

APPROXIMATE AND CHARACTER AMENABILITY OF VECTOR-VALUED LIPSCHITZ ALGEBRAS

EMAMGHOLI BIYABANI AND ALI REJALI

ABSTRACT. For a metric space (X, d) and $\alpha > 0$. We study the structure and properties of vector-valued Lipschitz algebra $Lip_\alpha(X, E)$ and $lip_\alpha(X, E)$ of order α . We investigate the approximate and Character amenability of vector-valued Lipschitz algebras.

1. Introduction and preliminaries

Let (X, d) be a metric space and $B(X)$ (resp. $C_b(X)$) indicates the Banach space consisting of all bounded complex valued functions on X , endowed with the norm

$$\|f\|_{sup} = \sup_{x \in X} |f(x)| \quad (f \in B(X)).$$

Take $\alpha \in \mathbb{R}$ with $\alpha > 0$. Then $Lip_\alpha X$ is the subspace of $B(X)$, consisting of all bounded complex-valued functions f on X such that

$$p_\alpha(f) := \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)^\alpha} : x, y \in X, x \neq y \right\} < \infty.$$

It is known that $Lip_\alpha X$ endowed with the norm $\|\cdot\|_\alpha$ given by

$$\|f\|_\alpha = p_\alpha(f) + \|f\|_{sup};$$

and pointwise product is a unital commutative Banach algebra, called Lipschitz algebra.

Let (X, d) be a metric space with at least two elements and $(E, \|\cdot\|)$ be a Banach space over the scalar field \mathbb{F} ($= \mathbb{R}$ or \mathbb{C}) for a constant $\alpha > 0$ and a function $f : X \rightarrow E$, set

$$p_{\alpha, E}(f) := \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{d(x, y)^\alpha},$$

which is called the Lipschitz constant of f . Define

$$Lip_\alpha(X, E) = \{f : X \rightarrow E : f \text{ is bounded and } p_\alpha(f) < \infty\}.$$

Received July 11, 2017; Revised November 30, 2017; Accepted February 27, 2018.

2010 *Mathematics Subject Classification*. Primary 46B28, 46M05, 46J10.

Key words and phrases. approximately amenability, character amenability, Lipschitz algebra, metric space, vector-valued functions.

The Lipschitz algebra $lip_\alpha(X, E)$ is the subalgebra of $Lip_\alpha(X, E)$ defined by

$$lip_\alpha(X, E) = \{f : X \rightarrow E : \frac{\|f(x) - f(y)\|}{d(x, y)^\alpha} \rightarrow 0 \text{ as } d(x, y) \rightarrow 0\}.$$

Finally, if X is a locally compact metric space, then $lip_\alpha^0(X, E)$ is the subalgebra of $lip_\alpha(X, E)$ consisting of those functions whose norm tend to zero at infinity. The elements of $Lip_\alpha(X, E)$ and $lip_\alpha(X, E)$ are called big and little Lipschitz operators and for each element f of $Lip_\alpha(X, E)$, define

$$\|f\|_{\alpha, E} = p_{\alpha, E}(f) + \|f\|_{\infty, E}.$$

Let $C_b(X, E)$ be the set of all bounded continuous functions from X into E and for each $f \in C_b(X, E)$, define

$$\|f\|_{\infty, E} = \sup_{x \in X} \|f(x)\|,$$

and for $f, g \in C_b(X, E)$ and $\lambda \in \mathbb{F}$, define

$$(f + g)(x) = f(x) + g(x), (\lambda f)(x) = \lambda f(x), (x \in X).$$

It is easy to see that $(C_b(X, E), \|\cdot\|_{\infty, E})$ becomes a Banach space over \mathbb{F} and $Lip_\alpha(X, E)$ is a linear subspace of $C_b(X, E)$ [6]. In their papers [6, 7] Cao, Zhang and Xu proved that $(Lip_\alpha(X, E), \|\cdot\|_{\alpha, E})$ is a Banach space over \mathbb{F} and $lip_\alpha(X, E)$ is a closed linear subspace of $(Lip_\alpha(X, E), \|\cdot\|_{\alpha, E})$. If E is a Banach algebra. Then $(Lip_\alpha(X, E), \|\cdot\|_{\alpha, E})$ is a Banach algebra under pointwise and scalar multiplication and $lip_\alpha(X, E)$ is a closed linear subalgebra of $(Lip_\alpha(X, E), \|\cdot\|_{\alpha, E})$. The spaces $Lip_\alpha(X, E)$ and $lip_\alpha(X, E)$ are called big and little Lipschitz operators algebra. It is clear that the Lipschitz algebras $Lip_\alpha(X, E)$ contains the space $Cons(X, E)$ consisting of all constant E -valued functions on X . The Lipschitz algebras were first considered by Sherbert [22]; see also Bishop [5]. There are valuable works related to some notions of amenability of Lipschitz algebras. Gourdeau [11] discussed amenability of vector-valued Lipschitz algebras. Also he proved that if a Banach algebra A is amenable, then $\Delta(A)$ is uniformly discrete with respect to norm topology induced by A^* ; see also Bade, Curtis, and Dales [3], Gourdeau [10] and Zhang [23]. Moreover Hu, Monfared and Traynor investigated character amenability of Lipschitz algebras, see [13]. They showed that if X is an infinite compact metric space and $0 < \alpha < 1$, then $Lip_\alpha X$ is not character amenable. Moreover, recently C -character amenability of Lipschitz algebras was studied by Dashti, Nasr Isfahani and Soltani for each $\alpha > 0$ see [9]. In fact as a generalization of [20], they showed that for $\alpha > 0$ and any locally compact metric space X , the algebra $Lip_\alpha X$ is C -character amenable, for some $C > 0$, if and only if X is uniformly discrete. Also a necessary and sufficient condition for amenability and character amenability of Lipschitz algebras was provided; see [9]. Sherbert [22], Honary, Nikou and Sanatpour [12] and Bade [3], studied some properties of Lipschitz algebras.

In this paper, we study some structure and properties of vector-valued Lipschitz algebra $Lip_\alpha(X, E)$, $lip_\alpha(X, E)$ and $lip_\alpha^0(X, E)$ of order α . Furthermore we obtain some necessary and sufficient conditions for amenability, C -character amenability and approximately amenability of vector-valued Lipschitz operators algebras.

2. Preliminaries and some basic results

Let (X, d) be a metric space and $\alpha > 0$. It is easy to show that $Lip_\alpha(X, E)$, $lip_\alpha(X, E)$ and $lip_\alpha^0(X, E)$ is a vector space, Banach space and Banach algebra, whenever E is so, respectively. Also it is easy to see that if (X, d) is a metric space and E be a Banach algebra. Then $Lip_\alpha(X, E)$ is a commutative(unital) Banach algebra if and only if E is a commutative(unital) Banach algebra. Let E be a $*$ -Banach algebra, $f^*(x) = (f(x))^*$ for $x \in X$ and $f \in Lip_\alpha(X, E)$. Then $p_\alpha(f^*) = p_\alpha(f)$ and $\|f^*\|_{\infty, E} = \|f\|_{\infty, E}$, so $Lip_\alpha(X, E)$ is a $*$ -Banach algebra. We first bring some preliminary result and definitions. The first one, which appeared in [12] for compact metric space and $0 < \alpha \leq 1$. We generalized for arbitrary metric space and for each $\alpha > 0$.

