Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b170633
pISSN: 1015-8634 / eISSN: 2234-3016

A GENERALIZATION OF GAUSS' TRIANGULAR THEOREM

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Abstract. A quadratic polynomial $\Phi_{a,b,c}(x,y,z) = x(ax+1) + y(by+1)+z(cz+1)$ is called universal if the diophantine equation $\Phi_{a,b,c}(x,y,z) = n$ has an integer solution x, y, z for any nonnegative integer n. In this article, we show that if $(a,b,c) = (2,2,6), (2,3,5)$ or $(2,3,7)$, then $\Phi_{a,b,c}(x,y,z)$ is universal. These were conjectured by Sun in [8].

1. Introduction

A triangular number is a number represented as dots or pebbles arranged in the shape of an equilateral triangle. More precisely, the n-th triangular number is defined by $T_n = \frac{n(n+1)}{2}$ for any nonnegative integer n.

In 1796, Gauss proved that every positive integer can be expressed as a sum of three triangular numbers, which was first asserted by Fermat in 1638. This follows from the Gauss-Legendre theorem, which states that every positive integer which is not of the form $4^k(8l+7)$ with nonnegative integers k and l, is a sum of three squares of integers.

In general, a ternary sum $aT_x + bT_y + cT_z$ $(a,b,c > 0)$ of triangular numbers is called *universal* if for any nonnegative integer n, the diophantine equation

$$aT_x + bT_y + cT_z = n$$

has an integer solution x, y, z. In 1862, Liouville generalized Gauss' triangular theorem by proving that a ternary sum $aT_x + bT_y + cT_z$ of triangular numbers is universal if and only if (a,b,c) is one of the following triples:

$$(1,1,1),\quad (1,1,2),\quad (1,1,4),\quad (1,1,5),\quad (1,2,2),\quad (1,2,3),\quad (1,2,4).$$

Recently, Sun in [8] gave another generalization of the Gauss' triangular theorem. He noticed that since $\{T_x : x \in \mathbb{N} \cup \{0\}\} = \{T_x : x \in \mathbb{Z}\} = \{x(2x+1) : x \in \mathbb{Z}\}$, Gauss' triangular theorem implies that the equation $x(2x+1) + y(2y+$

Received July 18, 2017; Revised February 28, 2018; Accepted March 8, 2018.
2010 *Mathematics Subject Classification.* Primary 11E12, 11E20.
Key words and phrases. triangular theorem, universal polynomials.
This work of the first author was supported by BK21 PLUS SNU Mathematical Sciences Division.
This work of the second author was supported by the National Research Foundation of Korea (NRF-2017R1A2B4003758).

$1) + z(2z+1) = n$ has an integer solution x, y, z for any nonnegative integer n. Inspired by this motivation, he defined in [8] that for positive integers $a \leqslant b \leqslant c$, a ternary sum $\Phi_{a,b,c}(x,y,z) = x(ax+1) + y(by+1) + z(cz+1)$ is *universal* if the equation

$$x(ax+1) + y(by+1) + z(cz+1) = n$$

has an integer solution for any nonnegative integer n. He showed that if $\Phi_{a,b,c}(x,y,z)$ is universal, then (a,b,c) is one of the following 17 triples:

$$\begin{array}{llllll} (1,1,2), & (1,2,2), & (1,2,3), & (1,2,4), & (1,2,5), & (2,2,2), \\ (2,2,3), & (2,2,4), & (2,2,5), & (2,2,6), & (2,3,3), & (2,3,4), \\ (2,3,5), & (2,3,7), & (2,3,8), & (2,3,9), & (2,3,10) & \end{array}$$

and proved the universalities of some candidates. In fact, the universality of each ternary sum was proved except the following 6 triples:

$$\textbf{(2,2,6)},\quad \textbf{(2,3,5)},\quad \textbf{(2,3,7)},\quad (2,3,8),\quad (2,3,9),\quad (2,3,10). \tag{1.1}$$

He also conjectured that for each of these six triples, $\Phi_{a,b,c}(x,y,z)$ is universal.

In this article, we prove that if (a,b,c) is one of the triples written in boldface in the above candidates, then $\Phi_{a,b,c}(x,y,z)$ is, in fact, universal. To prove the universality, we use the method developed in [5]. We briefly review this method in Section 2 for those who are unfamiliar with it.

