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PULLBACKS OF $\mathcal{C}$-HEREDITARY DOMAINS

Yongyan Pu, Gaohua Tang, and Fanggui Wang

Abstract. Let $(RDTF, M)$ be a Milnor square. In this paper, it is proved that R is a $\mathcal{C}$-hereditary domain if and only if both D and T are $\mathcal{C}$-hereditary domains; R is an almost perfect domain if and only if D is a field and T is an almost perfect domain; R is a Matlis domain if and only if T is a Matlis domain. Furthermore, to give a negative answer to Lee's question, we construct a counter example which is a $\mathcal{C}$-hereditary domain R with $w.gl.\dim(R) = \infty$.

Throughout this paper all rings are commutative with identity element and all modules are unitary. For a ring R and an R-module M, we use $\mathrm{pd}_R M$ and $\mathrm{fd}_R M$ to denote, respectively, the classical projective and flat dimension of M. We use $gl.\dim(R)$ and $w.gl.\dim(R)$ to denote, respectively, the classical global and weak dimension of R. If R is an integral domain, we denote its quotient field by Q.

The motivation for this paper was the following question posed by Sang B́um Lee in [9]: is it true that, for an integral domain R, if all flat R-modules are of projective dimension ≤ 1, then $\mathrm{pd}_R Q \leq 1$ and $w.gl.\dim(R) \leq 2$? First of all, recall that a domain R is called a *Matlis domain* if the projective dimension of R-module Q is 1. Then R is a Matlis domain if and only if all divisible R-modules are h-divisible modules (see [9, Lemma 2.4], [8, p. 252]). (Recall: an R-module M is called *h-divisible* if it is an epic image of an injective R-module.) Next, recall that an R-module M is said to be *cotorsion* if $\mathrm{Ext}^1_R(F, M) = 0$ for all flat R-modules F (see [4]). The class of all cotorsion modules is denoted by $\mathcal{C}$. In [10], Mao and Ding introduce the cotorsion dimension of modules and rings, which are defined as follows. The cotorsion dimension of an R-module M, denoted by $\mathrm{cd}_R M$, is the least positive integer n for which $\mathrm{Ext}^{n+1}_R(F, M) = 0$ for all flat R-modules F. The *global cotorsion dimension* of R, denoted by $\mathrm{CD}(R)$, is the quantity:

$$\mathrm{CD}(R) = \sup\{\mathrm{cd}_R M \mid M \in \mathfrak{M}_R\}.$$

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On the basis of the classical homological theory, in this paper, a domain R is said to be a $\mathcal{C}$-*hereditary domain* if $\mathrm{CD}(R) \leq 1$. In [9,10], we have the following proposition.

Proposition 1 ([9, Theorem 3.2], [10, Theorem 7.2.8])**.** *For a domain R, the following are equivalent*:
(1) *R is a $\mathcal{C}$-hereditary domain.*
(2) *All h-divisible R-modules are cotorsion.*
(3) *All flat R-modules are of projective dimension ≤ 1.*

Therefore, by Proposition 1, $\mathcal{C}$-hereditary domains coincide with 1-perfect domains. (Recall: a domain R is said to be n-*perfect* if every flat R-module has projective dimension less or equal than n (see [5]).) Obviously, we have the following.

Corollary 2. (1) *If R is a $\mathcal{C}$-hereditary domain, then R is a Matlis domain.*
(2) *A domain R is $\mathcal{C}$-hereditary if and only if all divisible R-modules are cotorsion.*

So Lee's question can be simply stated that: is it true that if R is a $\mathcal{C}$-hereditary domain, then $w.gl.\dim(R) \leq 2$? To give a negative answer to this open problem, we should investigate properties of $\mathcal{C}$-hereditary domains in Milnor squares. A commutative square of ring homomorphisms $(RDTF)$

$$\begin{array}{ccc} R & \xrightarrow{\lambda_2} & T \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ D & \xrightarrow{\lambda_1} & F \end{array}$$