Lemma 2.1. *Let (X, d) be a metric space, $\alpha > 0$ and E be a Banach algebra. Then the following statements are equivalent.*

- (i) $f \in Lip_\alpha(X, E)$.
- (ii) $\sigma \circ f \in Lip_\alpha X$ for all $\sigma \in E^*$.

Proof. Suppose that $f \in Lip_\alpha(X, E)$ and $\sigma \in E^*$. Then

$$|\sigma \circ f(x) - \sigma \circ f(y)| \leq \|\sigma\| \|f(x) - f(y)\| \leq \|\sigma\| p_{\alpha, E}(f) d(x, y)^\alpha, \quad (x, y \in X).$$

Hence $p_\alpha(\sigma \circ f) \leq \|\sigma\| p_{\alpha, E}(f) < \infty$ and $\|\sigma \circ f\|_\infty \leq \|\sigma\| \|f\|_{\infty, E} < \infty$. Thus $\sigma \circ f \in Lip_\alpha X$. Conversely, Suppose that $T_\sigma : Lip_\alpha(X, E) \rightarrow Lip_\alpha X$, where $f \mapsto \sigma \circ f$ for all $\sigma \in E^*$ and $f \in Lip_\alpha(X, E)$. Then $\{T_\sigma\}_{\sigma \in E^*}$ is a family of continuous linear functions such that

$$\sup_{\|\sigma\| \leq 1} \|T_\sigma(f)\|_\alpha \leq \|f\|_{\alpha, E} < \infty.$$

So by the Principle of Uniform Boundedness, we have $\sup_{\|\sigma\| \leq 1} \|T_\sigma\|_\alpha < M$ for some $M > 0$. So $\|\sigma \circ f\|_\alpha \leq M$. Hence, $p_\alpha(\sigma \circ f) \leq M$ and $\|\sigma \circ f\|_\infty \leq M$ for all $f \in Lip_\alpha(X, E)$. Thus,

$$\|f(x) - f(y)\| = \sup_{\|\sigma\| \leq 1} \{|\sigma(f(x) - f(y))|\} \leq M d(x, y)^\alpha.$$

Also, $\|f(x)\| = \sup_{\|\sigma\| \leq 1} |\sigma(f(x))| \leq M$ and $f \in Lip_\alpha(X, E)$. \square

Remark 2.2. Let (X, d) be a metric space, $\alpha > 0$ and E be a Banach algebra. Then,

$$\|f\|_{\alpha, E} = \sup\{\|\sigma \circ f\|_\alpha : \sigma \in E^*, \|\sigma\| \leq 1\}.$$

Corollary 2.3. *Let (X, d) be a metric space, $0 < \alpha \leq \beta$ and E be Banach algebra. Then*

- (i) $Lip_\beta(X, E) \subseteq Lip_\alpha(X, E)$.
- (ii) $lip_\beta(X, E) \subset Lip_\beta(X, E) \subset lip_\alpha(X, E) \subset Lip_\alpha(X, E)$.

Proof. Suppose that $f \in Lip_\beta(X, E)$, then $\sigma \circ f \in Lip_\beta X$ and since $Lip_\beta X \subseteq Lip_\alpha X$ and so $\sigma \circ f \in Lip_\beta X$, then by Lemma 2.1 $f \in Lip_\alpha(X, E)$ for each $\sigma \in E^*$.

Suppose that $f \in Lip_\beta(X, E)$, then for all $\sigma \in E^*$ we have $\sigma \circ f \in Lip_\beta X \subset lip_\alpha X$. Hence $f \in lip_\alpha(X, E)$. \square

In this section we study the structure of Lipschitz algebra $Lip_\alpha(X, E)$. The following examples show that the algebraic and topological properties of $Lip_\alpha(X, E)$ depend the metric space (X, d) order α and the structure of E . By using Lemma 2.1, for Banach space, $C_0(Y)$, Hilbert space H and $L_p(Y, \mu)$ for $1 \leq p < \infty$ and σ -finite measure μ . The following is immediate.

Example 2.4. Let (X, d) be a metric space and $\alpha > 0$.

(1) Let Y be a locally compact Hausdorff space and $E = C_0(Y)$, $E^* = M_b(Y)$, the space of all complex-valued bounded, regular measures of Y . Thus $f \in Lip_\alpha(X, E)$ if and only if for each $\mu \in M_b(Y)$ there exists $M > 0$ such that

$$\left| \int_Y (f(a) - f(b))(y) d\mu(y) \right| \leq Md(a, b)^\alpha, \quad (a, b \in X).$$

(2) Let H be a Hilbert space and $E = H$. Then $E^* = E$, so $f \in Lip_\alpha(X, E)$ if and only if for each $h \in H$, there exist $M > 0$ and $k \in H$ such that

$$|\langle f(x) - f(y), k \rangle| \leq Md(x, y)^\alpha, \quad (x, y \in X).$$

(3) Let $E = L_p(Y, \mu)$. Then $E^* = L_q(Y, \mu)$ for which $\frac{1}{p} + \frac{1}{q} = 1$. Thus $f \in Lip_\alpha(X, E)$ if and only if for each $g \in L_q(Y, \mu)$ there exists $M > 0$ such that

$$\left| \int_Y (f(a) - f(b))(y) g(y) d\mu(y) \right| \leq Md(a, b)^\alpha, \quad (a, b \in X).$$

(4) Let G be a locally compact Hausdorff topological group and $E = (L^1(G), *)$. Then $E^* = L^\infty(G)$, so $f \in Lip_\alpha(X, E)$ if and only if $g \in L^\infty(G)$ there exist $g \in L^\infty(G)$ and $M > 0$ such that

$$\left| \int_G g(x)(f(a)(x) - f(b)(x)) d\lambda(x) \right| \leq Md(a, b)^\alpha, \quad (a, b \in G).$$

Where λ is the left Haar measure of G .

(5) Let G be a locally compact Hausdorff topological group and $E = A(G)$, the Fourier algebra of G . Then $E^* = VN(G)$, the Von-Neumann algebra of G . Thus $f \in Lip_\alpha(X, E)$ if and only if $\sigma \in VN(G)$ there exists $M > 0$ such that

$$|\sigma(f(a)) - \sigma(f(b))| \leq Md(x, y)^\alpha, \quad (a, b \in G).$$

Let $\sigma \in VN(G)$, $g = g_1 * \tilde{g}_2 \in A(G)$ for $g_1, g_2 \in L^2(G)$. Then

$$\langle \sigma, g_1 * \tilde{g}_2 \rangle := \langle \sigma(g_2), g_1 \rangle = \int_G \sigma(g_2) \overline{g_1(x)} d\lambda(x).$$

Example 2.5. Let (X, d) is a normed space, $\alpha > 1$ and E be Banach algebra. Then $Lip_\alpha(X, E) = cons(X, E)$.

