\ &= \left(TP_{T_1}\right)^{\dagger} CP_{S_1^{\perp}} + P_{T_1\cap N_r(A)}U_2P_{S_1^{\perp}} + P_{T_1}V_2P_{S_1^{\perp}}, \end{aligned} \tag{13}$$

$$Y = Q_{S_2^{\perp}}\left(E - AX\right)\left(Q_{T_2}B\right)^{\dagger} + Q_{S_2^{\perp}}V_3R_{Q_{T_2}B}Q_{T_2}. \tag{14}$$

(b) *Suppose that E_1^*, K_1^*, E_1 and K_2 are full column rank matrices such that*

$$T_1 = \mathcal{N}_r\left(E_1\right),\ T_1\cap\mathcal{N}_r\left(T\right) = \mathcal{R}_r\left(K_1^*\right),\ T_2 = \mathcal{N}_l\left(E_2\right),\ T_2\cap\mathcal{N}_r\left(B\right) = \mathcal{R}_l\left(K_2^*\right).$$

Then X, Y posses the following determinantal expressions

$$x_{ij} = \frac{\operatorname{cdet}_i\left(A^*A + E_1^*E_1 + K_1^*K_1\right)_{.i}\left(d_{.j}\right)}{\det\left(A^*A + E_1^*E_1 + K_1^*K_1\right)}, \tag{15}$$

$$y_{kl} = \frac{\operatorname{rdet}_l\left(BB^* + E_2E_2^* + K_2K_2^*\right)_{l.}\left(d_{k.}\right)}{\det\left(BB^* + E_2E_2^* + K_2K_2^*\right)}, \tag{16}$$

with $d_{.j}$ is the jth column vector of

$$T\left(A^*R_{Q_{S_2^{\perp}}}E + A^*EL_{Q_{T_2}B} + A^*R_{Q_{S_2^{\perp}}}EL_{Q_{T_2}B} + Y_1\right) + K_1^*K_1Z_1,$$

$d_{k.}$ is the ith row vector of $(E-AX)B^*+Z_2K_2K_2^*$ for all $i=1,\ldots,n$, $j=1,\ldots,q$, $k=1,\ldots,m$, $l=1,\ldots,t$, where Y_1 is an arbitrary solution of the system of matrix equations

$$A^*R_{Q_{S_2^\perp}}AT^\dagger Y_1=A^*AT^\dagger A^*R_{Q_{S_2^\perp}}E \text{ and } Y_1L_{Q_{T_2}B}=A^*EL_{Q_{T_2}B},$$

Z_1 and Z_2 are arbitrary matrix over $\mathbb{H}$ with appropriate dimensions.

Proof. Note that

$$\mathcal{R}_r(X)\subseteq T_1,\ \mathcal{N}_r(X)\supseteq S_1\Leftrightarrow X=P_{T_1}W_1P_{S_1^\perp} \text{ and}$$
$$\mathcal{R}_l(Y)\subseteq T_2,\ \mathcal{N}_l(Y)\supseteq S_2\Leftrightarrow Y=Q_{S_2^\perp}W_2Q_{T_2},$$

then the restricted quaternion matrix equation (2) can be changed into

$$AP_{T_1}W_1P_{S_1^\perp}+Q_{S_2^\perp}W_2Q_{T_2}B=E, \tag{17}$$

without any restricted conditions on the unknown variables W_1 and W_2. It follows from Lemma 3.1 that (2) is consistent if and only if (12) is satisfied. Moreover, we can get the expression of the general solution (X,Y) of (2) by solving (W_1,W_2) in (17). However, it is a hard work for us to prove the determinantal expressions of (X,Y) through (W_1,W_2). In order to derive the determinantal expression of (X,Y), we need to find a system of matrix equations which not only have the same solution with (2) but also can be solved by Cramer's rules. By reducing the restricted condition of Y, the equation (2) can be written as

$$AX+Q_{S_2^\perp}W_2Q_{T_2}B=E,\ \mathcal{R}_r(X)\subseteq T_1,\ \mathcal{N}_r(X)\supseteq S_1. \tag{18}$$

Recall that the restricted quaternion matrix equation (18) is consistent relate to W_2 if and only if there exist matrix X such that

$$R_{Q_{S_2^\perp}}(E-AX)=0,(E-AX)L_{Q_{T_2}B}=0,\mathcal{R}_r(X)\subseteq T_1,\mathcal{N}_r(X)\supseteq S_1. \tag{19}$$

Thus the equation (2) and the system (19) have the same solution relate to X. If these equalities in (12) are all satisfied, then (2) is consistent and (19) is consistent too. In addition, if (19) is consistent then it has the same solution with

$$\begin{cases} A^*R_{Q_{S_2^\perp}}AX=A^*R_{Q_{S_2^\perp}}E \\ A^*AXL_{Q_{T_2}B}=A^*EL_{Q_{T_2}B}, \end{cases} \mathcal{R}_r(X)\subseteq T_1,\ \mathcal{N}_r(X)\supseteq S_1. \tag{20}$$

