Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b170607
pISSN: 1015-8634 / eISSN: 2234-3016

A CHARACTERIZATION OF HYPERBOLIC SPACES

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Abstract. Let M be a complete spacelike hypersurface in the $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1}$. Suppose that every unit speed curve $X(s)$ on M satisfies $\langle X''(s), X''(s)\rangle \geq -1/r^2$ and there exists a point $p \in M$ such that for every unit speed geodesic $X(s)$ of M through the point p, $\langle X''(s), X''(s)\rangle = -1/r^2$ holds. Then, we show that up to isometries of $\mathbb{L}^{n+1}$, M is the hyperbolic space $H^n(r)$.

1. Introduction

We consider an n-dimensional hypersurface M in the $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1} = \mathbb{R}^{n+1}_1$ with the canonical flat metric $ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2$, where $x = (x_1, \ldots, x_{n+1}) \in \mathbb{L}^{n+1}$. Let us denote by $H^n(r) \subset \mathbb{L}^{n+1}$ the spacelike hyperquadric of radius r defined by $\langle x, x\rangle = -r^2$ with $x_{n+1} > 0$. Then $H^n(r)$ is a Riemannian space form with constant sectional curvature $K = -1/r^2$, which is called the standard imbedding of the hyperbolic space of curvature $K = -1/r^2$, or simply the hyperbolic space ([3]).

We have the following property of the hyperbolic space $H^n(r)$.

Proposition 1.1. *The hyperbolic space $H^n(r)$ satisfies the following condition.*
(C) Every unit speed curve $X(s)$ on $H^n(r)$ satisfies $\langle X''(s), X''(s)\rangle \geq -1/r^2$.

Proof. Let $X(s)$ be a unit speed curve in the hyperbolic space $H^n(r)$, so we have $\langle X(s), X(s)\rangle = -r^2$ and $\langle X'(s), X'(s)\rangle = 1$ for all s. For the unit normal $U(s) = X(s)/r$ along $X(s)$, we consider the following orthogonal decomposition:

$$X''(s) = \mathbf{k}_g(s) - \langle X''(s), U(s)\rangle\, U(s), \tag{1.1}$$

Received July 11, 2017; Accepted January 1, 2018.
2010 *Mathematics Subject Classification.* 53B25, 53B30.
Key words and phrases. Minkowski space, hyperbolic space, normal curvature, spacelike hypersurface.
The first named author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2015R1D1A3A01020387).
The second named author was supported by Kyungpook National University Bokhyeon Research Fund, 2015.

where $\mathbf{k}_g(s)$ is the tangential part of $X''(s)$. Together with $\langle X'(s), X'(s)\rangle = 1$, differentiating $\langle X(s), X(s)\rangle = -r^2$ twice yields $\langle X''(s), X(s)\rangle = -1$. Hence we get from (1.1)

$$X''(s) = \mathbf{k}_g(s) + \frac{1}{r}U(s), \tag{1.2}$$

which shows that

$$\langle X''(s), X''(s)\rangle = \langle \mathbf{k}_g(s), \mathbf{k}_g(s)\rangle - \frac{1}{r^2}. \tag{1.3}$$

Since $\langle \mathbf{k}_g(s), \mathbf{k}_g(s)\rangle \geq 0$, (1.3) completes the proof. □

The hyperbolic spaces of radius exceeding r also satisfy the condition (C). Thus it is natural to ask what conditions should be added to the property (C) in order to characterize the hyperbolic space $H^n(r)$.

In this note, we prove the following:

Theorem 1.2. *Let M be a complete spacelike hypersurface in the $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1}$. Then, up to isometries of $\mathbb{L}^{n+1}$, M is the hyperbolic space $H^n(r)$ if and only if it satisfies*

(C) Every unit speed curve $X(s)$ on M satisfies $\langle X''(s), X''(s)\rangle \geq -1/r^2$.

(G) There exists a point $p \in M$ such that for every unit speed geodesic $X(s)$ of M through the point p, $\langle X''(s), X''(s)\rangle = -1/r^2$ holds.

With the help of Lemma 2.1 and (2.4) in Section 2, we obtain a characterization of the hyperbolic space $H^n(r)$ in terms of normal curvatures as follows.