Let f be a positive definite integral ternary quadratic form. A symmetric matrix corresponding to f is denoted by M_f. For a diagonal form $f(x,y,z) = a_1x^2 + a_2y^2 + a_3z^2$, we simply write $M_f = \langle a_1, a_2, a_3 \rangle$. The genus of f is the set of equivalence classes of all integral quadratic forms that are locally isometric to f, which is denoted by $\text{gen}(f)$. The number of isometry classes in $\text{gen}(f)$ is called the class number of f. For the computations of class numbers of positive definite ternary quadratic forms, one may use the computer program MAGMA [3]. The set of all integers that are represented by the genus of f (f itself, respectively) is denoted by $Q(\text{gen}(f))$ ($Q(f)$, respectively). For an integer a, we define

$$R(a,f) = \{(x,y,z) \in \mathbb{Z}^3 : f(x,y,z) = a\} \quad \text{and} \quad r(a,f) = |R(a,f)|.$$

For a set S, we define $\pm S = \{s : s \in S \text{ or } -s \in S\}$. For two $v = (v_1, v_2, v_3), v' = (v_1', v_2', v_3') \in \mathbb{Z}^3$ and a positive integer s, we write $v \equiv v' \pmod{s}$ if $v_i \equiv v_i' \pmod{s}$ for any $i \in \{1,2,3\}$.

Any unexplained notation and terminology can be found in [2] or [7].

2. General tools

Let $a \leqslant b \leqslant c$ be positive integers. Recall that a ternary sum

$$\Phi_{a,b,c}(x,y,z) = x(ax+1) + y(by+1) + z(cz+1)$$

is said to be universal if the equation $\Phi_{a,b,c}(x,y,z) = n$ has an integer solution for any nonnegative integer n. It can be directly checked that $\Phi_{a,b,c}(x,y,z)$ is universal if and only if the equation

$$bc(2ax+1)^2 + ac(2by+1)^2 + ab(2cz+1)^2 = 4abcn + bc + ac + ab =: \Psi_{a,b,c}(n)$$

has an integer solution for any nonnegative integer n. This is equivalent to the existence of an integer solution X, Y and Z of the diophantine equation

$$bcX^2 + acY^2 + abZ^2 = \Psi_{a,b,c}(n)$$

satisfying the following congruence condition

$$X \equiv 1 \pmod{2a},\ Y \equiv 1 \pmod{2b} \text{ and } Z \equiv 1 \pmod{2c}.$$

In some particular cases, representations of quadratic forms with some congruence condition correspond to representations of a subform which is suitably taken (for details, see [1]).

For the representation of an arithmetic progression by a ternary quadratic form, we use the method developed in [1], [4] and [5]. We briefly introduce this method for those who are unfamiliar with it.

Let d be a positive integer and let a be a nonnegative integer ($a \leqslant d$). We define

$$S_{d,a} = \{dn + a \mid n \in \mathbb{N} \cup \{0\}\}.$$

For integral ternary quadratic forms f, g, we define

$$R(g, d, a) = \{v \in (\mathbb{Z}/d\mathbb{Z})^3 \mid vM_g v^t \equiv a \pmod{d}\}$$

and

$$R(f, g, d) = \{T \in M_3(\mathbb{Z}) \mid T^t M_f T = d^2 M_g\}.$$

A coset (or, a vector in the coset) $v \in R(g, d, a)$ is said to be *good* with respect to f, g, d and a if there is a $T \in R(f, g, d)$ such that $\frac{1}{d} \cdot vT^t \in \mathbb{Z}^3$. The set of all good vectors in $R(g, d, a)$ is denoted by $R_f(g, d, a)$. Every vector in $R(g, d, a) \backslash R_f(g, d, a)$ is said to be *bad*. If there does not exist a bad vector, we write $g \prec_{d,a} f$. If $g \prec_{d,a} f$, then by Lemma 2.2 of [4], we have

$$S_{d,a} \cap Q(g) \subset Q(f).$$

Note that the converse is not true in general.

In general, if d is large, then it is not easy to compute the set

$$R(g, d, a) \backslash R_f(g, d, a)$$

of bad vectors exactly by hand. A MAPLE based computer program for this set is available upon request to the authors.

