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WEAKLY TRIPOTENT RINGS

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Abstract. We study the class of rings R with the property that for $x \in R$ at least one of the elements x and $1 + x$ are tripotent. We prove that a commutative ring has this property if and only if it is a subring of a direct product $R_0 \times R_1 \times R_2$ such that $R_0/J(R_0) \cong \mathbb{Z}_2$, for every $x \in J(R_0)$ we have $x^2 = 2x$, R_1 is a Boolean ring, and R_3 is a subring of a direct product of copies of $\mathbb{Z}_3$.

1. Introduction

An element x of a ring R is called *tripotent* if $x^3 = x$, and a ring R is *tripotent* if all its elements are tripotent. Hirano and Tominaga proved in [4, Theorem 1] that a ring R is tripotent if and only if every element of R is a sum of two commuting idempotents. The class of these rings was extended in [7, Section 4] to the class of rings R such that every element is a sum or a difference of two commuting idempotents. This is a natural approach since similar methods were applied for other classes of rings. For instance, the classes of clean rings or nil-clean rings were extended in [1] and [2] to the classes of weakly clean, respectively weakly nil-clean in the following ways: a ring R is weakly (nil-)clean if and only if every element of R is a sum or a difference of a unit (nilpotent) element and an idempotent.

Apparently the natural question which asks if we can properly extend in a similar way the classes of rings which satisfy identities which involve tripotents instead of idempotents has a negative answer, since $-u$ is tripotent whenever u is tripotent. However, we want to propose such an extension, starting by the following remarks:

(1) every element of a ring R is a sum or a difference of two (commuting) idempotents if and only if for every $x \in R$ at least one of the elements x or $1 + x$ is a sum of two (commuting) idempotents;
(2) a ring R is weakly clean if and only if for every $x \in R$ at least one of the elements x or $1 + x$ is a sum of a unit and an idempotent.

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(3) a ring R is weakly nil-clean if and only if for every $x \in R$ at least one of the elements x or $1+x$ is a sum of a nilpotent and an idempotent.

Therefore, we will say that a ring R is *weakly tripotent* if for every $x \in R$ at least one of the elements x or $1+x$ is tripotent. Two important differences between the properties of tripotent rings and weakly tripotent rings are the following: while tripotent rings are always commutative and the class of tripotent rings is closed under direct products, these properties are not valid for weakly tripotent rings. One of the main result of the present paper is Theorem 14 where we provide a characterization of commutative weakly tripotent rings as subrings of rings of the form $R_0 \times R_1 \times R_2$, where R_0 is a weakly tripotent ring without nontrivial idempotents such that 3 is a unit (these rings are described in Corollary 9), R_1 is a Boolean ring, and R_2 is a tripotent ring of characteristic 3. Moreover, we will obtain connections between the class of weakly tripotent rings and the rings studied in [7]. In particular, it is proved in the end of the paper that the class of rings such that every element is a sum or a difference of two commuting idempotents is a strict subclass of the class of weakly tripotent rings (Corollary 18).

2. Preliminaries and examples

Before we will study the connections between these classes, we state some basic properties of weakly tripotent rings.

Lemma 1. *Let R be a weakly tripotent ring. Then*

(1) *every subring of R is weakly tripotent*;
(2) *every homomorphic image of R is weakly tripotent*;
(3) $24=0$*, hence R has a decomposition $R=R_1\times R_2$ such that R_1 and R_2 are weakly tripotent rings, $8R_1=0$, and $3R_2=0$.*

For the case R is of characteristic 3, it is easy to see that the identity $(1+x)^3=1+x$ implies $x^3=x$. Therefore, weakly tripotent rings of characteristic 3 are tripotent, hence we can restrict our study to rings on characteristic 2^k, $k\in\{1,2,3\}$. We refer to [3] for other extensions of other generalizations of tripotent rings of characteristic 3.

We start with the following useful remark.

Proposition 2. *The following are equivalent for a ring R*:

(a) *R is weakly tripotent*;
(b) *for every element $x \in R$ at least one of the elements x or $1-x$ is tripotent.*

Proof. (a)⇒(b) Let $x\in R$ such that $x^3\neq x$. It follows that $(-x)^3\neq -x$, hence $(1-x)^3=1-x$.