Denote $T=A^*R_{Q_{S_2^\perp}}A+A^*A$ and $S=I+L_{Q_{T_2}B}$, then we can prove the system (20) and the following matrix equation

$$\begin{aligned} &TXS=\left(A^*R_{Q_{S_2^\perp}}E+A^*EL_{Q_{T_2}B}+A^*R_{Q_{S_2^\perp}}EL_{Q_{T_2}B}+Y_1\right),\\ &\mathcal{R}_r(X)\subseteq T_1,\ \mathcal{N}_r(X)\supseteq S_1 \end{aligned} \tag{21}$$

have the same solution relate to X, where Y_1 is an arbitrary solution of the system of matrix equations

$$\left\{\begin{array}{c} A^*R_{Q_{S_2^\perp}}AT^\dagger Y_1 = A^*AT^\dagger A^*R_{Q_{S_2^\perp}}E \\ Y_1 L_{Q_{T_2}B} = A^*EL_{Q_{T_2}B}. \end{array}\right. \tag{22}$$

Suppose that X_0 is an arbitrary solution of (20), then

$$A^*R_{Q_{S_2^\perp}}AX_0 = A^*R_{Q_{S_2^\perp}}E \quad \text{and} \quad A^*AX_0L_{Q_{T_2}B} = A^*EL_{Q_{T_2}B}.$$

Setting $Y_1 = A^*AX_0$, it is easy to prove it satisfies the system (22). Moreover,

$$\begin{aligned} &\left(A^*R_{Q_{S_2^\perp}}A + A^*A\right)X_0\left(I + L_{Q_{T_2}B}\right) \\ &= A^*R_{Q_{S_2^\perp}}E + A^*EL_{Q_{T_2}B} + A^*R_{Q_{S_2^\perp}}EL_{Q_{T_2}B} + Y_1, \end{aligned}$$

which is saying that every solution of (20) satisfies (21). For the other direction, note that $L_{Q_{T_2}B}$ is an idempotent matrix, then $I + L_{Q_{T_2}B}$ is a positive matrix and (21) can be written as

$$\begin{aligned} &TX = \left(A^*R_{Q_{S_2^\perp}}E + A^*EL_{Q_{T_2}B} + A^*R_{Q_{S_2^\perp}}EL_{Q_{T_2}B} + Y_1\right)S^{-1}, \\ &\mathcal{R}_r(X) \subseteq T_1,\ \mathcal{N}_r(X) \supseteq S_1. \end{aligned} \tag{23}$$

Suppose that X_1 is an arbitrary solution of (21), then it can be expressed as

$$X_1 = T^\dagger\left(A^*R_{Q_{S_2^\perp}}E + A^*EL_{Q_{T_2}B} + A^*R_{Q_{S_2^\perp}}EL_{Q_{T_2}B} + Y_{10}\right)S^{-1} + L_T Z,$$

where Y_{10} a special solution of (22) and Z is an arbitrary matrix with proper size. Taking it into the first equation in (20) gives

$$\begin{aligned} &A^*R_{Q_{S_2^\perp}}AX_1 \\ &= A^*R_{Q_{S_2^\perp}}A\left(T^\dagger\left(A^*R_{Q_{S_2^\perp}}E + A^*EL_{Q_{T_2}B} + A^*R_{Q_{S_2^\perp}}EL_{Q_{T_2}B} + Y_1\right)S^{-1} + L_T Z\right) \\ &= A^*R_{Q_{S_2^\perp}}AT^\dagger\left(A^*R_{Q_{S_2^\perp}}E + A^*EL_{Q_{T_2}B} + A^*R_{Q_{S_2^\perp}}EL_{Q_{T_2}B} + Y_1\right)S^{-1} \\ &= A^*R_{Q_{S_2^\perp}}AT^\dagger A^*R_{Q_{S_2^\perp}}ES^{-1} + A^*R_{Q_{S_2^\perp}}AT^\dagger A^*EL_{Q_{T_2}B}S^{-1} \\ &\quad + A^*AT^\dagger A^*R_{Q_{S_2^\perp}}ES^{-1} + A^*R_{Q_{S_2^\perp}}AT^\dagger A^*R_{Q_{S_2^\perp}}EL_{Q_{T_2}B}S^{-1} \\ &= A^*R_{Q_{S_2^\perp}}AT^\dagger A^*R_{Q_{S_2^\perp}}E\left(I + L_{Q_{T_2}B}\right)S^{-1} + A^*AT^\dagger A^*R_{Q_{S_2^\perp}}E\left(I + L_{Q_{T_2}B}\right)S^{-1} \\ &= TT^\dagger A^*R_{Q_{S_2^\perp}}ESS^{-1} = A^*R_{Q_{S_2^\perp}}E. \end{aligned}$$