Theorem 2.1. *Under the same notation given above, assume that there is a partition* $R(g, d, a) \backslash R_f(g, d, a) = (P_1 \cup \cdots \cup P_k) \cup (\widetilde{P}_1 \cup \cdots \cup \widetilde{P}_{k'})$ *satisfying the following properties: for each* P_i*, there is a* $T_i \in M_3(\mathbb{Z})$ *such that*

(i) T_i *has an infinite order*;
(ii) $T_i^t M_g T_i = d^2 M_g$;
(iii) *for any vector* $v \in \mathbb{Z}^3$ *such that* $v \pmod{d} \in P_i$, $\frac{1}{d} \cdot vT_i^t \in \mathbb{Z}^3$ *and* $\frac{1}{d} \cdot vT_i^t \pmod{d} \in P_i \cup R_f(g, d, a)$,

and for each $\widetilde{P}_j$*, there is a* $\widetilde{T}_j \in M_3(\mathbb{Z})$ *such that*

(iv) $\widetilde{T}_j^t M_g \widetilde{T}_j = d^2 M_g$;

(v) *for any vector* $v \in \mathbb{Z}^3$ *such that* $v \pmod d \in \widetilde{P}_j$, $\frac{1}{d} \cdot v\widetilde{T}_j^t \in \mathbb{Z}^3$ *and* $\frac{1}{d} \cdot v\widetilde{T}_j^t \pmod d \in P_1 \cup \cdots \cup P_k \cup R_f(g,d,a)$.

Then we have

$$(S_{d,a} \cap Q(g)) \setminus \bigcup_{i=1}^{k} g(z_i)\mathcal{S} \subset Q(f),$$

where the vector z_i *is a primitive eigenvector of* T_i*,* $\mathcal{S}$ *is the set of squares of integers, and* $g(z_i)\mathcal{S} = \{g(z_i)n^2 \mid n \in \mathbb{Z}\}$.

Proof. See Theorem 2.3 in [1]. □

3. Universality of $x(ax+1)+y(by+1)+z(cz+1)$

In this section, we prove that if $(a,b,c) = (2,2,6), (2,3,5)$ or $(2,3,7)$, then $\Phi_{a,b,c}(x,y,z)$ is universal. For the list of all candidates, see (1.1) in the introduction.

Lemma 3.1. *For any positive integer* n*, there are integers* a, b *and* c *such that* $a^2+3b^2+3c^2 = 24n+7$ *and* $a \equiv b \equiv c \pmod 4$*, whereas*

$$a \not\equiv b \pmod 8 \quad or \quad a \not\equiv c \pmod 8.$$

Proof. Since the class number of $f(x,y,z) = x^2+3(x-4y)^2+3(x-4z)^2$ is one and $S_{24,7} \subset Q(\mathrm{gen}(f))$ (for details, see [6]), there is an integer solution $x^2+3y^2+3z^2 = 24n+7$ with $x \equiv y \equiv z \pmod 4$ for any nonnegative integer n. If we define $g(x,y,z) = x^2+3(x-8y)^2+3(x-8z)^2$, then it suffices to show that for any positive integer n,

$$r(24n+7,f) - r(24n+7,g) > 0.$$

Note that the genus of g consists of

$$M_g = \begin{pmatrix} 7 & -1 & 3 \\ -1 & 55 & 27 \\ 3 & 27 & 111 \end{pmatrix},\ M_2 = \begin{pmatrix} 15 & 6 & 3 \\ 6 & 28 & 14 \\ 3 & 14 & 103 \end{pmatrix} \text{ and } M_3 = \begin{pmatrix} 28 & 10 & 2 \\ 10 & 31 & -13 \\ 2 & -13 & 55 \end{pmatrix}.$$

Furthermore, since f is isometric to g over $\mathbb{Z}_p$ for any odd prime p, one may easily show by using the Minkowski-Siegel formula that

$$r(24n+7,f) = r(24n+7,\mathrm{gen}(f)) = 4r(24n+7,\mathrm{gen}(g)),$$

where

$$r(24n+7,\mathrm{gen}(g)) = \frac{1}{4}r(24n+7,g) + \frac{1}{4}r(24n+7,M_2) + \frac{1}{2}r(24n+7,M_3).$$

Therefore we have

$$r(24n+7,f) - r(24n+7,g) = r(24n+7,M_2) + 2r(24n+7,M_3).$$

Hence it suffices to show that every integer of the form $24n+7$ that is represented by g is also represented by M_2 or M_3.