(b)⇒(a) This can be proved in a similar way. □

Remark 3. A similar equivalence is not valid neither for Boolean rings nor for nil-clean conditions. More precisely:

(i) a ring is Boolean (clean, respectively nil-clean) if and only if for every element $x \in R$ at least one of the elements x or $1 - x$ is idempotent (clean, respectively nil-clean);
(ii) in the ring $\mathbb{Z}_3$ for every element $x \in R$ at least one of the elements x or $1 + x$ is idempotent (resp. nil-clean), but $\mathbb{Z}_3$ is not Boolean or nil-clean, while $\mathbb{Z}_{(15)}$ (i.e., the ring of rational numbers with denominator coprime with 15) is a weakly clean ring which is not clean, [1].

In the following result we describe a connection between the class of weakly tripotent rings and one of the classes studied in [7]. A second connection will be established in Corollary 18.

Corollary 4. *The class of all weakly tripotent ring is a strict subclass of the class of rings such that all elements are sums of an idempotent and a tripotent that commute.*

Proof. The inclusion is obvious since for every element x of a weakly tripotent ring such that $x^3 \neq x$ we have that $x - 1$ is tripotent. To produce an example, we observe that in the ring $\mathbb{Z}_4 \times \mathbb{Z}_4$ every element is a sum of a idempotent and a tripotent, but the elements $x = (1, 2)$ and $(1, 1) + x$ from $\mathbb{Z}_4 \times \mathbb{Z}_4$ are not tripotents. □

Moreover, let us remark that it is easy to see that the class of weakly tripotent rings is contained in the class, studied in [7, Section 2], of those rings such that every element is a sum of two commuting tripotents.

The proof of the following proposition is easy.

Proposition 5. *If a ring R is weakly tripotent, then it can be embedded as a subring of a direct product of subdirectly irreducible weakly tripotents rings.*

However $\mathbb{Z}_4 \times \mathbb{Z}_4$ is not weakly tripotent, hence a direct product of weakly tripotent ring is not necessarily weakly tripotent. The proof of the following proposition is standard. We refer to the similar property stated for weakly nil-clean rings in [1, Theorem 1.7].

Proposition 6. *A direct product $\prod_{i \in I} R_i$ of weakly tripotent rings is weakly tripotent if and only if there exists $i_0 \in I$ such that R_{i_0} is weakly tripotent and for all $i \in I \setminus \{i_0\}$ the rings R_i are tripotent.*

The following result was proved in [7, Theorem 3.6] by using the fact that if R is a ring such that every element of R is a sum of an idempotent and a tripotent that commute, then R is strongly nil-clean, hence $R/J(R)$ is Boolean and $J(R)$ is nil, [5, Theorem 5.6]. We present, for reader's convenience, a proof which uses the general strategy described in [6, Section 12].

Theorem 7 ([7, Theorem 3.6])**.** *Let R be a ring of characteristic 2^k. The following are equivalent:*

(1) *every element of R is a sum of an idempotent and a tripotent that commute;*

(2) $x^4 = x^6$ *for every* $x \in R$;
(3) $R/J(R)$ *is Boolean and* $U(R)$ *is a group of exponent* 2;
(4) $R/J(R)$ *is Boolean, and for every* $x \in J(R)$ *we have* $x^2 = 2x$.

Consequently, if R *satisfies the above conditions, then* $J(R)$ *is the set of all nilpotent elements of* R.

Proof. (1)⇒(2) This is the implication (2)⇒(3) from [7, Theorem 3.6].

(2)⇒(3) First, we observe that for every ring R which satisfies (2) and every $x \in U(R)$ we have $x^2 = 1$.

The class of rings which satisfies (2) is closed with respect to subrings and factor rings. Using [6, Theorem 12.5] together with its proof we conclude that we can embed $R/J(R)$ as a subring of a direct product of left primitive rings which are factor rings of R. We will prove that every left primitive ring which satisfies (2) is isomorphic to a division ring.

In order to prove this, we first observe that if K is a division ring of characteristic 2, then for every $n \geq 2$, there exists an $n \times n$ matrix A with coefficients in K such that $A^4 \neq A^6$. For instance, we can take $A = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$. Then for every $n \geq 2$ and every division ring K of characteristic 2, the ring $M_n(K)$ of $n \times n$ matrices over K does not satisfy (2).

