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A question on ∗-regular rings

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Abstract. A ∗-ring R is called ∗-regular if every principal one-sided ideal of R is generated by a projection. In this note, several characterizations of ∗-regular rings are provided. In particular, it is shown that a matrix ring $M_n(R)$ is ∗-regular if and only if R is regular and $1+x_1^*x_1+\cdots+x_{n-1}^*x_{n-1}$ is a unit for all x_i of R; which answers a question raised in the literature recently.

1. Introduction

Regular rings were invented by von Neumann in order to coordinatize certain lattices of projections. Recall that a ring R is *regular* if for every $a \in R$ the principal right ideal aR is generated by an idempotent, or equivalently there exists $b \in R$ such that $a = aba$; and in addition, if b can be chosen as a unit, then R is said to be *unit regular*. It is well known that a regular ring R is unit regular if and only if R possesses the stable range one (see [4]). Regularity of rings plays an important role in ring theory and module theory, one may refer to [4, 8] for more general theory.

A ring R is a ∗*-ring* (or *ring with involution*) if there exists a map $* : R \to R$ such that for all $x,\ y \in R$

$$(x+y)^* = x^* + y^*, \quad (xy)^* = y^*x^*, \quad \text{and } (x^*)^* = x.$$

An involution ∗ of R is *proper* if $x^*x = 0$ implies $x = 0$ for all $x \in R$. Recall that an element p of a ∗-ring R is a *projection* if $p^2 = p = p^*$. Due to [1, Proposition 3], a ∗-ring R is called ∗*-regular* if for every a in R there exists a projection $p \in R$ such that $aR = pR$, or equivalently R is a regular ring and the involution is proper. Clearly the property of being ∗-regularity is left-right symmetric. Recently, the authors studied properties of ∗-regular rings in [3], and proved that if R is unit regular, then a matrix ring $M_n(R)$ is ∗-regular if and only if R is regular and $1 + x_1^*x_1 + \cdots + x_{n-1}^*x_{n-1}$ is a unit for all x_i of R; but it is a question that whether the words 'R is unit regular' can be removed.

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In this note, we answer the above question, and prove that a matrix ring $M_n(R)$ is $*$-regular if and only if R is regular and $1+x_1^*x_1+\cdots+x_{n-1}^*x_{n-1}$ is a unit for all x_i of R. Some basic properties of $*$-regular rings are also considered.

Rings considered are associative with unity. The set of all idempotents, all projections and all units of a ring R are denoted by $Id(R)$, $P(R)$ and $U(R)$, respectively. The symbol $M_n(R)$ stands for the $n\times n$ matrix ring over R. For a $*$-ring R, $M_n(R)$ has a natural involution inherited from R : if $A=(a_{ij})\in M_n(R)$, A^* equals (a_{ji}^*). Henceforth we consider $M_n(R)$ as a $*$-ring with respect to this natural involution.

2. Main results

We begin with the following result.

Proposition 1. *Let R be a $*$-ring. The following are equivalent*:
(1) *R is $*$-regular.*
(2) *For every $a\in R$, $Ra=Ra^*a$.*
(3) *R is regular and $Re^*e=Re$ for every idempotent e of R.*

Proof. (1) $\Rightarrow$ (2) follows from [3, Lemma 2.1].

(2) $\Rightarrow$ (3). It suffices to show that R is regular. Let $a\in R$. By hypothesis, there exists $r\in R$ such that $a=r^*a^*a$. Then we have $ar=r^*a^*ar=(ar)^*ar$. It follows that $ar=(ar)^*$ and $ara=(ar)^*a=r^*a^*a=a$, as required.

(3) $\Rightarrow$ (1). Let $x\in R$ with $x^*x=0$. Since R is regular, $x=xyx$ for some $y\in R$. Write $e=xy$. It is clear that $e\in Id(R)$ and $e^*e=y^*x^*xy=0$. So $Re=Re^*e=0$. Thus $e=0$ and $x=ex=0$, which implies that the involution of R is proper. Therefore, R is a $*$-regular ring. $\square$

Recall that an element a of a $*$-ring R is called *Moore-Penrose invertible* [7] if there exists $b\in R$ such that $a=aba, b=bab, (ab)^*=ab$ and $(ba)^*=ba$, where b is called the *Moore-Penrose inverse* of a and denoted by $b=a^\dagger$. The following result is known in literature, we give the proof for a convenience.