Proof. Suppose that $f \in Lip_\alpha(X, E)$ so by Lemma 2.1 $\sigma \circ f \in Lip_\alpha X$ for all $\sigma \in E^*$ and for $a, b \in X$, define

$$F(t) := \sigma \circ f(ta + (1-t)b), \quad t \in \mathbb{R}.$$

Then, $F : \mathbb{R} \rightarrow \mathbb{C}$ such that there is $M > 0$ such that

$$\begin{aligned} |F(t_1) - F(t_2)| &= |\sigma \circ f(t_1a + (1-t_1)b) - \sigma \circ f(t_2a + (1-t_2)b)| \\ &\leq M \|t_1a + (1-t_1)b - t_2a - (1-t_2)b\|^\alpha \\ &\leq M |t_1 - t_2|^\alpha \|a - b\|^\alpha. \end{aligned}$$

So $F \in Lip_\alpha(\mathbb{R})$. But $Lip_\alpha(\mathbb{R}) = cons(\mathbb{R})$. Hence $F(1) = F(0)$, so for all $\sigma \in E^*$ $\sigma \circ f(a) = \sigma \circ f(b)$. Thus f is a constant. \square

Recall that (X, d) is called uniformly discrete if there exists $\varepsilon > 0$ such that $d(x, y) \geq \varepsilon$ for all $x, y \in X$ with $x \neq y$.

Example 2.6. If (X, d) is not uniformly discrete, and $0 < \alpha \leq 1$, then $cons(X, E) \subset Lip_\alpha(X, E) \subset l_\infty(X, E)$.

Proof. Since (X, d) is not uniformly discrete, then $Lip_\alpha(X, E) \subset l_\infty(X, E)$. Also, define

$$f_\alpha(x) = \frac{d^\alpha(x, x_1)}{d^\alpha(x, x_1) + d^\alpha(x, x_2)}, \quad (x_1, x_2 \in X, x_1 \neq x_2).$$

Then $f_\alpha \in Lip_\alpha(X, E) \setminus cons(X, E)$. \square

Let (X, d) be a compact metric space and $0 < \alpha \leq 1$ and E be Banach algebra. Then $\Delta(C(X, E)) = \{\Delta_{x, \sigma} : x \in X, \sigma \in \Delta(E)\}$, where

$$\Delta_{x, \sigma}(f) = \sigma(f(x)), \quad (f \in Lip_\beta(X, E), x \in X).$$

Define $\varphi : X \times \Delta(E) \rightarrow \Delta(C(X, E))$ where $(x, \sigma) \rightarrow \Delta_{x, \sigma}$. Then φ is a bijection and we denote, $\Delta(C(X, E)) = X \times \Delta(E)$.

Let $(A, \|\cdot\|_A)$ in $(B, \|\cdot\|_B)$ be a Banach algebras such that $B \subset A$ and B in A be dense and the inclusion map $i : B \rightarrow A$ is continuous. Then, $\Delta(B) = \{\varphi|_B : \varphi \in \Delta(A)\}$. In particular, let (X, d) be a compact metric space and $0 < \alpha \leq 1$ and E be Banach algebra. Then $Lip_\alpha(X, E)$ and $lip_\alpha(X, E)$ are dense in $C(X, E)$ and for all $f \in Lip_\alpha(X, E)$, $\|f\|_{\infty, E} \leq \|f\|_{\alpha, E}$, see [4]. Thus, $lip_\alpha(X, E)$ and $Lip_\alpha(X, E)$ are Segal algebra in $C(X, E)$. Thus,

$$\Delta(lip_\alpha(X, E)) = \{\varphi|_{lip_\alpha(X, E)} : \varphi \in X \times \Delta(E)\} = \{\Delta_{x, \sigma}^l : x \in X, \sigma \in \Delta(E)\}.$$

Also

$$\Delta(Lip_\alpha(X, E)) = \{\varphi|_{Lip_\alpha(X, E)} : \varphi \in X \times \Delta(E)\} = \{\Delta_{x, \sigma}^L : x \in X, \sigma \in \Delta(E)\}.$$

Let A be a commutative Banach algebra. Then the radical of A , denoted by $Rad(A)$, is defined by

$$Rad(A) = \bigcap_{\varphi \in \Delta(A)} \ker \varphi.$$

Clearly, $Rad(A)$ is a closed ideal of A . Also A is called semisimple if

$$Rad(A) = \{0\}.$$

Theorem 2.7. *Let (X, d) be a metric space, E be a commutative Banach algebra and $0 < \alpha \leq 1$. Then the following statements are equivalent.*

- (i) $C_b(X, E)$ is semisimple.
- (ii) $Lip_\alpha(X, E)$ is semisimple.
- (iii) $lip_\alpha(X, E)$ is semisimple.
- (iv) E is semisimple.

Proof. (iv) \implies (i) Let $x \in X$ and $\theta_x : C_b(X, E) \rightarrow E$, define by $\theta_x(f) = f(x)$. Then θ_x is linear, continuous and epimorphism. Thus,

$$\theta_x(Rad(C_b(X, E))) \subseteq Rad(E) = \{0\}.$$

Thus

$$Rad(C_b(X, E)) \subseteq \ker(\theta_x) = \{f : f(x) = 0\}.$$

Hence $Rad(C_b(X, E)) \subseteq \bigcap_{x \in X} \ker(\theta_x) = \{0\}$. So $C_b(X, E)$ is semisimple.

(i) \implies (iv) Let $\varphi : E \rightarrow C_b(X, E)$ define by $\varphi(z) = \varphi_z$ where $\varphi_z(x) = z$ for $x \in X$. Then φ is linear, isometric and homomorphism. Hence

$$\varphi(Rad(E)) \subseteq Rad(C_b(X, E)) = \{0\}.$$

But φ is one to one, so $Rad(E) = \{0\}$.

(ii) \implies (iv) Let $\varphi : E \rightarrow Lip_\alpha(X, E)$ define by $\varphi(z) = f_z$ where $f_z(x) = z$ for $x \in X$. Thus $\|f_z\|_{\alpha, E} = \|z\| = \|f_z\|_{\infty, E}$ for each $z \in E$. so

$$\varphi(Rad(E)) \subseteq Rad(Lip_\alpha(X, E)) = \{0\}.$$

Then $Rad(E) = \{0\}$.