Suppose that R is left primitive and satisfies (2). If R is not left artinian, then we use [6, Theorem 11.19] to conclude that there exist a subring S of R and a division ring (of characteristic 2) such that there exists an onto ring homomorphism from S to the ring of $M_2(K)$. This is not possible since $M_2(K)$ does not satisfy (2). Therefore, there exists a positive integer n and a division ring K such that $R \cong M_n(K)$. Using again the above remarks we conclude that this is possible only if $n = 1$, hence K is a division ring of characteristic 2.

Moreover, for every $x \in K$ we have $x^2 = 1$, and it follows that $K \cong \mathbb{Z}_2$ (if we have a nonzero element $x \neq 1$ in K, then the subfield generated by x has 4 elements, and it is easy to see that $x^2 \neq 1$).

Coming back to the general case, it follows that if R is a ring which satisfies (2), then the ring $R/J(R)$ is Boolean since it is a subring of a direct product of copies of $\mathbb{Z}_2$. □

Corollary 8. *If every element of* R *is a sum of an idempotent and a tripotent that commute, then*

(i) *the ideal* $J(R)$ *is nil and it contains all nilpotent elements of* R;
(ii) *for every* $x \in J(R)$ *we have* $x^2 = 2x$, $4x = 0$, *and* $x^3 = 0$;
(iii) $xy = yx$ *for all* $x, y \in J(R)$;
(iv) $2J(R)^2 = 0$.

Proof. (i) is obvious.

(ii) For every $x \in J(R)$ the elements $1 \pm x$ are invertible. Hence $(1 \pm x)^2 = 1$, and it follows that $x^2 = 2x = -2x$. Hence $0 = 4x = x^3$.

(iii) Since $1+x$ and $1+y$ are invertible, and the group $U(R)$ is commutative, we have $(1+x)(1+y) = (1+y)(1+x)$, hence $xy = yx$.

(iv) Observe that if $x, y \in J(R)$, then $2(x+y) = (x+y)^2 = x^2+xy+yx+y^2 = 2x + 2y + xy + yx$, hence $xy = -yx$. Therefore, $2xy = 0$. □

In the following corollary we have a characterization of weakly tripotent rings without nontrivial idempotents. This will be useful in order to describe commutative weakly tripotent rings, since in this case every subdirectly irreducible ring is without nontrivial idempotents.

Corollary 9. *The following are equivalent for a ring R without nontrivial idempotents such that $3 \in U(R)$:*

(1) *R is weakly tripotent;*
(2) *R is a local ring such that $R/J(R) \cong \mathbb{Z}_2$, and $x^2 = 1$ for all $x \in U(R)$;*
(3) *$R/J(R) \cong \mathbb{Z}_2$, and for every $x \in J(R)$ we have $x^2 = 2x$.*

Consequently, every weakly tripotent ring without nontrivial idempotents is commutative.

Proof. (1)⇒(2) If R is weakly tripotent, then every element of R is a sum of an idempotent and a tripotent that commute. Then (2) follows from Theorem 7 since every idempotent of $R/J(R)$ can be lifted to an idempotent of R.

(2)⇒(3) This can be proved as in Corollary 8(ii).

(3)⇒(1) Let $x \in R \setminus J(R)$. Then $1+x \in J(R)$, and it follows that $(1+x)^2 = 2(1+x)$. This implies that $x^2 = 1$. If $x \in J(R)$, then $1 + x \notin J(R)$, hence $(1+x)^2 = 1$, hence $1+x$ is tripotent.

For the last statement, it is enough to observe that every element of $R \setminus J(R)$ is of the form $1 + x$ with $x \in J(R)$, and to apply Corollary 8. □

We recall that a ring is *abelian* if every idempotent of R is central.