Note that

$$R(g,8,7) = \{(a,b,c) \in (\mathbb{Z}/8\mathbb{Z})^3 \mid a \equiv 1 \pmod 2\}.$$

In fact, one may use the computer program MAPLE to show that the set $R(M_3,g,8)$ consists of 208 matrices and in particular, it contains the following 6 matrices:

$$\begin{pmatrix} 4 & 20 & 8 \\ 0 & -16 & 16 \\ 0 & -8 & 0 \end{pmatrix}, \quad \begin{pmatrix} 4 & 8 & 20 \\ 0 & 16 & -16 \\ 0 & 0 & -8 \end{pmatrix}, \quad \begin{pmatrix} 4 & 10 & 20 \\ 0 & 8 & -16 \\ 0 & 12 & -8 \end{pmatrix},$$
$$\begin{pmatrix} 4 & 20 & 10 \\ 0 & -16 & 8 \\ 0 & -8 & 12 \end{pmatrix}, \quad \begin{pmatrix} 4 & 13 & 10 \\ 0 & 4 & 8 \\ 0 & -10 & 12 \end{pmatrix}, \quad \begin{pmatrix} 4 & 10 & 13 \\ 0 & 8 & 4 \\ 0 & 12 & -10 \end{pmatrix}.$$

One may easily show that for any $(a,b,c) \in R(g,8,7)$, if $(a,b,c) \neq \pm(1,0,0)$, $\pm(3,0,0)$, then there is a matrix, say T, among the above six matrices such that

$$\frac{1}{8} \cdot (a,b,c)T^t \in \mathbb{Z}^3.$$

As a sample, for $(1,1,0) \in R(g,8,7)$, one may easily show that

$$\frac{1}{8} \cdot (1,1,0) \cdot \begin{pmatrix} 4 & 20 & 8 \\ 0 & -16 & 16 \\ 0 & -8 & 0 \end{pmatrix}^t = (3,-2,-1).$$

Note that if $(a',b',c') = \pm(1,0,0)$ or $\pm(3,0,0)$, then $\frac{1}{8} \cdot (a',b',c')S^t$ is not an integer vector for any $S \in R(M_3,g,8)$. Therefore we have

$$R(g,8,7) \backslash R_{M_3}(g,8,7) = \{\pm(1,0,0), \pm(3,0,0)\}.$$

Let $P_1 = R(g,8,7) \backslash R_{M_3}(g,8,7)$ and $T_1 = \left(\begin{smallmatrix} 8 & -4 & 0 \\ 0 & -2 & -12 \\ 0 & 6 & 4 \end{smallmatrix}\right)$. Since P_1 and T_1 satisfy all conditions in Theorem 2.1, and the eigenvector of T_1 is $(1,0,0)$ and $g(1,0,0) = 7$,

$$(S_{24,7} \cap Q(g)) \backslash 7\mathcal{S} \subset Q(M_3).$$

Assume that $24n + 7 = 7t^2$ for some positive integer t. Then t has a prime divisor p greater than 3. Since $M_2, M_3 \in \mathrm{spn}(M_g)$, and M_g represents 7, following Lemma 2.4 of [1], $7t^2$ is represented by M_2 or by M_3. Therefore we have $r(24n+7, M_2) + 2r(24n+7, M_3) > 0$ for any positive integer n, which completes the proof. □

Theorem 3.2. *The ternary sum $x(2x+1) + y(2y+1) + z(6z+1)$ is universal.*

Proof. As explained at the beginning of Section 2, it suffices to show that for any positive integer n,

$$3(4x+1)^2 + 3(4y+1)^2 + (12z+1)^2 = 24n + 7$$

has an integer solution x, y, z. By Lemma 3.1, there are integers a, b, c such that

$$a^2 + 3b^2 + 3c^2 = 24n + 7, \quad a \equiv b \equiv c \pmod 4 \quad \text{and} \quad a \not\equiv b \pmod 8$$

for any positive integer n. By changing signs suitably, we may assume that $a \equiv b \equiv c \equiv 1 \pmod 4$. If $a \equiv 1 \pmod 3$, then everything is trivial. Assume that $a \equiv 2 \pmod 3$. In this case, note that

$$\left(\frac{a-3b}{2}\right)^2 + 3\left(\frac{a+b}{2}\right)^2 + 3c^2 = a^2 + 3b^2 + 3c^2 = 24n + 7$$

and

$$\frac{a-3b}{2} \equiv 1 \pmod 3 \quad \text{and} \quad \frac{a-3b}{2} \equiv 1 \pmod 4.$$