is said to be a pullback square, if given $(x, y) \in D \times T$ and $\lambda_1(x) = \pi(y)$, there exists a unique element $r \in R$ such that $\pi_1(r) = x$ and $\lambda_2(r) = y$. The ring R is called a pullback of D and T over F. We shall refer to the diagram $(RDTF)$ as a pullback square of type $(RDTF)$. In this pullback square, if λ_1 is a monomorphism and π is surjective, the pullback diagram $(RDTF)$ is called a *Cartesian square*. So, in a Cartesian square $(RDTF)$, we can think that R is a subring of T, D is a subring of F, $M = \ker(\pi)$ is a common ideal of R and T. To simplify the notation, we write $F = T/M$ and $D = R/M$. Therefore, we also express a Cartesian square as $(RDTF, M)$. Accordingly, if F is a field, D and T are integral domains, then the Cartesian square $(RDTF, M)$ is called a *Milnor square*. Especially, in a Milnor square $(RDTF, M)$, if M is a nonzero maximal ideal of T and $T = F + M$, then $F \cap M = 0$. Each subring D of F determines a subring $R = D + M$ of T. This construction is the well-known $D + M$-*construction*.

Let $(RDTF, M)$ be a Milnor square, P be a D-module, and N be a T-module. If there exists an F-isomorphism $h : F \otimes_D P \to F \otimes_T N$, we can make a pullback (P, N, h) of P and N over the F-isomorphism h:

$$A := (P, N, h) = \{(x, y) \in P \times N \mid h(1 \otimes x) = 1 \otimes y\}.$$

More precisely, if $\lambda_1 : P \to F \otimes_D P$ and $\lambda_2 : N \to F \otimes_T N$ are the natural homomorphism, then A is a pullback of R-modules in the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{p_2} & N \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \lambda_2} \\ P & \xrightarrow{h\lambda_1} & F \otimes_T N, \end{array}$$

where p_1, p_2 are the projective maps, that is, $p_1(x_1, x_2) = x_1$, $p_2(x_1, x_2) = x_2$, $(x_1, x_2) \in A$. For additional information on pullbacks, we refer the reader to [14, Chapter 8].

To study the properties of a $\mathcal{C}$-hereditary domain in Milnor squares, we should recall some properties of Milnor squares.

Lemma 3 ([14, Theorem 8.2.1 and Theorem 8.2.3])**.** *Let $(RDTF, M)$ be a Cartesian square and A be an R-module.*

(1) *A is a flat R-module if and only if $T \otimes_R A$ is a flat T-module and $D \otimes_R A$ is a flat D-module.*

(2) *A is a projective R-module if and only if $T \otimes_R A$ is a projective T-module and $D \otimes_R A$ is a projective D-module.*

Lemma 4 ([14, Theorem 8.2.2 and Theorem 8.2.4])**.** *Let $(RDTF, M)$ be a Cartesian square, P be a D-module, N be a T-module. If there exists an F-isomorphism $h : F \otimes_D P \to F \otimes_T N$, we can make a pullback (P, N, h) of P and N over the F-isomorphism. Let $A = (P, N, h)$.*

(1) *If P is a flat D-module and N is a flat T-module, then A is a flat R-module.*

(2) *If P is a projective D-module and N is a projective T-module, then A is a projective R-module.*

Lemma 5 ([14, Proposition 8.3.1, Proposition 8.2.8 and Theorem 8.3.10])**.** *Let $(RDTF, M)$ be a Milnor square. Then:*

(1) *R and T have the same quotient field Q.*

(2) *If A is an R-submodule of some T-module, then A/MA is a torsion-free D-module.*

(3) *T is a flat R-module if and only if F is the quotient field of D.*

Next, we can give another characterization of a $\mathcal{C}$-hereditary domain in Milnor squares.

Theorem 6. *Let $(RDTF, M)$ be a Milnor square. Then R is a $\mathcal{C}$-hereditary domain if and only if both D and T are $\mathcal{C}$-hereditary domains.*