(iv) \implies (ii) Let $\sigma \in \Delta(E)$ and $\varphi_\sigma : Lip_\alpha(X, E) \rightarrow Lip_\alpha X$ define by $\varphi_\sigma(f) = \sigma \circ f$. Then φ_σ is linear, continuous and epimorphism. Thus,

$$\varphi_\sigma(Rad(Lip_\alpha(X, E))) \subseteq Rad(Lip_\alpha X) \subseteq \bigcap_{x \in X} \delta_x = \{0\},$$

where $\delta_x(g) = g(x)$ for $g \in Lip_\alpha X$. Hence

$$\begin{aligned} Rad(Lip_\alpha(X, E)) &\subseteq \bigcap_{\sigma \in \Delta(E)} \ker \varphi_\sigma = \{f : \sigma \circ f(x) = 0, \sigma \in \Delta(E), x \in X\} \\ &= \{f : f(x) \in \bigcap_{\sigma \in \Delta(E)} \ker \sigma, x \in X\} \\ &= \{f : f(x) \in Rad(E), x \in X\} = \{0\}. \end{aligned}$$

(i) \implies (iv) Let $\varphi : E \rightarrow Lip_\alpha(X, E)$ define by $\varphi(z) = f_z$, where $f_z(x) = z$ for $x \in X$. Then $\|f_z\|_{\alpha, E} = \|z\| = \|f_z\|_{\infty, E}$ for each $z \in E$. Thus φ is well-defined. Also;

$$\varphi(Rad(E)) \subseteq Rad(Lip_\alpha(X, E)) = \{0\}.$$

and φ is 1-1, so $Rad(E) = \{0\}$.

(iii) \iff (iv) is similar to (ii) \iff (iv). □

Suppose that $x, y \in X$ and $x \neq y$. Let $f(w) = \min\{1, d(w, y)^\alpha\}$. Then

$$f(x) \neq f(y).$$

Let $0 \neq z \in E$ and $g(x) = f(x)z$. Then $g(x) \in E$ and $g(x) \neq 0$, $g(y) = 0$. Hence,

$$p_\alpha(g) = \sup_{a \neq b} \frac{\|f(a)z - f(b)z\|}{d(a, b)^\alpha} = \|z\|p_\alpha(f) < \infty,$$

so $g \in Lip_\alpha(X, E)$. Thus $Lip_\alpha(X, E)$ is a $*$ -Banach algebra such that separates the points of X .

Lemma 2.8. *Let (X, d) be a metric space and E is a Banach algebra. Then the followings holds.*

- (i) $E^* \circ B(X, E) = B(X)$.
- (ii) $E^* \circ Lip_\alpha(X, E) = Lip_\alpha X$.
- (iii) $E^* \circ lip_\alpha(X, E) = lip_\alpha X$.
- (iv) *If X is a locally compact metric space, then $E^* \circ lip_\alpha^0(X, E) = lip_\alpha^0 X$.*
- (v) $E^* \circ l^\infty(X, E) = l^\infty(X)$.
- (vi) $E^* \circ C_b(X, E) = C_b(X)$.

Proof. i) Suppose that $\sigma \in E^*$ and $f \in B(X, E)$. Then

$$\|\sigma \circ f(x)\| \leq \|\sigma\| \|f(x)\| \leq \|\sigma\| \|f\|_\infty, \quad (x \in X).$$

Thus $\sigma \circ f \in B(X)$.

ii) Let $g \in Lip_\alpha X$, $0 \neq z \in E$. Then there exists $\sigma \in E^*$ such that $\sigma(z) = 1$. Put $f(x) := g(x)z$ for $x \in X$. Then $f \in Lip_\alpha(X, E)$ and $g = \sigma \circ f$.

iii) Suppose that $\sigma \in E^*$ and $f \in lip_\alpha(X, E)$. Then

$$\frac{|\sigma \circ f(x) - \sigma \circ f(y)|}{d(x, y)^\alpha} \leq \frac{\|\sigma\| \|f(x) - f(y)\|}{d(x, y)^\alpha} \rightarrow 0.$$

Therefore $\sigma \circ f \in lip_\alpha X$. Conversely, if $g \in lip_\alpha X$, $0 \neq \sigma \in E^*$ and $0 \neq z \in E$. Define $f : X \rightarrow E$ with $f(x) = g(x)z$. Then $f \in lip_\alpha(X, E)$ such that $g = \sigma \circ f$. Hence $lip_\alpha X \subseteq E^* \circ lip_\alpha(X, E)$.

iv) Suppose that $\sigma \in E^*$ and $f \in lip_\alpha^0(X, E)$ so, $f \in lip_\alpha(X, E)$ and $f \in C_0(X, E)$. Hence $\sigma \circ f \in lip_\alpha X \cap C_0(X) = lip_\alpha^0 X$. Conversely, if $g \in lip_\alpha^0 X$ and $0 \neq z \in E$. Then there exist $0 \neq \sigma \in E^*$ such that $\sigma(z) = 1$. Define $f(x) := g(x)z$. Then $f \in lip_\alpha^0(X, E)$ and $\|f(x)\| = \|g(x)z\| = |g(x)| \|z\| < \varepsilon$, $x \in X \setminus K$ for some compact space $K \subseteq X$. Hence $f \in C_0(X, E) \cap lip_\alpha(X, E) = lip_\alpha^0(X, E)$. These complete the proof of lemma. \square

Corollary 2.9. *Let (X, d) be a metric space, $0 < \alpha \leq 1$ and E be a Banach algebra. Then the following statements are equivalent.*

- (i) $Lip_\alpha(X, E) = Cons(X, E)$.
- (ii) $Lip_\alpha X = Cons(X)$.

Proof. (i) \implies (ii) By Lemma 2.8 we have

$$Lip_\alpha X = E^* \circ Lip_\alpha(X, E) = E^* \circ Cons(X, E) = Cons(X).$$

(ii) \implies (i) Let $f \in Lip_\alpha(X, E)$ and $\sigma \in E^*$. Then by Lemma 2.1, $\sigma \circ f \in Lip_\alpha X = Cons(X)$, so $f \in Cons(X, E)$ \square

Theorem 2.10. *Let (X, d) be a metric space, $\alpha > 0$ and E is a Banach algebra. Then the following statements are equivalent.*

- (i) $Lip_\alpha(X, E) = B(X, E)$ with equivalent norms.
- (ii) $Lip_\alpha X = B(X)$ with equivalent norms.
- (iii) (X, d) is uniformly discrete space.
- (iv) $lip_\alpha(X, E) = B(X, E)$ with equivalent norms.
- (v) $lip_\alpha X = B(X)$ with equivalent norms.

Proof. (i) \implies (ii) This is clear by Lemma 2.8,

$$Lip_\alpha X = E^* \circ Lip_\alpha(X, E) = E^* \circ B(X, E) = B(X).$$

(ii) \implies (iii) By using [15] and [1, Proposition 1.1] is immediate.