Theorem 1.3. *Let M be a complete spacelike hypersurface in the $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1}$. Then, up to isometries of $\mathbb{L}^{n+1}$, M is the hyperbolic space $H^n(r)$ if and only if it satisfies*

(N) For every unit tangent vector v to M, the normal curvature $\kappa_n(v)$ in the direction of v satisfies $|\kappa_n(v)| \leq 1/r$.

(G') There exists a point $p \in M$ such that for every unit speed geodesic $X(s)$ of M through the point p, the normal curvature $\kappa_n(X'(s))$ in the direction of $X'(s)$ satisfies $|\kappa_n(X'(s))| = 1/r$.

For closed hypersurfaces in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$, the following characterizations were established ([1]).

Proposition 1.4. *Suppose that M is a closed hypersurface in $\mathbb{E}^{n+1}$ satisfying the following two conditions*:

$(C1)$ every curve on M has curvature ≥ 1;

$(C2)$ on M there exists a curve γ_0 of length π with constant curvature 1.

Then M is the unit sphere.

In [2], the first two authors and D. W. Yoon provide several characterizations for the standard imbeddings of hyperbolic spaces in the $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1}$ under suitable intrinsic and extrinsic assumptions on quantities such as the n-dimensional area of the sections cut off by hyperplanes, the $(n+1)$-dimensional volume of regions between parallel hyperplanes, and the n-dimensional surface area of regions between parallel hyperplanes.

Throughout this note, all objects are smooth and connected, unless otherwise mentioned.

2. Proofs

Suppose that M is a spacelike hypersurface (that is, the induced metric is positive definite) in the $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1}$. We consider a timelike unit normal vector U to the hypersurface M. Then we have $\langle U, U\rangle = -1$. For a point $p \in M$, we denote by $S_p : T_pM \to T_pM$ the shape operator defined by

$$S_p(v) = -\nabla_v U, \tag{2.1}$$

where v is a tangent vector to M at p and ∇ the usual connection of $\mathbb{L}^{n+1}$. For a unit tangent v at $p \in M$ the normal curvature $\kappa_n(v)$ in the direction of v is given by

$$\kappa_n(v) = \langle S_p(v), v\rangle . \tag{2.2}$$

First, we give a condition that is equivalent to (C).

Lemma 2.1. *For a spacelike hypersurface M in $\mathbb{L}^{n+1}$, the following conditions are equivalent*:

(C) *Every unit speed curve $X(s)$ on M satisfies $\langle X''(s), X''(s)\rangle \geq -1/r^2$.*

(N) *For every unit tangent vector v to M, the normal curvature $\kappa_n(v)$ in the direction of v satisfies $|\kappa_n(v)| \leq 1/r$.*

Proof. For any unit tangent vector v to M at p, we consider the geodesic X with $X(0) = p$ and $X'(0) = v$. Then for a given local unit normal vector field U to M around p we have $X''(0) = -\langle X''(0), U\rangle U$ because $X''(0)$ is normal to the hypersurface M. Hence the normal curvature satisfies

$$\kappa_n(v) = \langle S_p(v), v\rangle = \langle X''(0), U\rangle . \tag{2.3}$$

In the direction of v, we have

$$\kappa_n(v)^2 = -\langle X''(0), X''(0)\rangle . \tag{2.4}$$

Thus, (N) follows from (C).

For any unit speed curve $X(s)$ on M, we consider the following orthogonal decomposition:

$$X''(s) = \mathbf{k}_g(s) - \langle X''(s), U\rangle U, \tag{2.5}$$

where $\mathbf{k}_g(s)$ is the tangential part of $X''(s)$. Then, it follows from (2.3) that the normal curvature $\kappa_n(X'(s))$ of M in the direction of $X'(s)$ at $X(s)$ satisfies

$$X''(s) = \mathbf{k}_g(s) - \kappa_n(X'(s))U. \tag{2.6}$$

Hence we get

$$\langle X''(s), X''(s)\rangle = \langle \mathbf{k}_g(s), \mathbf{k}_g(s)\rangle - \kappa_n(X'(s))^2 \geq -\kappa_n(X'(s))^2, \tag{2.7}$$

where the inequality follows from $\langle \mathbf{k}_g(s), \mathbf{k}_g(s)\rangle \geq 0$. Thus, (N) together with (2.7) implies (C). $\square$

Next, we show that the curve $X(s)$ in Condition (G) of the main theorem is nothing but a hyperbola as follows.