Proof. To prove the sufficiency, we can assume that A is a flat R-module and P is a projective R-module such that $P \to A \to 0$ is exact. We have only to show that $\mathrm{pd}_R A \leq 1$ by Proposition 1. The exactness of the sequence $0 \to B \to P \to A \to 0$ induces the exact sequence

$$0 \to D \otimes_R B \to D \otimes_R P \to D \otimes_R A \to 0, \text{ and}$$

$$0 \to T \otimes_R B \to T \otimes_R P \to T \otimes_R A \to 0,$$

where $\mathrm{Tor}_1^R(D, A) = \mathrm{Tor}_1^R(T, A) = 0$. Clearly, by Lemma 3, $D \otimes_R A$ is a flat D-module and $T \otimes_R A$ is a flat T-module respectively. So, by hypothesis, $\mathrm{pd}_R(D \otimes_R A) \leq 1$ and $\mathrm{pd}_R(T \otimes_R A) \leq 1$. Then $D \otimes_R B$ is a projective D-module and $T \otimes_R B$ is a projective T-module respectively. Thus, by Lemma 3, B is a projective R-module. Consequently, $\mathrm{pd}_R A \leq 1$, completing the proof.

Conversely, let N be a flat T-module. Since F is a field, $F \otimes_R N$ is a free F-module. Then there exists an isomorphism $h : F \otimes_D P \to F \otimes_T N$ for some free D-module P. We can make a pullback of an R-module $A = (P, N, h)$ of P and N over the F-isomorphism h. Consequently, $D \otimes_R A \cong P$ and $T \otimes_R A \cong N$, A is a flat R-module by Lemma 4. Obviously, $\mathrm{pd}_R A \leq 1$ by hypothesis. Let A_0 and A_1 be a projective.The exactness of the sequence $0 \to A_0 \to A_1 \to A \to 0$ induces the exact sequence $0 \to T \otimes_R A_0 \to T \otimes_R A_1 \to N \to 0$. Thus $\mathrm{pd}_T N \leq 1$ and T is a $\mathcal{C}$-hereditary domain by Proposition 1. By the same way, we can prove that D is a $\mathcal{C}$-hereditary domain, establishing the result. □

Now we study almost perfect domains which are special $\mathcal{C}$-hereditary domains. Recently, the notion of almost perfect domains has been introduced by Bazzoni and Sacle [3]. A ring R is said to be almost perfect if all its proper homomorphic images are perfect (see [2, 3]). And they proved that an almost perfect ring which is not a domain is always perfect (see [3, 12]). Consequently, they focused their work on almost perfect domains. In [1], Bass linked perfect rings with the finitistic projective dimension of rings. Recall that the *finitistic projective dimension* of a ring R, denoted by $\mathrm{FPD}(R)$, is defined by:

$$\mathrm{FPD}(R) = \sup\{\mathrm{pd}_R M \mid M \text{ is an } R\text{-module and } \mathrm{pd}_R M < \infty\}.$$

In [7], an almost perfect domain R is used to relate $\mathrm{FPD}(R)$ as follows:

Lemma 7. *For a domain R, the following are equivalent*:

(1) *R is an almost perfect domain.*

(2) $\mathrm{FPD}(R) \leq 1$.

(3) $\mathrm{FPD}(R/(u)) = 0$, *for any nonzero nonunit $u \in R$. Namely, $R/(u)$ is a perfect ring.*

(4) *Flat R-submodules of projective R-modules are again projective.*

Proof. (1) ⇔ (2) ⇔ (4) See [7, Theorem 6.3, Corollary 6.4].

(1) ⇒ (3) Trivial.

(3) ⇒ (1) Let I be a nonzero proper ideal of R, $u \in I$ and $u \neq 0$. $R/(u)$ is a perfect ring by hypothesis. Since R/I is an epic image of $R/(u)$, then R/I is a perfect ring by [14, Corollary 3.10.23], completing the proof. □

Recall that R is an almost perfect domain if and only if every R-module of flat dimension ≤ 1 has projective dimension ≤ 1 (see [7, Corollary 6.4]). An immediate consequence is the following.

Proposition 8. *If R is an almost perfect domain, then R is a $\mathcal{C}$-hereditary domain.*

Almost perfect domains can be characterized in several ways, one of which is using the $D+M$ construction. Let us note a theorem by Bazzoni and Salce [3], i.e., let $T = F[[Y]]$ be the power series ring in the indeterminate Y with coefficients in the field F, and let every local subring D of F with maximal ideal P consider the domain $R = D + M$, where $M = YT$, then R is an almost perfect domain if and only if D is a field (see [3, Lemma 3.1]). Next, we turn to a question involving an almost perfect domain in Milnor squares. Namely, we give a generalization of [3, Lemma 3.1].