Corollary 10. *If R is an abelian ring such that every element of R is a sum of an idempotent and a tripotent, then R is commutative. In particular, every abelian weakly tripotent ring is commutative.*

Proof. Using [7, Theorem 3.10], we observe that we can assume that R is of characteristic 2^k. Since $J(R)$ is nil, it follows that all elements from the Boolean ring $R/J(R)$ can be lifted to idempotents in R. Therefore, it is enough to prove that for every $x, y \in J(R)$ we have $xy = yx$. But this is proved in Corollary 8, hence the proof for the first statement is complete. The second statement is obvious. □

But, let us remark that there exists non-commutative subdirectly irreducible weakly tripotent rings which are not local rings:

Example 11. The ring $T_2(\mathbb{Z}_2)$ of upper triangular matrices over $\mathbb{Z}_2$ is subdirectly irreducible and weakly tripotent.

The above corollaries allow us to provide nontrivial examples of weakly tripotent rings:

Example 12. (1) If N is an abelian group such that $2^k N = 0$, with $k \leq 3$, then the idealization of N by $\mathbb{Z}_{2^k}$, i.e., the ring of all matrices $\left(\begin{smallmatrix} x & n \\ 0 & x \end{smallmatrix}\right)$ with $x \in \mathbb{Z}_{2^k}$ and $n \in N$, is weakly tripotent if and only if $2N = 0$.

(2) Let $2 \leq k \leq 3$ be a positive integer. If $N = \mathbb{Z}_4$, then the multiplication defined on the abelian group $R = \mathbb{Z}_{2^k} \times N$ by $(x, n) * (y, m) = (xy, xm + ny + 2mn)$ induces a structure of weakly tripotent ring on the abelian group $\mathbb{Z}_{2^k} \times Z_4$.

(3) The ring $R = \mathbb{Z}_4[X, Y]/(X^2, Y^2, XY - 2, 2X + 2Y)$ is a subdirectly irreducible weakly tripotent ring. In order to prove that R is subdirectly irreducible, we observe that if I is an ideal in R such that $X \in I$ or $Y \in I$, then $2 \in I$. Moreover, if $2 + X \in I$, then $2X = X(2 + X) \in I$ and $2Y + 2 = Y(2 + X) \in I$. Therefore, $2 = 2X + 2Y + 2 \in I$. It follows that the ideal $\{0, 2\}$ is the smallest ideal of R.

3. Commutative weakly tripotent rings

In order to provide a characterization for commutative weakly tripotent rings, we need the following lemma.

Lemma 13. *Let R be a commutative weakly tripotent ring of characteristic 2^k ($k \leq 3$), and let N be the nil radical of R. Then for every idempotent e in R one of the following properties is true*:

(a) *$en = 0$ for all $n \in N$;*
(b) *$en = n$ for all $n \in N$.*

Proof. We will denote $f = 1 - e$. If n is a nilpotent element, then $(e+n)^3 = e+n$ or $(f - n)^3 = f - n$.

Suppose that $(e + n)^3 = e + n$. Using the identities from Corollary 8, it follows that $(e + n)^3 = e + en$, hence $en = n$. This also implies that $fn = 0$.

If $(f - n)^3 = f - n$ it follows in the same way that $fn = n$, hence $en = 0$.

Now, suppose that there exist n_1 and n_2 nonzero nilpotents such that $en_1 = 0$ and $fn_2 = 0$. Since $n_1 + n_2$ is nilpotent, we can assume without loss the generality that $e(n_1 + n_2) = 0$. But this implies that $en_2 = 0$. Therefore $n_2 = 0$, a contradiction, and the proof is complete. □

We have the following characterization which shows that weakly tripotent commutative rings are exactly the subrings of direct products $R_0 \times R_1 \times R_2$ such that R_0 is a ring as those described in Corollary 9, R_1 is Boolean and R_2 is a tripotent ring.

Theorem 14. *A commutative ring R is weakly tripotent if and only if $R = R' \times R''$ such that*:

(1) *R'' is a tripotent ring of characteristic 3 or $R'' = 0$;*

(2) *$R' = 0$ or $3 \in U(R')$ and R' can be embedded as a subring of a direct product $R_0 \times \left(\prod_{i\in I} R_i\right)$ such that R_0 is a weakly tripotent ring without nontrivial idempotents, and all R_i are Boolean rings.*

Proof. We only have to prove that if R is a weakly tripotent ring of characteristic 2^k, then it can be embedded in a direct product as in (2).

Let J be the Jacobson radical of R. Using Corollary 8, it follows that $J = N(R)$. We fix an ideal L which is maximal with the property $J \cap L = 0$. Then $L \cong (L+J)/J$ can be embedded in R/J. Therefore all elements of L are idempotents and $2L = 0$.