Lemma 2.2. *Suppose that a spacelike hypersurface M in $\mathbb{L}^{n+1}$ satisfies Condition (C). If a unit speed curve $X(s)$ on M satisfies $\langle X''(s), X''(s)\rangle = -1/r^2$, then it is a hyperbola given by*

$$X(s) = a\cosh\frac{s}{r} + b\sinh\frac{s}{r} + c \tag{2.8}$$

for some vectors $a, b, c \in \mathbb{L}^{n+1}$.

Proof. Suppose that a unit speed curve $X(s)$ on M satisfies $\langle X''(s), X''(s)\rangle = -1/r^2$. We consider the orthogonal decomposition of $X''(s)$ given by (2.6). Then, we get

$$\begin{aligned} \langle X''(s), X''(s)\rangle &= |\mathbf{k}_g(s)|^2 - \kappa_n(X'(s))^2 \\ &\geq |\mathbf{k}_g(s)|^2 - \frac{1}{r^2}, \end{aligned} \tag{2.9}$$

where the inequality follows from Lemma 2.1.

Together with (2.9), the hypothesis $\langle X''(s), X''(s)\rangle = -1/r^2$ implies that $X(s)$ is a geodesic with $\kappa_n(X'(s)) = 1/r$ (after replacing U with $-U$ if necessary). Hence we obtain from (2.6)

$$X''(s) = -\frac{1}{r}U(s). \tag{2.10}$$

Since $\kappa_n(X'(s)) = 1/r$ is a maximum value of normal curvatures at $X(s)$, $X'(s)$ is a principal direction. Hence we get

$$S_{X(s)}(X'(s)) = \frac{1}{r}X'(s). \tag{2.11}$$

By the definition of S, we have $S_{X(s)}(X'(s)) = -U'(s)$. Thus, it follows from (2.10) and (2.11) that

$$X'''(s) = \frac{1}{r^2}X'(s). \tag{2.12}$$

This completes the proof. $\square$

Proof of Theorem 1.2. Suppose that M is a complete spacelike hypersurface in the $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1}$ satisfying Conditions (C) and (G). Then, by isometries of $\mathbb{L}^{n+1}$, we may assume that $p = (0, \ldots, 0, r)$ and the unit normal $U(p)$ at p is given by $U(p) = (0, \ldots, 0, -1)$.

For any arbitrary point $q \in M$, the completeness of M implies that there exists a unit speed geodesic $X(s)$ of M connecting p and q with $X(0) = p$.

Together with the hypotheses, Lemma 2.2 shows that for some vectors $a, b, c \in \mathbb{L}^{n+1}$ the geodesic $X(s)$ is given by (2.8). Hence we get

$$X'(s) = \frac{a}{r}\sinh\frac{s}{r} + \frac{b}{r}\cosh\frac{s}{r}. \tag{2.13}$$

Since the geodesic $X(s)$ is of unit speed, it follows from (2.13) that

$$\langle a, b\rangle = 0, \quad \langle a, a\rangle + \langle b, b\rangle = 0, \quad \langle b, b\rangle - \langle a, a\rangle = 2r^2, \tag{2.14}$$

which shows that

$$\langle a, b\rangle = 0, \quad -\langle a, a\rangle = \langle b, b\rangle = r^2. \tag{2.15}$$

It follows from (2.10) and the assumption $U(p) = (0, \ldots, 0, -1)$ that $a = (0, \ldots, 0, r)$. Hence, from (2.15) we see that $b = ru$ for some unit vector $u = (u_1, \ldots, u_n, 0)$. Together with the initial condition $X(0) = p$, this shows that $X(s)$ is given by

$$X(s) = (0, \ldots, 0, r)\cosh\frac{s}{r} + ru\sinh\frac{s}{r}. \tag{2.16}$$

Thus, $X(s)$ satisfies

$$\langle X(s), X(s)\rangle = -r^2. \tag{2.17}$$

This completes the proof of the if part of Theorem 1.2.

The converse follows from Proposition 1.1 and (1.3). □

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