Theorem 9. *Let $(RDTF, M)$ be a Milnor square. Then R is an almost perfect domain if and only if D is a field and T is an almost perfect domain, viz.* $\mathrm{FPD}(R) \leq 1$ *if and only if D is a field and* $\mathrm{FPD}(T) \leq 1$.

Proof. To prove the sufficiency, we can assume that A is a flat R-submodule of a projective R-module P. We have only to prove that A is a projective R-module by Lemma 7. Denote $C = P/A$. So $\mathrm{fd}_R C \leq 1$. Consider an exact sequence $0 \to A \to P \to C \to 0$. Note that T is a torsion-free R-module. So $\mathrm{Tor}_1^R(T, C) = 0$. Consequently, it induces the exact sequence $0 \to T \otimes_R A \to T \otimes_R P \to T \otimes_R C \to 0$. By Lemma 3, $T \otimes_R A$ is a flat T-submodule of a projective T-module $T \otimes_R P$. Clearly, $\mathrm{FPD}(T) \leq 1$ by Lemma 7. Then $T \otimes_R A$ is a projective T-module. Obviously, by hypothesis, $D \otimes_R A$ is a projective D-module. Then A is a projective R-module by Lemma 3. Thus R is an almost perfect domain by Lemma 7.

Conversely, we first note that $D \cong R/M$ is a proper homomorphic image of R. Then D is a perfect domain. Therefore D is a field. To prove that T is an almost perfect domain, by Lemma 7, we have only to prove that T/uT is a perfect ring for any nonzero nonunit $u \in T$. Let $a \in M$ be a nonzero nonunit. Clearly, $auT \subseteq uT$. So a natural homomorphism $T/auT \to T/uT$ is surjective. Consequently, we should have only to prove that $\overline{T} = T/auT$ is a perfect ring. So we may assume that $u \in M$ and $\overline{T} = T/uT$. Meanwhile, $uT \subseteq M \subseteq R$. Denote $\overline{R} = R/uT$. Then the commutative diagram

$$\begin{array}{ccc} \overline{R} & \longrightarrow & \overline{T} \\ \downarrow & & \downarrow \\ D & \longrightarrow & F \end{array}$$

is a Cartesian square. Let N be a flat $\overline{T}$-module. Since F is a field, $F \otimes_{\overline{T}} N$ is a free F-module. Then there exists an isomorphism $h : F \otimes_D P \to F \otimes_{\overline{T}} N$ for some free D-module P. As a result, we can make a pullback $A = (P, N, h)$ of P and N over the F-isomorphism h. So $\overline{T} \otimes_{\overline{R}} A \cong N$ by [14, Theorem 8.1.9] and A is a flat $\overline{R}$-module by Lemma 4. Since $\overline{R}$ is a perfect ring, then

A is a projective $\overline{R}$-module. Thus N is a perfective $\overline{T}$-module, completing the proof. □

To construct a counter example to the Lee's question, we should study the finitistic flat dimension in Milnor squares. In [1], recall that the *finitistic flat dimension* of a ring R, denoted by $\mathrm{FFD}(R)$, is defined as follows:

$$\mathrm{FFD}(R) = \sup\{\mathrm{fd}_R M \mid M \text{ is an } R\text{-module and } \mathrm{fd}_R M < \infty\}.$$

Obviously, $\mathrm{FFD}(R) \leq \mathrm{FPD}(R)$. Meanwhile, if $w.gl.\dim(R) < \infty$, then $\mathrm{FFD}(R) = w.gl.\dim(R)$. Recall that R is a Prüfer domain if and only if $w.gl.\dim(R) \leq 1$ if and only if every torsion-free R-module is flat (see [6, p.194], [14, Theorem 3.7.13]). In Milnor squares, the finitistic flat dimension is used to relate a Prüfer domain as follows:

Theorem 10. *Let $(RDTF, M)$ be a Milnor square and F be the field of quotients of D. If $\mathrm{FFD}(T) \leq 1$ and D is a Prüfer domain, then $\mathrm{FFD}(R) \leq 1$.*