Proposition 2. *Let R be a $*$-ring. Then R is $*$-regular if and only if every element of R is Moore-Penrose invertible.*

Proof. Suppose that R is $*$-regular. Given $a\in R$. Then $Ra=Rp$ for some $p\in P(R)$. So there is an element $r\in R$ such that $p=ra$ and $a=ap=ara$. Similarly, $aR=qR$ for some $q\in P(R)$, which implies that there exists $s\in R$ satisfying $q=as$ and $a=qa=asa$. Let $b=ras$. Then $aba=(ara)sa=asa=a$ and $bab=r(asa)ras=r(ara)s=ras=b$. Further, $(ab)^*=(aras)^*=(as)^*=as=(ara)s=ab$ and $(ba)^*=(rasa)^*=(ra)^*=ra=r(asa)=ba$. This proves that a is Moore-Penrose invertible and $b=a^\dagger$.

Conversely, let $a\in R$. Then there exists $b\in R$ such that $a=aba$ and $(ab)^*=ab$. So $a=(ab)^*a=b^*a^*a$. It follows that $Ra\subseteq Ra^*a$. Clearly, $Ra^*a\subseteq Ra$. In view of Proposition 1, the result follows. $\square$

Due to [3], a $*$-ring R is said to have the *k-GN property* if $1 + x_1^* x_1 + \cdots + x_k^* x_k \in U(R)$ for all $x_1, \ldots, x_k$ in R; if $k = 1$, then R is known as a $*$-ring which possesses the *Gelfand–Naimark property* [6] (written *GN property*).

Lemma 3. (1) [3, Proposition 3.7] *If R has the k-GN property, then R has the l-GN property for any integer $1 \le l \le k$.*

(2) [3, Lemma 2.5] *If R is a regular $*$-ring with the GN property, then the involution of R is proper.*

In [3, Theorem 3.8], it was shown that for a $*$-ring R and an integer $n \ge 2$, $M_n(R)$ is $*$-regular and unit regular if and only if R is unit regular and $1 + x_1^* x_1 + \cdots + x_{n-1}^* x_{n-1} \in U(R)$ for all $x_i \in R$. The property of being unit regularity is Morita invariant ([4, Corollary 4.7]), so it is a question that whether the words 'unit regular' be weakened as 'regular'. We give an affirmative answer.

Theorem 4. *Let R be a $*$-ring and an integer $n \ge 2$. The following are equivalent*:

(1) *$M_n(R)$ is $*$-regular.*

(2) *R is regular with the $(n-1)$-GN property.*

(3) *R is regular and $x_1^* x_1 + x_2^* x_2 + \cdots + x_n^* x_n = 0$ implies $x_i = 0$ for all $x_i \in R$.*

Proof. (1) $\Rightarrow$ (2). Write $S = M_n(R)$. Since S is $*$-regular, it is regular. By [4, Theorem 1.7], R is regular. Take any $x_1, x_2, \ldots, x_{n-1} \in R$. Let $E = \begin{pmatrix} 1 & 0 \\ \alpha & O_{n-1} \end{pmatrix}$ be a 2×2 block matrix with $\alpha = (x_1, x_2, \ldots, x_{n-1})^T$ and O_{n-1} the $(n-1) \times (n-1)$ zero matrix. Clearly, $E^2 = E \in S$. In view of Proposition 1, $SE^*E = SE$. Note that $E^*E = \begin{pmatrix} 1 + \sum_{i=1}^{n-1} x_i^* x_i & 0 \\ 0 & O_{n-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & O_{n-1} \end{pmatrix} \in SE$. So there exists a 2×2 block matrix $Y = \begin{pmatrix} y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \in S$ such that $YE^*E = \begin{pmatrix} 1 & 0 \\ 0 & O_{n-1} \end{pmatrix}$, which yields $y_1(1 + \sum_{i=1}^{n-1} x_i^* x_i) = 1$. It follows that $(1 + \sum_{i=1}^{n-1} x_i^* x_i) y_1^* = [y_1(1 + \sum_{i=1}^{n-1} x_i^* x_i)]^* = 1$. So $1 + \sum_{i=1}^{n-1} x_i^* x_i \in U(R)$, and therefore, R has the $(n-1)$-GN property.

(2) $\Rightarrow$ (3). Assume (2) holds. In view of Lemma 3, R has the GN property. Then R is a $*$-regular ring since it is regular. By Proposition 2, every element of R is Moore-Penrose invertible. Let $x_1, x_2, \ldots, x_n \in R$ with $x_1^* x_1 + x_2^* x_2 + \cdots + x_n^* x_n = 0$. Multiplying the above equation by $x_n^\dagger$ on the right and by $(x_n^\dagger)^*$ on the left, we obtain

$$\begin{aligned} 0 &= (x_n^\dagger)^* x_1^* x_1 x_n^\dagger + (x_n^\dagger)^* x_2^* x_2 x_n^\dagger + \cdots + (x_n^\dagger)^* x_n^* x_n x_n^\dagger \\ &= (x_1 x_n^\dagger)^* x_1 x_n^\dagger + (x_2 x_n^\dagger)^* x_2 x_n^\dagger + \cdots + (x_n x_n^\dagger)^* x_n x_n^\dagger. \end{aligned}$$