(iii) \implies (i) Suppose that (X, d) is uniformly discrete. Thus there exists $\varepsilon > 0$ such that for all $x, y \in X$ with $x \neq y$ we have

$$d(x, y) \geq \varepsilon.$$

Suppose that $f \in B(X, E)$. We have

$$p_\alpha(f) = \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{d(x, y)^\alpha} \leq \frac{1}{\varepsilon^\alpha} \sup_{x \neq y} \|f(x) - f(y)\| \leq \frac{2}{\varepsilon^\alpha} \|f\|_\infty < \infty.$$

It follows that $f \in Lip_\alpha(X, E)$. Moreover,

$$\|f\|_\infty \leq \|f\|_\alpha \leq \left(1 + \frac{2}{\varepsilon^\alpha}\right) \|f\|_\infty,$$

and consequently $B(X, E) = Lip_\alpha(X, E)$ with equivalent norms. By a similar argument an above one can show that the statements (iii), (iv) and (v) are equivalent. \square

Theorem 2.11. *Let (X, d) be a metric space, E be a Banach algebra and $\alpha, \beta > 0$ and $\Delta(E) \neq \emptyset$. Then the following statements are equivalent.*

- (i) $Lip_\alpha(X, E) = Lip_\beta(X, E)$ with equivalent norms.
- (ii) $Lip_\alpha X = Lip_\beta X$ with equivalent norms.
- (iii) (X, d) is uniformly discrete or $\alpha = \beta$.
- (iv) $lip_\alpha(X, E) = lip_\beta(X, E)$ with equivalent norms.
- (v) $lip_\alpha X = lip_\beta X$ with equivalent norms.

Proof. (i) \implies (ii) This is obtained by Lemma 2.1. We have

$$Lip_\alpha X = E^* \circ Lip_\alpha(X, E) = E^* \circ Lip_\beta(X, E) = Lip_\beta X.$$

(ii) \iff (iii). This is clear by [2, Lemma 1.5].

(iii) \implies (i) This is clear, since if (X, d) is uniformly discrete. Then by Lemma 2.8 we have $Lip_\alpha(X, E) = B(X, E) = Lip_\beta(X, E)$.

By a similar argument is used in above, one can show that the statements (iii), (iv) and (v) are equivalent. \square

Theorem 2.12. *Let (X, d) be a metric space, E be a Banach algebra, where $\Delta(E) \neq \emptyset$ and $\alpha, \beta > 0$. Then the following statements are equivalent.*

- (i) $Lip_\alpha(X, E)$ is a $Lip_\beta(X, E)$ -module Banach.
- (ii) $Lip_\alpha X$ is a $Lip_\beta X$ -module Banach.
- (iii) $lip_\alpha(X, E)$ is a $lip_\beta(X, E)$ -module Banach.
- (iv) $lip_\alpha X$ is a $lip_\beta X$ -module Banach.
- (v) (X, d) is uniformly discrete or $\alpha \leq \beta$.

Proof. (i) \implies (ii) Suppose that $Lip_\alpha(X, E)$ is a $Lip_\beta(X, E)$ -module Banach. Then

$$Lip_\alpha(X, E)Lip_\beta(X, E) \subset Lip_\alpha(X, E),$$

and

$$Lip_\beta(X, E)Lip_\alpha(X, E) \subset Lip_\alpha(X, E).$$

Let $\sigma \in \Delta(E)$ and $z \in E$ such that $\sigma(z) = 1$, $g \in Lip_\beta X$ and $f \in Lip_\alpha X$. Put $\bar{f}(x) := f(x)z$ and $\bar{g}(x) := g(x)z$. Then $\bar{f} \in Lip_\alpha(X, E)$ and $\bar{g} \in Lip_\beta(X, E)$ and so

$$\bar{f} \cdot \bar{g} \in Lip_\alpha(X, E) \text{ and } \bar{g} \cdot \bar{f} \in Lip_\alpha(X, E).$$

Hence,

$$\sigma \circ \bar{f} \cdot \bar{g} \in Lip_\alpha X \text{ and } \sigma \circ \bar{g} \cdot \bar{f} \in Lip_\alpha X.$$

So, for all $x \in X$ we have

$$\sigma \circ \bar{f} \cdot \bar{g}(x) = \sigma(\bar{f}(x) \cdot \bar{g}(x)) = \sigma(f(x)z \cdot g(x)z) = f(x)\sigma(z)g(x)\sigma(z) = f \cdot g(x).$$

Therefore $f \cdot g = \sigma \circ \bar{f} \cdot \bar{g} \in Lip_\alpha X$. Similarly, $g \cdot f = \sigma \circ \bar{g} \cdot \bar{f} \in Lip_\alpha X$. Let $f \in Lip_\alpha X$ and $g_n \rightarrow g$ in $Lip_\beta X$. Then

$$\|f \cdot g_n - f \cdot g\|_\alpha = \|\sigma \circ \bar{f} \cdot \bar{g}_n - \sigma \circ \bar{f} \cdot \bar{g}\|_\alpha \leq \|\sigma\| \|\bar{f} \cdot \bar{g}_n - \bar{f} \cdot \bar{g}\|_{\alpha, E} \rightarrow 0.$$

Thus $Lip_\alpha X$ is a $Lip_\beta X$ -module Banach.

The implication, (iii) \implies (iv) is similar to (i) \implies (ii).

By using [2, Proposition 1.6] the implication (ii) \leftrightarrow (v) and (iv) \implies (v) is immediate.

(v) \implies (i) Suppose that (X, d) is uniformly discrete. Then

$$Lip_\alpha(X, E) = B(X, E) = Lip_\beta(X, E),$$

since

$$B(X, E)B(X, E) \subset B(X, E).$$

Thus $Lip_\alpha(X, E)$ is a $Lip_\beta(X, E)$ -module Banach. If $\alpha \leq \beta$, clearly

$$Lip_\beta(X, E) \subseteq Lip_\alpha(X, E).$$

Hence

$$Lip_\alpha(X, E)Lip_\beta(X, E) \subseteq Lip_\alpha(X, E)Lip_\alpha(X, E) \subseteq Lip_\alpha(X, E).$$

Therefore, $Lip_\alpha(X, E)$ is a $Lip_\beta(X, E)$ -module Banach.

The implication (v) \implies (iii) is similar to (v) \implies (i). \square

3. Character ameanability of vector-valued Lipschitz algebras

Let A be a Banach algebra and $\Delta(A)$ denotes the spectrum of A consisting of all nonzero multiplicative linear functionals on A . In [16, 17], Kaniuth, Lau and Pym introduced and studied the concept of φ -amenability for Banach algebras, where $\varphi \in \Delta(A)$. In fact, a Banach algebra A is called φ -amenable if there exists a bounded linear functional m on A^* , satisfying $m(\varphi) = 1$ and $m(fa) = m(f).\varphi(a)$ for all $a \in A$ and $f \in A^*$ where $fa \in A^*$ is defined by $(fa)(b) = f(ab)$, ($b \in A$). Let φ be the identity map, then φ -amenability is the same as amenability, see Lau [18]. Moreover, for some $C > 0$, A is called C - φ -amenable if m is bounded by C ; see Hu, Monfared and Traynor [13]. The notion of (right) character amenability was introduced and studied by Monfared [20]. character amenability of A is equivalent to A being φ -amenable for all $\varphi \in \Delta(A)$, and A having a bounded right approximate identity. The concept of character amenability is defined similarly see [17] for more details in this field.