The theorem follows directly from this. □

Theorem 3.3. *The ternary sum* $x(2x+1) + y(3y+1) + z(5z+1)$ *is universal.*

Proof. As explained at the beginning of Section 2, it suffices to show that the diophantine equation

$$15(4x+1)^2 + 10(6y+1)^2 + 6(10z+1)^2 = 120n + 31 \tag{3.1}$$

has an integer solution x, y, z for any nonnegative integer n. Firstly, assume that

$$f(x, y, z) = 6x^2 + 10(2y+z)^2 + 15z^2 = 120n + 31$$

has an integer solution $(x, y, z) = (a, b, c)$ for any nonnegative integer n. Since $c \equiv 1 \pmod 2$, we have $10(2b+c)^2 + 15c^2 \equiv 1 \pmod 8$. Hence a should be odd. Furthermore, since $6a^2 \equiv 1 \pmod 5$, $a \equiv \pm 1 \pmod 5$. Therefore either a or $-a$ is congruent to 1 modulo 10. By using a similar argument used above, we can also prove that either $2b + c$ or $-2b - c$ is congruent to 1 modulo 6, and either c or $-c$ is congruent to 1 modulo 4. Therefore if $f(x, y, z) = 120n + 31$ has an integer solution, so is Equation (3.1).

Note that the genus of f consists of

$$M_f = \langle 6 \rangle \perp \begin{pmatrix} 25 & 5 \\ 5 & 25 \end{pmatrix}, \ M_2 = \langle 1, 30, 120 \rangle \ \text{ and } \ M_3 = \langle 10, 15, 24 \rangle.$$

Note also that $S_{120,31} \subset Q(\text{gen}(f))$ (for details, see [6]). Since every integer that is locally represented by f is represented by a quadratic form in the genus of f, it suffices to show that every integer of the form $40n + 31$ that is represented by either M_2 or M_3 is represented by f.

By a direct computation, one may easily show that $M_3 \prec_{40,31} M_2$. Hence it suffices to show that every integer of the form $40n + 31$ that is represented by M_2 is also represented by f.

Note that $R(M_2,40,31)\backslash R_f(M_2,40,31)$ consists of the followings:

$$\begin{array}{llll}
\pm(1,\pm7,10), & \pm(1,\pm7,30), & \pm(1,\pm13,0), & \pm(1,\pm13,20),\\
\pm(9,\pm3,0), & \pm(9,\pm3,20), & \pm(9,\pm17,10), & \pm(9,\pm17,30),\\
\pm(11,\pm3,10), & \pm(11,\pm3,30), & \pm(11,\pm17,0), & \pm(11,\pm17,20),\\
\pm(19,\pm7,0), & \pm(19,\pm7,20), & \pm(19,\pm13,10), & \pm(19,\pm13,30).
\end{array}$$

Now, we define a partition of $R(M_2,40,31)\backslash R_f(M_2,40,31)$ as follows:

$$\begin{aligned}
P_1 &= \{(m_1,m_2,m_3) \mid 2m_1+m_2-m_3 \equiv 0 \pmod 5\},\\
\widetilde{P}_1 &= \{(m_1,m_2,m_3) \mid 3m_1+m_2-m_3 \equiv 0 \pmod 5\},
\end{aligned}$$

and we also define

$$T_1 = \begin{pmatrix} -20 & -180 & 120\\ -6 & 22 & 12\\ -1 & -3 & -38 \end{pmatrix},\ \widetilde{T}_1 = \begin{pmatrix} -20 & 180 & -120\\ -6 & -22 & -12\\ -1 & 3 & 38 \end{pmatrix}.$$

Then one may easily show that it satisfies all conditions in Theorem 2.1. Note that the eigenvector of T_1 is $(-3,1,0)$ and $120n+31$ is not of the form $39t^2$ for any positive integer t. Therefore for any nonnegative integer n, the equation $f(x,y,z) = 120n+31$ has an integer solution. This completes the proof. □

Theorem 3.4. *The ternary sum $x(2x+1)+y(3y+1)+z(7z+1)$ is universal.*

Proof. As explained at the beginning of Section 2, it suffices to show that the diophantine equation

$$21(4x+1)^2+14(6y+1)^2+6(14z+1)^2 = 168n+41 \tag{3.2}$$

has an integer solution x,y,z for any nonnegative integer n. Assume that

$$f(x,y,z) = 6x^2+14y^2+21z^2 = 168n+41$$

has an integer solution $(x,y,z)=(a,b,c)$ for any nonnegative integer n. Then one may easily show that $a \equiv b \equiv c \equiv 1 \pmod 2$, $b \not\equiv 0 \pmod 3$ and $a \equiv \pm1 \pmod 7$. Therefore if $f(x,y,z) = 168n+41$ has an integer solution, so is Equation (3.2).