Let $\overline{e}$ be an idempotent in R/L. Then $R/L \cong \overline{e}R/L \times \overline{f}R/L$, where $\overline{f} = 1-\overline{e}$. Since $(J+L)/L \subseteq J(R/L)$, we can suppose by Lemma 13 that $(J+L)/L \subseteq \overline{e}R$. Therefore, if K is an ideal in R such that $L \subseteq K$ and $K/L \cong \overline{f}R/L$, then $K \cap J = 0$. By the maximality of L it follows that $K = L$, hence R/L has no nontrivial idempotents.

In order to complete the proof, we proceed as in the proof of Birkhoff's Theorem, [6, Theorem 12.3]. For every $0 \neq x \in L$ we choose an ideal I_x which is maximal with the property $x \notin I_x$. By Zorn's Lemma we can choose I_x such that $J \subseteq I_x$. Therefore R/I_x is Boolean, and there exists an injective ring homomorphism $R \to R/L \times \left(\prod_{0\neq x\in L} R/I_x\right)$. This injective ring homomorphism provides a subdirect decomposition for R. □

We complete this theorem with an example of a weakly tripotent ring R as in Theorem 14(2) such that it has no direct factors isomorphic to R_0.

Example 15. Let R_0 be a subdirectly irreducible weakly tripotent commutative ring such that $R_0 \ncong \mathbb{Z}_2$. For every positive integer k we fix a ring $R_k \cong \mathbb{Z}_2$. In the direct product $\prod_{k\geq 0} R_k$ we consider the ring R generated by $1 = (1_k)_{k\geq 0}$ and the ideal $J(R_0) \oplus (\oplus_{k\geq 1} R_k)$ of all families of finite support in $\prod_{k\geq 0} R_k$ such that the 0-th component is in $J(R_0)$. It is easy to see that R has the desired properties.

Remark 16. Let us remark that the ring R_0 from Theorem 14 is not necessarily subdirectly irreducible. In order to see this, we consider the ring R from Example 12(1) with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$. This ring is not subdirectly irreducible since for every subgroup $S \leq N$ the set $I_S = \{\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) \in R \mid x \in S\}$ is a minimal ideal of R. Therefore, if we view R as a subdirect product of a family of subdirectly irreducible rings, at least two factors of this product are not tripotent rings.

Using the same proof as in Theorem 14 we obtain:

Proposition 17. *The following are equivalent for a ring R:*

(a) *every element of R is a sum or a difference of two commuting idempotents;*
(b) *R is isomorphic to a subring of a direct product $R_0 \times R_1 \times R_2$ such that R_0 is 0 or $\mathbb{Z}_4$, R_1 is Boolean, and R_2 is a subdirect product of a family of copies of $\mathbb{Z}_3$.*

Proof. (a)⇒(b) Let R be a ring such that every element is a sum or a difference of two commuting idempotents. From [7, Lemma 4.2(5)] it follows that R is abelian, hence R is commutative by Corollary 10.

By [7, Theorem 4.4] we have $R = R_1 \times R_2$ such that $R_1/J(R_1)$ is Boolean such that $J(R_1) = 0$ or $J(R_1) = \{0, 2\}$ and R_2 is a tripotent ring, so it is enough to assume that $R/J(R)$ is Boolean such that $J(R) = \{0, 2\}$. In this hypothesis, we observe that for every idempotent $e \in R$ we have $2e = 0$ or $2(1 - e) = 0$. Therefore, we can apply the same technique as in the proof of Theorem 14 to conclude that R is isomorphic to a subring of a direct product of a subdirectly irreducible factor ring of R and a Boolean ring. Since in this case subdirectly irreducible factor rings of R are isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_4$, the proof is complete.

(b)⇒(a) This is obvious. □

Corollary 18. *If every element of a ring R is a sum or a difference of two commuting idempotents, then R is a commutative weakly tripotent ring. The converse is not true.*

Proof. The first statement follows from Proposition 17 and Theorem 14.

In order to see that the converse is not true, let us observe that $\mathbb{Z}_8$ is weakly tripotent, but not all elements of R are sums or differences of two commuting idempotents. □

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