There are valuable works related to some notions of amenable of Lipschitz algebras. Gourdeau [11], discussed amenability of Lipschitz algebras and proved that if a Banach algebra A is amenable, then $\Delta(A)$ is uniformly discrete with respect to norm topology induced by A^* ; see also Bade, Curtis and Dales [3], Gourdeau [10] and Zhang [23]. Hu, Monfared and Traynor [13] investigated character amenability of Lipschitz algebras. They showed that if X is an infinite compact metric space and $0 < \alpha < 1$, then $Lip_\alpha X$ is not character amenable. Moreover, recently C -character amenability of Lipschitz algebras was studied by Dashti, Nasr Isfahani and Soltani for each $\alpha > 0$. In fact as a generalization of [9], they showed that for $\alpha > 0$ and any locally compact metric space X , $Lip_\alpha X$ is C -character amenable for some $C > 0$, if and only if X is uniformly discrete. In this section, we characterized C -character amenability of vector valued Lipschitz algebras.

Lemma 3.1. *Let (X, d) be metric space and $\alpha > 0$ such that $Lip_\alpha(X, E)$ (resp. $lip_\alpha(X, E)$) separates the points of X and E be a Banach algebra with $\Delta(E) \neq \emptyset$. If $Lip_\alpha(X, E)$ (resp. $lip_\alpha(X, E)$) is C -character amenable for some $C > 0$. Then (X, d) is uniformly discrete.*

Proof. Let A be any of the Lipschitz algebras above. Define for all $f \in A$, $x \in X$ and $\psi \in \Delta(E)$. $\phi_x : A \rightarrow \mathbb{C}$ where $\phi_x(f) = \psi \circ f(x)$. Then ϕ_x is a character of A . Since A is C -character amenable for some $C > 0$. It follows from [9, Corollary 2.2] and [10] that $\Delta(A)$ is uniformly discrete. Thus there is

$\varepsilon > 0$ such that for all distinct elements ϕ_x, ϕ_y in $\Delta(A)$,

$$\|\phi_x - \phi_y\|_{A^*} > \varepsilon.$$

We have,

$$\begin{aligned} \varepsilon < \|\phi_x - \phi_y\|_{A^*} &= \sup_{\|f\|_{\alpha, E} \leq 1} \|\phi_x(f) - \phi_y(f)\|_{A^*} \\ &= \sup_{\|f\|_{\alpha, E} \leq 1} |\psi \circ f(x) - \psi \circ f(y)| \\ &= \|\psi\| \sup_{\|f\|_{\alpha, E} \leq 1} \frac{\|f(x) - f(y)\|}{d(x, y)^\alpha} d(x, y)^\alpha \leq \|\psi\| d(x, y)^\alpha \end{aligned}$$

for all $x, y \in X$. Therefore $d(x, y)^\alpha > \frac{\varepsilon}{\|\psi\|}$ and so $d(x, y) > \left(\frac{\varepsilon}{\|\psi\|}\right)^{\frac{1}{\alpha}}$ and (X, d) is uniformly discrete. \square

It should be noted that $Lip_\alpha(X, E)$ does not separate the points of X in general, whenever $\alpha > 1$. For example consider \mathbb{R}^n endowed with the usual Euclidean. Let $f \in Lip_\alpha X$. Then

$$\frac{\|f(x) - f(y) - 0(x - y)\|}{\|x - y\|} \leq M \|x - y\|^{\alpha-1}, \quad (x, y \in X)$$

for some $M > 0$. Thus $Df(z) = 0$ for each $z \in X = \mathbb{R}^n$. Also,

$$\|f(x) - f(y)\| \leq \|Df(z)\| \cdot \|x - y\|$$

for some $z \in X$, so f is constant. Therefore $Lip_\alpha(X) = Cons(X)$. Thus $Lip_\alpha(\mathbb{R}^n, E) = Cons(\mathbb{R}^n, E)$ for all $\alpha > 1$ by Corollary 2.9. In this situation, $Lip_\alpha(\mathbb{R}^n, E)$ does not separate the points of X .

Corollary 3.2. *Let (X, d) be metric space, $0 < \alpha \leq 1$ and E be a commutative Banach algebra. If $Lip_\alpha(X, E)$ (resp. $lip_\alpha(X, E)$) is C -character amenable for some $C > 0$. Then (X, d) is uniformly discrete.*

Let (X, d) be a metric space and $C_c(X, E)$ be $C_b(X, E)$ with compact open topology. In [19, Theorem 1.1], the author showed that $C_c(X, E) \cong C_c(X) \hat{\otimes} E$ is an algebra isomorphism and isometric. If (X, d) is discrete, then compact open topology is the same with discrete topology. Thus $C_c(X, E) = l_\infty(X, E)$ and $C_c(X) = l_\infty(X)$. Hence $l_\infty(X, E) = l_\infty(X) \hat{\otimes} E$ is an algebra isomorphism with equivalent norm. Let $\psi : l_\infty(X) \hat{\otimes} E \rightarrow l_\infty(X, E)$ defined by $\psi(f \otimes a) = f \cdot a$. Then ψ is linear, one to one, homomorphism with dense range. If E is amenable, then $l_\infty(X) \hat{\otimes} E$ is amenable [21]. Thus $l_\infty(X, E) = Lip_\alpha(X, E)$ is amenable.

Theorem 3.3. *Let (X, d) be metric space, $\alpha > 0$ and $Lip_\alpha(X, E)$, $lip_\alpha(X, E)$ separates the points of X and E is Banach algebra. Then the following statements are equivalent.*

- (i) $Lip_\alpha(X, E)$ (resp. $lip_\alpha(X, E)$) is amenable.
- (ii) (X, d) is uniformly discrete and E is amenable.

Proof. (i) \implies (ii) If $Lip_\alpha(X, E)$ is amenable, then (X, d) is uniformly discrete by [11, Theorem, 6]. Let $x_0 \in X$ and $\varphi : Lip_\alpha(X, E) \rightarrow E$ defined by $\varphi(f) = f(x_0)$. Then φ is linear, homomorphism and onto. Thus E is amenable by [21].

(ii) \implies (i) Let (X, d) be uniformly discrete. Then $l_\infty(X, E) = \overline{Lip_\alpha(X, E)}$ is Banach algebra with equivalent norm and $l_\infty(X, E) = \widehat{l_\infty(X)} \widehat{\otimes} E$. Thus $Lip_\alpha(X, E)$ is amenable by [21]. Similarly for $lip_\alpha(X, E)$. \square

Theorem 3.4. *Let (X, d) be metric space, $\alpha > 0$, $Lip_\alpha(X, E)$ and $lip_\alpha(X, E)$ separates the points of X , E is amenable Banach algebra, and $\Delta(E)$ is non-empty. Then the following statements are equivalent.*

- (i) $Lip_\alpha(X, E)$ (resp. $lip_\alpha(X, E)$) is C -character amenable for some $C > 0$.
- (ii) $Lip_\alpha(X, E)$ (resp. $Lip_\alpha(X, E)$) is amenable.
- (iii) (X, d) is uniformly discrete.
- (iv) $Lip_\alpha X$ (resp. $lip_\alpha X$) is C -character amenable for some $C > 0$.
- (v) $Lip_\alpha(X, E)$ (resp. $Lip_\alpha(X, E)$) is amenable.