Set $p = x_n x_n^\dagger$. Clearly, $p^* = p$ and $p^2 = p$. Since R has the $(n-1)$-GN property, $1 + (x_1 x_n^\dagger)^* x_1 x_n^\dagger + (x_2 x_n^\dagger)^* x_2 x_n^\dagger + \cdots + (x_{n-1} x_n^\dagger)^* x_{n-1} x_n^\dagger = 1 - (x_n x_n^\dagger)^* x_n x_n^\dagger = 1 - p \in Id(R) \cap U(R) = \{1\}$. Hence $p = 0$ and $x_n = x_n x_n^\dagger x_n = p x_n = 0$.

So $x_1^*x_1 + x_2^*x_2 + \cdots + x_{n-1}^*x_{n-1} = 0$. By Lemma 3, for each positive integer $l \leq n-1$, R possesses the l-GN property. Repeating the above procedure, we will get $x_{n-1} = x_{n-2} = \cdots = x_2 = x_1 = 0$.

(3) ⇒ (1) follows from [5, Theorem 1]. □

Corollary 5. *Let R be a $*$-ring. Then $M_2(R)$ is $*$-regular if and only if R is a regular ring with GN property.*

Corollary 6. *Let R be a $*$-ring. If $M_n(R)$ is $*$-regular, then $M_m(R)$ is $*$-regular for any positive integer $m \leq n$.*

By Theorem 4 and Corollary 6, we have the following result.

Corollary 7 ([3, Proposition 2.9])**.** *Let R be a $*$-ring. If $M_n(R)$ is $*$-regular, then $m \cdot 1 \in U(R)$ for any positive integer $m \leq n$.*

For a ring R, we use 1_R to denote the identity endomorphism of R.

Remark 8. (1) The property of being $*$-regularity relies on the choice of the involution. Let $\mathbb{C}$ be the field of complex numbers and $* = 1_{\mathbb{C}}$. Then $\mathbb{C}$ is regular. For any $n \geq 2$, $M_n(\mathbb{C})$ is not $*$-regular by Theorem 4 (indeed, $M_2(\mathbb{C})$ is not $*$-regular as $1 + i^*i = 0 \notin U(R)$). Nevertheless, if the involution $*$ of $\mathbb{C}$ is defined by $x \mapsto \bar{x}$ where $\bar{x}$ is the conjugation of x, then $M_n(\mathbb{C})$ is $*$-regular.

(2) Let $\mathbb{R}$ be the field of real numbers. Set $* = 1_{\mathbb{R}}$. Then $\mathbb{R}$ is regular with k-GN property for each $k \geq 1$. So $M_n(\mathbb{R})$ is $*$-regular for every integer $n \geq 2$.

(3) Let $R = \mathbb{Z}_3$ be the ring of integers modulo 3 and $* = 1_R$. Clearly, R is regular with the GN property. So $M_2(R)$ is $*$-regular. But $M_3(R)$ is not $*$-regular since R does not possess the 2-GN property.

It is well known that C^*-algebras possess the GN property (see also [6]). So we have the following result immediately.

Example 9. If R is a regular C^*-algebra, then $M_2(R)$ is $*$-regular.

Let I be an ideal of a ring R. Recall that I is called *regular* provided that for each $x \in I$, there exists $y \in I$ such that $x = xyx$ (see [4, Definition, p. 2]).

Lemma 10 ([4, Lemma 1.3])**.** *Let I be an ideal of a ring R. Then R is regular if and only if I and the factor ring R/I are both regular.*

Let R be a $*$-ring. An ideal I of R is called $*$-*invariant* if $I^* \subseteq I$. In this way, I is a $*$-ring (possibly without the identity of R) and the involution of R can be extended to the factor ring R/I which is still denoted by $*$. We call an $*$-invariant ideal I is $*$-*regular* if I is regular and the involution of I is proper.

Lemma 11. *Let I be an $*$-invariant ideal of R. Then R is $*$-regular if and only if I and R/I are both $*$-regular.*

Proof. Assume that R is $*$-regular. In view of Lemma 10, I and R/I are regular. Let $x \in I$ with $x^*x = 0$. As $I \subseteq R$ and the involution of R is proper, $x = 0$. So I is $*$-regular. To show that the involution of R/I is proper. Take

$\bar{x} = x + I \in R/I$. If $\bar{x}^*\bar{x} = 0$, then $x^*x \in I$. By Proposition 2, we may write $p = xx^\dagger$. Then $p \in P(R)$ and $x = xx^\dagger x = px = p^*x = (x^\dagger)^*x^*x \in I$ as I is an ideal, whence $\bar{x} = 0 \in R/I$, which implies that R/I is $*$-regular.