Note that the genus of f consists of

$$M_f = \langle 6,14,21\rangle,\ M_2 = \begin{pmatrix} 5 & 1 & 2\\ 1 & 17 & -8\\ 2 & -8 & 26 \end{pmatrix},\ \text{and}\ M_3 = \langle 3,14,42\rangle.$$

Note also that $S_{168,41} \subset Q(\text{gen}(f))$ (for details, see [6]). One may easily show by a direct computation that $M_3 \prec_{21,20} M_2$. Hence it suffices to show that every integer of the form $21n+20$ that is represented by M_2 is also represented by f.

One may show by a direct computation that the set

$$R(M_2,21,20)\backslash R_f(M_2,21,20)$$

consists of 52 vectors, which is the union of the following sets:

$$\begin{aligned} &P_1 = \pm\{(0,0,2),(0,0,5)\}, && P_2 = \pm\{(0,4,0),(0,10,0)\},\\ &P_3 = \pm\{(3,2,0),(3,16,0)\}, && P_4 = \pm\{(2,0,0),(5,0,0)\},\end{aligned}$$

$$\begin{aligned} &\widetilde{P}_1 = \pm\{(6,9,10),(6,9,17)\}, && \widetilde{P}_2 = \pm\{(2,19,19),(5,16,16)\},\\ &\widetilde{P}_3 = \pm\{(2,13,4),(5,1,10)\}, && \widetilde{P}_4 = \pm\{(2,1,10),(5,13,4)\},\\ &\widetilde{P}_5 = \pm\{(3,7,12),(3,14,12)\}, && \widetilde{P}_6 = \pm\{(6,5,12),(6,19,12)\},\\ &\widetilde{P}_7 = \pm\{(3,0,5),(3,0,19)\}, && \widetilde{P}_8 = \pm\{(1,2,2),(8,16,16)\},\\ &\widetilde{P}_9 = \pm\{(9,9,10),(9,9,17)\}. && \end{aligned}$$

We also define

$$T_1 = \begin{pmatrix} -13 & -24 & -21 \\ -6 & -3 & 21 \\ 4 & -12 & 21 \end{pmatrix},\ T_2 = \begin{pmatrix} -3 & -40 & 16 \\ 6 & 3 & 24 \\ -6 & 4 & 11 \end{pmatrix},\quad T_3 = \begin{pmatrix} -9 & 30 & -39 \\ 12 & 9 & 3 \\ 6 & -6 & -9 \end{pmatrix},$$
$$T_4 = \begin{pmatrix} -1 & 36 & 0 \\ -12 & -9 & 0 \\ -2 & -12 & 21 \end{pmatrix},\quad T_5 = \begin{pmatrix} -9 & 0 & -45 \\ 4 & -21 & 13 \\ 10 & 0 & 1 \end{pmatrix},\ T_6 = \begin{pmatrix} -1 & -30 & -15 \\ 6 & -9 & 27 \\ 10 & 6 & 3 \end{pmatrix},$$
$$T_7 = \begin{pmatrix} -9 & 24 & 12 \\ 4 & 15 & -24 \\ 10 & 6 & 3 \end{pmatrix},\quad T_8 = \begin{pmatrix} -3 & 36 & -3 \\ 6 & -9 & 27 \\ -6 & -12 & 15 \end{pmatrix},\quad T_9 = \begin{pmatrix} -21 & 0 & -21 \\ 0 & 7 & 21 \\ 0 & -14 & 21 \end{pmatrix}.$$

Note that for any $i = 1, 2, \ldots, 9$,

$$T_i^t M_2 T_i = 21^2 M_2 \quad \text{and} \quad (21^2 \cdot T_i^{-1})^t M_2 (21^2 \cdot T_i^{-1}) = 21^2 \cdot M_2.$$

Let $G = R_f(M_2, 21, 20)$, which is the set of good vectors. For any i, let Q_i be the set P_k or $\widetilde{P}_s$ for some k or s. Let U (and V) be an integral matrix $T_{k'}$ or $21^2 T_{s'}^{-1}$ for some k' or s'. For any vector $v \in \mathbb{Z}^3$ such that $v \pmod{21} \in Q_1$, if

$$\frac{1}{21} \cdot vU^t \in \mathbb{Z}^3 \quad \text{and} \quad \frac{1}{21} \cdot vU^t \pmod{21} \in G \cup Q_{j_1} \cup \cdots \cup Q_{j_k},$$

then we write

$$Q_1 \xrightarrow{U} G \cup Q_{j_1} \cup \cdots \cup Q_{j_k}.$$

We also use the notation

$$Q_1 \xrightarrow{U} G \cup Q_2 \cup B \xrightarrow{V} G \cup Q_{j_1} \cup \cdots \cup Q_{j_k},$$