Proof. (i) \implies (iii) By Lemma 3.1.

(iii) \implies (ii) This follows from [10, Theorem 6] and Theorem 3.3.

(ii) \implies (i) Since $Lip_\alpha(X, E)$ is amenable, it has an approximate diagonal bounded by some $C > 0$; see [21]. So, $Lip_\alpha(X, E)$ is C -character amenable for some by [13, Theorem 2.9].

Also, (iii), (iv) and (v) are equivalent by [9, Theorem 3.1]. \square

Corollary 3.5. *Let (X, d) be a metric space, $0 < \alpha \leq 1$ and E be a commutative amenable Banach algebra. Then the following statements are equivalent.*

- (i) $Lip_\alpha(X, E)$ (resp. $lip_\alpha(X, E)$) is C -character amenable for some $C > 0$.
- (ii) $Lip_\alpha(X, E)$ (resp. $lip_\alpha(X, E)$) is amenable.
- (iii) (X, d) is uniformly discrete.
- (iv) $Lip_\alpha X$ (resp. $lip_\alpha X$) is C -character amenable for some $C > 0$.
- (v) $Lip_\alpha X$ (resp. $lip_\alpha X$) is amenable.

Remark 3.6. Let (X, d) be a locally compact metric space. Then Lemma 3.1, Theorem 3.4 and corollary 3.5 holds for $lip_\alpha^0(X, E)$.

4. Approximately amenability of vector-valued Lipschitz algebras

Let A be a Banach algebra and let X be a Banach A -bimodule. A derivation is a bounded linear map $D : A \rightarrow X$ such that $D(ab) = aD(b) + D(a)b$ ($a, b \in A$). For $x \in X$, the map $ad_x : A \rightarrow X$ defined as $ad_x(a) = ax - xa$ ($a \in A$) is clearly a derivation on A called an inner derivation. A derivation D is called approximately inner if there is a net (x_α) in X such that

$$D(a) = \lim_{\alpha} ad_{x_\alpha}(a) \quad (a \in A).$$

The groundwork for amenability of Banach algebras was laid by Johnson [14]. The concept of amenability has occupied an important place in research in Banach algebras. In classic memoir [14] Johnson, initiated the theory of amenable Banach algebras. In fact a Banach algebra A is called approximately amenable if for each Banach A -bimodule X every continuous derivation $D : A \rightarrow X^*$ is approximately inner. For (X, d) compact metric space and $0 < \alpha \leq 1$ Choi, Ghahramani [8, Theorem, 3.4] shows that $lip_\alpha(X, d)$ is not approximately amenable. Now if (X, d) is a metric space, then we study approximately amenability of vector-valued Lipschitz Algebras.

Definition 4.1. A separated, unbounded, multiplier-bounded configuration (or SUM configuration for short) in A consists of two sequences $(u_n), (p_n) \subseteq A$ which satisfy the following properties.

- (i) (Separated) $u_n p_n = p_n u_n = u_n$ for all n and $u_j p_k = p_k u_j = 0$ whenever $j \neq k$.
- (ii) (Unbounded) $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$.
- (iii) (Multiplier-bounded) The sequences $(\|u_n\|_{mul})$ and $(\|p_n\|_{mul})$ are bounded.

Define the multiplier norm on A by $\|a\|_{mul} := \max(\|\lambda_a\|, \|\rho_a\|)$ where $\lambda_a : A \rightarrow A$, $x \mapsto ax$ is left multiplication by a and $\rho_a : A \rightarrow A$, $x \mapsto xa$ is right multiplication by a .

In [8] they proved that, if A be a Banach algebra, there exists an unbounded but multiplier-bounded sequence $(E_n)_{n \geq 1} \subseteq A$ such that $E_n E_{n+1} = E_n = E_{n+1} E_n$ for all n . Then A contains a SUM configuration. Also if A be a Banach algebra which contains a SUM configuration. Then A is not approximately amenable. We now state the main result of this section.

Theorem 4.2. *Let (X, d) be a compact metric space, E be a unital Banach algebra and $0 < \alpha \leq 1$. Then $lip_\alpha(X, E)$ and $Lip_\alpha(X, E)$ are not approximately amenable.*

Proof. (i) We show that $lip_\alpha(X, E)$ contains a SUM configuration. Suppose that E has a unit e . By [8, Theorem, 3.4] $lip_\alpha(X, d)$ contains a SUM configuration, so there exists an unbounded but multiplier-bounded sequence $(E_n)_{n \geq 1} \subseteq lip_\alpha(X, d)$, such that $E_n E_{n+1} = E_n = E_{n+1} E_n$ for all n . For any $x \in X$ define $F_n := E_n \cdot e$, hence

$$F_n \cdot F_{n+1}(x) = E_n(x) \cdot e E_{n+1}(x) \cdot e = E_n(x) E_{n+1}(x) \cdot e = E_n(x) \cdot e = F_n(x).$$

Thus $F_n \cdot F_{n+1} = F_n$. Similarly, $F_{n+1} \cdot F_n = F_n$. Now we shows that $(F_n) \subseteq lip_\alpha(X, E)$ is unbounded, we have

$$\begin{aligned} p_\alpha(F_n) &= p_\alpha(E_n \cdot e) = \sup_{x \neq y} \frac{\|E_n \cdot e(x) - E_n \cdot e(y)\|}{d(x, y)^\alpha} \\ &= \sup_{x \neq y} \frac{\|E_n(x) \cdot e - E_n(y) \cdot e\|}{d(x, y)^\alpha} \end{aligned}$$

$$= p_\alpha(E_n)\|e\| < \infty$$

and

$$\|E_n\|_\infty = \|E_n \cdot e\|_\infty = \sup_{x \in X} \|E_n \cdot e(x)\| = \sup_{x \in X} \|E_n(x) \cdot e\| \leq \|E_n\|_\infty \|e\|.$$

Hence,

$$\begin{aligned} \|F_n\|_{\alpha, E} &= \|E_n \cdot e\|_{\alpha, E} \\ &= \|E_n \cdot e\|_\infty + p_\alpha(E_n \cdot e) \\ &= \|E_n\|_\infty \|e\| + p_\alpha(E_n)\|e\| \\ &= \|E_n\|_\alpha \|e\| \rightarrow \infty. \end{aligned}$$