Proof. Let A be an R-module and $\mathrm{fd}_R A < \infty$. To prove $\mathrm{FFD}(R) \leq 1$, we have only to prove that $\mathrm{fd}_R A \leq 2$ implies $\mathrm{fd}_R A \leq 1$. We can assume that F_0, F_1 and F_2 are flat R-modules such that $0 \to F_2 \to F_1 \to F_0 \overset{g}{\to} A \to 0$ is exact. This exact sequence induces the exact $0 \to T \otimes_R F_2 \to T \otimes_R F_1 \to T \otimes_R F_0 \overset{1\otimes g}{\to} T \otimes_R A \to 0$, where T is a flat R-module by Lemma 5. Let $B = \ker(g)$. Consider an exact sequence $0 \to B \to F_0 \overset{g}{\to} A \to 0$. Clearly, $\mathrm{fd}_T(T \otimes_R A) \leq 1$ by hypothesis. So $T \otimes_R B$ is a flat R-module. Meanwhile, obviously, B is a torsion-free R-module and an R-submodule of a T-module $Q \otimes_R A$. Then B/MB is a torsion-free D-module by Lemma 5. By hypothesis, B/MB is a flat D-module. Thus B is a flat R-module by Lemma 3 and $\mathrm{fd}_R A \leq 1$, completing the proof. □

Lemma 11 ([6, Corollary 1.1.9]). *Let $(RDTF, M)$ be a Milnor square. Then R is a Prüfer domain if and only if D and T are Prüfer domains and F is the field of quotients of D.*

Theorem 12. *Let $(RDTF, M)$ be a Milnor square such that F is not the field of quotients of D. If T is an almost perfect domain and D is a Prüfer domain, then $w.gl.\dim(R) = \infty$.*

Proof. To prove $w.gl.\dim(R) = \infty$, by way of contradiction, we can assume that $w.gl.\dim(R) < \infty$. Let $(RDTF, M)$ be the Milnor square and L be the field of quotients of D, $T_1 = L + M$. So the Milnor square $(RDTF, M)$ can be divided into two Milnor squares which are the Milnor square (RDT_1L, M) and the Milnor square (T_1LTF, M). The commutative diagrams

$$\begin{array}{ccccc} R & \longrightarrow & T_1 & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & L & \longrightarrow & F \end{array}$$

are Milnor squares. Clearly, T_1 is an almost perfect domain by Theorem 9 and $\mathrm{FFD}(T_1) \leq \mathrm{FPD}(T_1) \leq 1$. So $\mathrm{FFD}(R) \leq 1$ by Theorem 10. Then $w.gl.\dim(R) \leq 1$ by hypothesis, and so R is a Prüfer domain. Obviously, in a Milnor square $(RDTF, M)$, F is the field of quotients of D by Lemma 11, a contradiction. Thus $w.gl\dim(R) = \infty$. □

Next, we construct a counter example which is a $\mathcal{C}$-hereditary domain R with $w.gl.\dim(R) = \infty$. Consequently, we give a negative answer to the Lee's question. Meanwhile, this concrete example shows that a $\mathcal{C}$-hereditary domain R is not almost perfect, which means that the converse of Proposition 8 is not true in general.

Example 13. Let $\mathbb{Z}$ denote the ring of integers, and let $\mathbb{R}$ denote the field of real numbers. Consider the power series ring $T = \mathbb{R}[[X]]$ in the indeterminate X with coefficients in the field $\mathbb{R}$. Then, for the pullback ring $R = \mathbb{Z} + XT$, we have: the domain R is $\mathcal{C}$-hereditary, $w.gl.\dim(R) = \infty$, and the domain R is not an almost perfect.

Proof. Use Theorem 6 and Theorem 12 and Theorem 9. □

Although the weak dimension of $\mathcal{C}$-hereditary domains can't be ascertained, the global dimension of $\mathcal{C}$-hereditary domains which are Prüfer domains can be calculated as follows:

Corollary 14. *Let R be a Prüfer domain. Then R is a $\mathcal{C}$-hereditary domain if and only if $gl.\dim(R) \leq 2$.*

Proof. To prove the necessity, we can assume that R is a Prüfer domain. So a torsion-free R-module M is flat. Then $\mathrm{pd}_R M \leq 1$ by Proposition 1. Thus $gl.\dim(R) \leq 2$ by [9, Corollary 3.4].