Conversely, R is regular by Lemma 10. It suffices to show that the involution of R is proper. Let $x^*x = 0$ with $x \in R$. Then $\bar{x}^*\bar{x} = x^*x + I = 0 \in R/I$. Since R/I is $*$-regular, $\bar{x} = 0$. So $x \in I$. Notice that the involution of I is proper. It follows from $x^*x = 0$ that $x = 0$. As desired. □

For an ideal I of a ring R, it is well known that $M_n(I)$ is an ideal of $M_n(R)$ and $M_n(R/I) \cong M_n(R)/M_n(I)$. So we may treat $M_n(R/I)$ and $M_n(R)/M_n(I)$ as the same.

Proposition 12. *Let I be an $*$-invariant ideal of a $*$-ring R. Then $M_n(R)$ is $*$-regular if and only if both of the following hold*:

(1) *I is regular and $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies $x_i = 0$ for all $x_i \in I$.*

(2) *R/I is regular and $y_1^*y_1 + y_2^*y_2 + \cdots + y_n^*y_n \in I$ implies $y_i \in I$ for all $y_i \in R$.*

Proof. We will use the following facts freely: (i) $M_n(R)$ is regular if and only if R is regular (by [4, Corollary 4.7]); (ii) $M_n(R)$ is regular if and only if I and R/I are regular if and only if $M_n(I)$ and $M_n(R/I)$ are regular (by Lemma 10).

Suppose that $M_n(R)$ is $*$-regular. Clearly, both I and R/I are regular. Let $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ with $x_i \in I \subseteq R$. By Theorem 4, we have $x_i = 0$ for all i. So (1) follows. Since $M_n(R)$ is $*$-regular, $M_n(R/I) \cong M_n(R)/M_n(I)$ is $*$-regular by Lemma 11. Now, given $y_1^*y_1 + y_2^*y_2 + \cdots + y_n^*y_n \in I$. Then $\bar{y_1}^*\bar{y_1} + \bar{y_2}^*\bar{y_2} + \cdots + \bar{y_n}^*\bar{y_n} = 0 \in R/I$. So one has $\bar{y_i} = 0 \in R/I$ by applying Theorem 4 again. Thus $y_i \in I$ for $i = 1, 2, \ldots, n$, and (2) follows.

Conversely, it is enough to verify both $M_n(I)$ and $M_n(R/I)$ are $*$-regular. Clearly, $M_n(I)$ and $M_n(R/I)$ are regular. Let $X = (x_{ij}) \in M_n(I)$ and $X^*X = O$. Then $x_{1j}^*x_{1j} + x_{2j}^*x_{2j} + \cdots + x_{nj}^*x_{nj} = 0$ for $j = 1, \ldots, n$. By (1), $x_{ij} = 0$ for all i and j. So $X = O$, which implies that $M_n(I)$ is $*$-regular. Next we show that the involution of $M_n(R/I)$ is proper. Let $Y = (y_{ij}) \in M_n(R)$ be such that $(\bar{Y})^*\bar{Y} = O \in M_n(R/I)$. Then we obtain $Y^*Y \in M_n(I)$. It follows that $y_{1j}^*y_{1j} + y_{2j}^*y_{2j} + \cdots + y_{nj}^*y_{nj} \in I$ for all j. By (2), $y_{ij} \in I$ for $i, j = 1, \ldots, n$. Hence $Y \in M_n(I)$, whence $\bar{Y} = O \in M_n(R/I)$. So the involution of $M_n(R/I)$ is proper, and therefore, $M_n(R/I)$ is $*$-regular. As desired. □

Recall that an element $e \in Id(R)$ is *left* (resp., *right*) *semicentral* in R if $re = ere$ (resp., $er = ere$) for all $r \in R$ (see [2]); e is central if and only if e is left and right semicentral. We now provide an application of $*$-regular rings.

Proposition 13. *If R is $*$-regular ring, then every left* (*right*) *semicentral idempotent of R is central.*

Proof. Without loss of generality, we suppose that $e \in Id(R)$ is left semicentral. Since R is $*$-regular, there exists $p \in P(R)$ such that $Re = Rp$. It follows that

$e = ep$ and $p = pe$. As e is left semicentral, $p = pe = epe = e^2 = e \in P(R)$. For any $r \in R$, we have $er = e^*r = (r^*e)^* = (er^*e)^* = e^*re^* = ere$, which implies that e is right semicentral. So e is central in R. □

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