if for any vector $v \in \mathbb{Z}^3$ such that $v \pmod{21} \in Q_1$, $\frac{1}{21} \cdot vU^t \in \mathbb{Z}^3$ and either $\frac{1}{21} \cdot vU^t \pmod{21} \in G \cup Q_2$ or

$$\frac{1}{21}\left(\frac{1}{21} \cdot vU^t\right) V^t \in \mathbb{Z}^3 \quad \text{and} \quad \frac{1}{21}\left(\frac{1}{21} \cdot vU^t\right) V^t \in G \cup Q_{j_1} \cup \cdots \cup Q_{j_k}.$$

Here B stands for the set of vectors of the form $\frac{1}{21} \cdot vU^t \pmod{21}$ that are not contained in $G \cup Q_2$.

Now, one may show by direct computations that

$$P_1 \xrightarrow{T_1} G \cup B \xrightarrow{T_2} G \cup \widetilde{P}_2, \qquad \widetilde{P}_2 \xrightarrow{T_3} G \cup B \xrightarrow{T_4} G \cup P_1,$$
$$P_2 \xrightarrow{T_5} G \cup P_2 \cup B \xrightarrow{T_6} G, \qquad P_4 \xrightarrow{T_9} G \cup P_4,$$
$$\widetilde{P}_1 \xrightarrow{T_1} G \cup B \xrightarrow{T_2} G, \qquad \widetilde{P}_3 \xrightarrow{21^2 T_1^{-1}} G \cup P_1 \cup \widetilde{P}_1,$$
$$\widetilde{P}_4 \xrightarrow{21^2 T_2^{-1}} G \cup \widetilde{P}_2 \cup \widetilde{P}_3, \qquad \widetilde{P}_5 \xrightarrow{T_5} G \cup P_2 \cup B \xrightarrow{T_6} G,$$
$$P_3 \xrightarrow{T_7} G \cup \widetilde{P}_3 \cup B \xrightarrow{T_8} G \cup P_3 \cup \widetilde{P}_5, \qquad \widetilde{P}_6 \xrightarrow{21^2 T_5^{-1}} G \cup P_2 \cup \widetilde{P}_5,$$
$$\widetilde{P}_7 \xrightarrow{21^2 T_8^{-1}} G \cup P_3 \cup \widetilde{P}_6, \qquad \widetilde{P}_8 \xrightarrow{21^2 T_9^{-1}} G \cup \widetilde{P}_7,$$
$$\widetilde{P}_9 \xrightarrow{21^2 T_8^{-1}} G \cup P_3 \cup \widetilde{P}_6.$$

As a sample of the above computations, assume that $v = (21x, 21y, 21z + 2)$ for some integers x, y, z. Note that $v \pmod{21} \in P_1$ and

$$w = \frac{1}{21} \cdot vT_1^t = (-13x - 24y - 21z - 2, -6x - 3y + 21z + 2, 4x - 12y + 21z + 2).$$

For any integers x, y, z, the vector $w \pmod{21}$ is contained in G or

$$w \pmod{21} = (16, 20, 11) \quad \text{or} \quad w \pmod{21} = (19, 2, 2).$$

If $w \pmod{21} = (16, 20, 11)$, then there are some integers a, b such that $(x, y, z) = (18 + 3a, a, b)$. Hence $w = (-236 - 63a - 21b, -106 - 21a + 21b, 74 + 21b)$. Now, one may show by a direct computation that

$$\frac{1}{21} \cdot wT_2^t \pmod{21} \in G \quad \text{or} \quad \frac{1}{21} \cdot wT_2^t \pmod{21} = (19, 2, 2) \in \widetilde{P}_2.$$

If $w \pmod{21} = (19, 2, 2)$, then there are some integers c, d such that $(x, y, z) = (3c, c, d)$. Hence $w = (-63c - 21d - 2, -21c + 21d + 2, 21d + 2)$. Now, one may show by a direct computation that

$$\frac{1}{21} \cdot wT_2^t \pmod{21} \in G \quad \text{or} \quad \frac{1}{21} \cdot wT_2^t \pmod{21} = (19, 2, 2) \in \widetilde{P}_2.$$

Now, from the above, one may easily show that

(i) every vector in $R(M_2, 21, 20) \backslash R_f(M_2, 21, 20)$ is transformed to a vector in either P_1, P_2, P_3, P_4 or G;