Similarly, we can get $A^*AX_1L_{Q_{T_2}B} = A^*EL_{Q_{T_2}B}$. Combine the above, we can derive that the system (20) and the equation (21) have the same solution. Moreover, if (12) is satisfied then for any solution Y_1 of the system (22) we can get

$$\mathcal{R}_r\left(A^*R_{Q_{S_2^\perp}}E + A^*EL_{Q_{T_2}B} + A^*R_{Q_{S_2^\perp}}EL_{Q_{T_2}B} + Y_1\right) \subseteq MT_1,$$

which is saying that the restricted matrix equation (21) is consistent. By Corollary 2.6, the general solution of (23) can be expressed as (13) which possessing the determinantal expressions as (15). Taking X into the equation (2) gives

$$YB = E - AX,\ \mathcal{R}_l(Y) \subseteq T_2,\ \mathcal{N}_l(Y) \supseteq S_2.$$

Then by Corollary 2.7, Y can be expressed as (14) and possessing the determinantal expression (16). □

Corollary 3.3. *Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{t\times q}$ and $E \in \mathbb{H}^{m\times q}$ be given such that*

$$AX + YB = E \tag{24}$$

is consistent. Suppose that K^*, M^* and L are full column rank matrices such that $\mathcal{N}_r(R_A) = \mathcal{R}_r(K^*)$, $\mathcal{N}_r(A) = \mathcal{R}_r(M^*)$ and $\mathcal{N}_r(B^*) = \mathcal{R}_r(L)$. Then its general solution can be expressed as

$$\begin{aligned} x_{ij} &= \frac{\mathrm{rdet}_j\left(R_A + K^*K\right)_{j.}\left(f_{i.}\right)}{\det\left(R_A + K^*K\right)}, \\ y_{kl} &= \frac{\mathrm{rdet}_j\left(BB^* + LL^*\right)_{j.}\left(c^A_{k.}\right)}{\det\left(R_A + K^*K\right)\det\left(BB^* + LL^*\right)}, \end{aligned} \tag{25}$$

where $f_{i.}$ is the ith column vector of $A^\dagger(E - XB) + M^*MH$ and

$$c^A_{k.} := \left[\ \mathrm{cdet}_i\left(R_A + K^*K\right)_{.i}\left(d_{.1}\right) \quad, \ldots, \quad \mathrm{cdet}_i\left(R_A + K^*K\right)_{.i}\left(d_{.n}\right)\ \right]$$

with $d_{i.}$ is the ith row vector of $R_AEB^* + (R_A + K^*K)V_2R_BLL^*$ for all $i = 1,\ldots,n$, $j = 1,\ldots,q$, $k = 1,\ldots,m$, $l = 1,\ldots,t$, V_2 and H are arbitrary matrices with proper sizes.

Proof. If (24) is consistent, then it has the same solution set of Y with the matrix equation $R_AYB = R_AE$. And by Lemma 2.2, Y can be expressed as $Y = R_AEB^\dagger + L_{R_A}V_1 + V_2R_B$, where V_1 an V_2 are arbitrary matrices with proper size. Denote $\mathcal{N}_r(R_A) = \mathcal{R}_r(K^*)$, $\mathcal{N}_r(B^*) = \mathcal{R}_r(L)$, then $R_A + K^*K$, $BB^* + LL^*$ are nonsingular and $R_AK^* = 0$, $B^*L = 0$. Moreover, it can be verified that

$$\left(R_A + K^*K\right)Y\left(BB^* + LL^*\right) = R_AEB^* + \left(R_A + K^*K\right)V_2R_BLL^*.$$

Thus by Lemma 2.1, Y can be written as (25). Taking Y into the equation (24) gives

$$AX = E - YB. \tag{26}$$

And the solution of (26) can be expressed as $X = A^\dagger(E - YB) + L_AH$, where H is an arbitrary matrix with proper size. Denote $\mathcal{N}_r(A) = \mathcal{R}_r(M^*)$, then $A^*A + M^*M$ is nonsingular and $M^*MA^\dagger = 0$. It follows that X satisfies the following matrix equation

$$\left(A^*A + M^*M\right)\left(A^\dagger(E - XB) + L_AH\right) = A^\dagger(E - XB) + M^*MH,$$

whose coefficient matrix is nonsingular. Thus by Lemma 2.1, X can be expressed as (25). □

Remark 3.1. Similarly, we can get the corresponding results relate to the following restricted quaternion matrix equation

$$AX + YB = E,\ \mathcal{R}_l(X) \subseteq T_1,\ \mathcal{N}_l(X) \supseteq S_1,\ \mathcal{R}_r(Y) \subseteq T_2,\ \mathcal{N}_r(Y) \supseteq S_2.$$

4. Conclusion

In this paper, we consider the determinantal representations for the general solution to (1) and (2), respectively. Corresponding results on some special cases are also given. Motivated by the work in this paper, it would be of interest to investigate the determinantal representation for the general solution to the following consistent system of quaternion matrix equations

$$\begin{cases} A_1XB_1 = C_1 \\ A_2XB_2 = C_2, \end{cases} \mathcal{R}_r(X) \subseteq T_1,\ \mathcal{N}_r(X) \supseteq S_1,$$

and

$$AXB + CYD = E,\ \mathcal{R}_r(X) \subseteq T_1,\ \mathcal{N}_r(X) \supseteq S_1,\ \mathcal{R}_r(Y) \subseteq T_2,\ \mathcal{N}_r(Y) \supseteq S_2,$$

respectively. We will show the results in the following paper.

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