Conversely, R is a Matlis domain by [13, Proposition 4.5]. Let M be a torsion-free module. So $\mathrm{pd}_R M \leq 1$ by [9, Corollary 3.4]. So all flat R-module are of projective dimension≤ 1. Thus R is a $\mathcal{C}$-hereditary domain by Proposition 1. □

Recall that a domain R is a Dedekind domain, then $gl.\dim(R) \leq 1$. Consequently, by Lemma 7, if R is a Dedekind domain, then R is an almost perfect domain. By pullbacks, we can construct a counter example that an almost perfect domain R is not a Dedekind domain.

Example 15. Let $\mathbb{Q}$ denote the field of rational numbers, and let $\mathbb{R}$ denote the field of real numbers. Consider the power series ring $T = \mathbb{R}[[X]]$ in the indeterminate X with coefficients in the field $\mathbb{R}$. Then, for the pullback ring $R = \mathbb{Q} + XT$, we have: R is an almost perfect domain, and R is not a Dedekind domain.

Proof. Use Theorem 9 and [14, Theorem 8.5.17]. □

By Corollary 2, any $\mathcal{C}$-hereditary domain is a Matlis domain. Followed that, we will construct the concrete example that a Matlis domain is not $\mathcal{C}$-hereditary, which means that the converse of this corollary is not true in general. To construct this counter example, we should study the properties of Matlis domains in Milnor squares.

Theorem 16. *Let $(RDTF, M)$ be a Milnor square. Then R is a Matlis domain if and only if T is a Matlis domain.*

Proof. R and T have the same field Q by Lemma 5. To prove the sufficiency, we can take an exact sequence $0 \to B \to P \to Q \to 0$, where P is a projective R-module. Obviously, $T \otimes_R P$ is a projective T-module and $D \otimes_R P$ is a projective D-module by Lemma 3. The exactness of the sequence $0 \to B \to P \to Q \to 0$ induces the following exact sequence

$$0 \to T \otimes_R B \to T \otimes_R P \to T \otimes_R Q = Q \to 0, \text{ and}$$
$$0 \to D \otimes_R B \to D \otimes_R P \to D \otimes_R Q = 0.$$

Then $T \otimes_R B$ is a projective T-module by hypothesis and $D \otimes_R B$ is a projective D-module. Thus B is a projective module by Lemma 3, establishing the result.

Conversely, suppose P and B are projective R-modules such that $0 \to B \to P \to Q \to 0$ is exact by hypothesis. This exact sequence induces the exact sequence $0 \to T \otimes_R B \to T \otimes_R P \to T \otimes_R Q = Q \to 0$, where $\mathrm{Tor}_1^R(T, Q) = 0$. By lemma 3, $T \otimes_R B$ and $T \otimes_R P$ are projective T-modules. Then $\mathrm{pd}_T Q \leq 1$. Thus T is a Matlis domain, completing the proof. □

Example 17. Let D be a valuation domain with $gl.\dim(D) = 3$, and let F denote the quotient field of D. Consider the power series ring $T = F[[X]]$ in the indeterminate X with coefficients in the field F. Then, for the pullback ring $R = D + XT$, we have: R is a Matlis domain, and R is not a $\mathcal{C}$-hereditary domain.

Proof. It is easy to construct such a D (see [11, Corollary 2]). By Corollary 14, the domain D is not $\mathcal{C}$-hereditary. The power series ring T is a discrete valuation. So, by [12, Corollary 5.2], T is an almost perfect domain. By Theorem 16, R is a Matlis domain. But, by Theorem 6, R is not a $\mathcal{C}$-hereditary domain. □

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Yongyan Pu
College of Mathematics and Software Science
Sichuan Normal University
Chengdu, 610068, P. R. China
and
College of Mathematics and Computer
Panzhihua University
Panzhihua, 617000, P. R. China
Email address: `puyongyan1982@qq.com`

Gaohua Tang
College of Mathematics and Statistics
Guangxi Teacher's Education University
Nanning, 530001, P. R. China
Email address: `tanggaohua@163.com`

Fanggui Wang
College of Mathematics and Software Science
Sichuan Normal University
Chengdu, 610068, P. R. China
Email address: `wangfg2004@163.com`