(ii) every vector in P_1 is transformed to a good vector or eventually is transformed to a vector in P_1 via $T_4T_3T_2T_1$. For example, if $x \in P_1$, then $x{T_1}^t \in G \cup B$. If $x{T_1}^t$ is not a good vector, then $x{T_1}^t{T_2}^t$ is a good vector or $x{T_1}^t{T_2}^t \in \tilde{P}_2$. If $x{T_1}^t{T_2}^t{T_3}^t$ is not a good vector, then $x{T_1}^t{T_2}^t{T_3}^t{T_4}^t$ is a good vector or $x{T_1}^t{T_2}^t{T_3}^t{T_4}^t \in P_1$;

(iii) every vector in P_2 is transformed to a good vector or eventually is transformed to a vector in P_2 via T_5;

(iv) every vector in P_3 is transformed to a good vector or eventually is transformed to a vector in P_3 via T_8T_7;

(v) every vector in P_4 is transformed to a good vector or eventually is transformed to a vector in P_4 via T_9.

Note that the order of each $T_4T_3T_2T_1, T_5, T_8T_7$ and T_9 is infinite. Therefore, though the situation is slightly different from Theorem 2.1, we may still apply it to this case. The primitive integral eigenvectors of these four matrices are $(0,0,1), (0,1,0), (3,2,0)$ and $(1,0,0)$, respectively. Therefore if $168n+41$ is not of the form $17t^2$ for some positive integer t, then $168n+41$ is represented by f. Assume that $168n+41 = 17t^2$ for some positive integer t. Then t has a prime divisor relatively prime to $2\cdot 3\cdot 7$. Since $M_2, M_3 \in \text{spn}(f)$, and both M_2 and M_3 represent 17, f represents $17t^2$ by Lemma 2.4 of [1]. Therefore the equation $f(x,y,z) = 168n+41$ has an integer solution for any nonnegative integer n. This completes the proof. □

Remark 3.5. The ternary sum $\Phi_{2,3,8}(x,y,z)$ is universal if and only if for any positive integer n, $12x^2+8y^2+3z^2 = 96n+23$ has an integer solution such that $x \equiv 1 \pmod 4$, $y \equiv 1 \pmod 6$ and $z \equiv 1 \pmod{16}$. Since the class number of $\langle 3,8,12\rangle$ is one, one may easily show that the above equation has always an integer solution $(x,y,z) = (a,b,c)$. Since a (c, respectively) is odd, either a (c, respectively) or $-a$ ($-c$, respectively) is congruent to 1 modulo 4. Now, by changing the sign of both a and c, if necessary, we may assume that $a \equiv c \equiv 1 \pmod 4$. Note that b is not divisible by 3. If b is odd, then one may easily show that c is congruent to 1 modulo 16 by comparing both sides modulo 32. Hence if b is odd, then the ternary sum $\Phi_{2,3,8}(x,y,z)$ is universal. Therefore, to prove the universality of $\Phi_{2,3,8}(x,y,z)$, it suffices to show that either $12x^2+8(z-2y)^2+3z^2$ or $12x^2+8(x-2y)^2+3z^2$ represents every integer of the form $96n+23$. Similarly, one may also show that $\Phi_{2,3,9}(x,y,z)$ is universal if and only if for any positive integer n, $9x^2+6y^2+2z^2 = 72n+17$ has an integer solution such that $y \equiv z \equiv 1 \pmod 2$, which is equivalent to the condition that either $2x^2+15y^2+6yz+15z^2$ or $6x^2+8y^2+8yz+11z^2$ represents every integer of the form $72n+17$.

For both cases, there are many bad vectors which cannot be transformed to the set of good vectors by an (rational) isometry with infinite order in any possible parameters of Theorem 2.1. So, it seems to be quite difficult to apply the theorem in each situation.

Finally, the ternary sum $\Phi_{2,3,10}(x,y,z)$ is universal if and only if for any positive integer n, $15x^2+10y^2+3z^2 = 120n+28$ has an integer solution such that $x \equiv 1 \pmod 4$, $y \equiv 1 \pmod 6$ and $z \equiv 1 \pmod{20}$. In this case, one may easily show that there is an integer solution $(x,y,z) = (a,b,c)$ such that $a \equiv b \equiv c \equiv 1 \pmod 2$. Hence $c \equiv 1, 9, 11$ or $19 \pmod{20}$. To prove the universality, we have to show that there is an integer solution such that $c \equiv \pm 1 \pmod{20}$, which seems to be difficult.

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