Also, $(F_n) \subseteq Lip_\alpha(X, E)$ and

$$\begin{aligned} \frac{\|F_n(x) - F_n(y)\|}{d(x, y)^\alpha} &= \frac{\|E_n \cdot e(x) - E_n \cdot e(y)\|}{d(x, y)^\alpha} \\ &= \frac{\|E_n(x) \cdot e - E_n(y) \cdot e\|}{d(x, y)^\alpha} \\ &= \|e\| \frac{\|E_n(x) - E_n(y)\|}{d(x, y)^\alpha} \rightarrow 0. \end{aligned}$$

Therefore $(F_n) \subseteq lip_\alpha(X, E)$ Also $\|F_n\|_{\alpha, E} \rightarrow \infty$. Now we define

$$\|F_n\|_{mul} = \max\{\|\lambda_{F_n}\|, \|\rho_{F_n}\|\}$$

where $\lambda_{F_n} : lip_\alpha(X, E) \rightarrow lip_\alpha(X, E)$ with $\lambda_{F_n}(f) = F_n \circ f$. Therefore

$$\begin{aligned} \|\lambda_{F_n}\| &= \sup_{\|f\|_{\alpha, E} \leq 1} \|\lambda_{F_n}(f)\|_{\alpha, E} \\ &= \sup_{\|f\|_{\alpha, E} \leq 1} \|F_n \circ f\|_{\alpha, E} \\ &= \sup_{\|f\|_{\alpha, E} \leq 1} \|(E_n \cdot e) \circ f\|_{\alpha, E} \\ &\leq \sup_{\|f\|_{\alpha, E} \leq 1} \|\lambda_{E_n}(f) \cdot e\|_{\alpha, E} \\ &\leq \|e\| \sup_{\|f\|_{\alpha, E} \leq 1} \|\lambda_{E_n}(f)\|_{\alpha, E} \\ &= \|e\| \|\lambda_{E_n}\| < \infty. \end{aligned}$$

Similarly, $\|\rho_{F_n}\| \leq \|\rho_{E_n}\| \|e\| < \infty$. Thus $\|F_n\|_{mul} < \infty$. Therefore $lip_\alpha(X, E)$ contains a SUM configuration. By [8, Theorem, 2.5] $lip_\alpha(X, E)$ is not approximately amenable.

(ii) Since $lip_\alpha(X, E) \subseteq Lip_\alpha(X, E)$. Thus $(F_n) \subseteq Lip_\alpha(X, E)$ is a SUM configuration. Therefore $Lip_\alpha(X, E)$ is not approximately amenable. \square

Question 4.3. When are the vector-valued Lipschitz algebras $lip_\alpha(X, E)$ and $Lip_\alpha(X, E)$ Arens regular?

Acknowledgment. The authors thank Dr Fatemeh Abtahi for her valuable comments and suggestions on the manuscript. The authors also thank the Banach algebra center of Excellence for Mathematics, University of Isfahan.

References

- [1] F. Abtahi, M. Azizi, and A. Rejali, *Character amenability of the intersection of Lipschitz algebras*, *Canad. Math. Bull.* **60** (2017), no. 4, 673–689.
- [2] F. Abtahi, E. Biyabani, and A. Rejali, *Some algebraic and homological of Lipschitz algebras and second dual*, submitted.
- [3] W. G. Bade, P. C. Curtis, Jr., and H. G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras*, *Proc. London Math. Soc.* (3) **55** (1987), no. 2, 359–377.
- [4] B. A. Barnes, *Banach algebras which are ideals in a Banach algebra*, *Pacific J. Math.* **38** (1971), 1–7; correction, *ibid.* **39** (1971), 828.
- [5] E. R. Bishop, *Generalized Lipschitz algebras*, *Canad. Math. Bull.* **12** (1969), 1–19.
- [6] H. X. Cao and Z. B. Xu, *Some properties of nonlinear Lipschitz- α operators*, *Acta Math. Sinica (Chin. Ser.)* **45** (2002), no. 2, 279–286.
- [7] H. X. Cao, J. H. Zhang, and Z. B. Xu, *Characterizations and extensions of Lipschitz- α operators*, *Acta Math. Sin. (Engl. Ser.)* **22** (2006), no. 3, 671–678.
- [8] Y. Choi and F. Ghahramani, *Approximate amenability of Schatten classes, Lipschitz algebras and second duals of Fourier algebras*, *Q. J. Math.* **62** (2011), no. 1, 39–58.
- [9] M. Dashti, R. Nasr-Isfahani, and S. Soltani Renani, *Character amenability of Lipschitz algebras*, *Canad. Math. Bull.* **57** (2014), no. 1, 37–41.
- [10] F. Gourdeau, *Amenability of Banach algebras*, *Math. Proc. Cambridge Philos. Soc.* **105** (1989), no. 2, 351–355.
- [11] F. Gourdeau, *Amenability of Lipschitz algebras*, *Math. Proc. Cambridge Philos. Soc.* **112** (1992), no. 3, 581–588.
- [12] T. G. Honary, A. Nikou, and A. H. Sanatpour, *On the character space of vector-valued Lipschitz algebras*, *Bull. Iranian Math. Soc.* **40** (2014), no. 6, 1453–1468.
- [13] Z. Hu, M. S. Monfared, and T. Traynor, *On character amenable Banach algebras*, *Studia Math.* **193** (2009), no. 1, 53–78.
- [14] B. E. Johnson, *Cohomology in Banach Algebras*, American Mathematical Society, Providence, RI, 1972.
- [15] J. A. Johnson, *Banach spaces of Lipschitz functions and vector-valued Lipschitz functions*, *Trans. Amer. Math. Soc.* **148** (1970), 147–169.
- [16] E. Kaniuth, A. T. Lau, and J. Pym, *On ϕ -amenability of Banach algebras*, *Math. Proc. Cambridge Philos. Soc.* **144** (2008), no. 1, 85–96.
- [17] E. Kaniuth, A. T. Lau, and J. Pym, *On character amenability of Banach algebras*, *J. Math. Anal. Appl.* **344** (2008), no. 2, 942–955.
- [18] A. T. M. Lau, *Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups*, *Fund. Math.* **118** (1983), no. 3, 161–175.
- [19] A. Mallios, *Topological Algebras*, 124, Science Direct April 1986.
- [20] M. S. Monfared, *Character amenability of Banach algebras*, *Math. Proc. Cambridge Philos. Soc.* **144** (2008), no. 3, 697–706.
- [21] V. Runde, *Lectures on Amenability*, *Lecture Notes in Mathematics*, **1774**, Springer-Verlag, Berlin, 2002.
- [22] D. R. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, *Trans. Amer. Math. Soc.* **111** (1964), 240–272.
- [23] Y. Zhang, *Weak amenability of a class of Banach algebras*, *Canad. Math. Bull.* **44** (2001), no. 4, 504–508.

EMAMGHOLI BIYABANI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ISFAHAN
ISFAHAN 81746-73441, IRAN
Email address: biyabani91@yahoo.com, biyabani@sci.ui.ac.ir

ALI REJALI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ISFAHAN
ISFAHAN 81746-73441, IRAN
Email address: rejali@sci.ui.ac.ir

Ahead of Print