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# APPROXIMATE CONTROLLABILITY OF SECOND-ORDER NONLOCAL IMPULSIVE FUNCTIONAL INTEGRO-DIFFERENTIAL SYSTEMS IN BANACH SPACES

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Abstract. This manuscript is involved with a category of second-order impulsive functional integro-differential equations with nonlocal conditions in Banach spaces. Sufficient conditions for existence and approximate controllability of mild solutions are acquired by making use of the theory of cosine family, Banach contraction principle and Leray-Schauder nonlinear alternative fixed point theorem. An illustration is additionally furnished to prove the attained principles.

## 1. Introduction

In this manuscript, we initially look at the next second order nonlocal impulsive functional integro-differential equations of the model

$$\begin{aligned} u''(t) &= \mathscr{A}u(t) + \mathscr{F}\left(t, u(\xi_1(t)), \dots, u(\xi_n(t)), \int_0^t k_1(t,s,u(\xi_{n+1}(s)))ds\right) \\ &\quad + \mathscr{G}\left(t, u(\zeta_1(t)), \dots, u(\zeta_p(t)), \int_0^t k_2(t,s,u(\zeta_{p+1}(s)))ds\right), \\ &\qquad t \in \mathscr{J} = [0,b],\ t \neq t_k,\ k = 1,2,\dots,m, \end{aligned} \tag{1.1}$$

$$u(0) = u_0 + q(u), \quad u'(0) = \widetilde{u}_0 + \widetilde{q}(u), \tag{1.2}$$

$$\Delta u(t_k) = I_k(u(t_k)), \quad \Delta u'(t_k) = \overline{I}_k(u(t_k)),\ k = 1,2,\dots,m, \tag{1.3}$$

where the unknown $u(\cdot)$ takes values in the Banach space $\mathbb{X}$, and $\mathscr{A}$ is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ defined on a Banach space $\mathbb{X}; 0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b$, are prefixed points and the symbol $\Delta u(t_k) = u(t_k^+) - u(t_k^-), \Delta u'(t_k) =$

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$u'(t_k^+) - u'(t_k^-)$, where $u(t_k^+), u(t_k^-)$ and $u'(t_k^+), u'(t_k^-)$ are represent the right and left limits of $u(t)$ and $u'(t)$ at $t = t_k$, respectively. $\mathscr{F}(\cdot)$, $\mathscr{G}(\cdot)$, $k_1(\cdot)$, $k_2(\cdot)$, $q(\cdot)$, $\widetilde{q}(\cdot)$, $I_k(\cdot)$, $\overline{I}_k(\cdot)$, $\xi_i$, $i = 1, 2, \ldots, n+1$ and $\zeta_l$, $l = 1, \ldots, p+1$, are apposite functions to be identified afterwards.

Differential equations (DEs) is often applied to design the dynamics of numerous real-world phenomena. Many dynamical processes are liable in accordance with unexpected changes in state. Frequently these perturbations is often periodic and also short length of time comparative to the evolving procedure. These kinds of phenomena are represented well by what are called to as impulsive differential equations (IDEs), systems of DEs coupled with discrete mappings in state space. Consequently, IDEs–that is, DEs involving impulse effects - appear as a natural information of noticed evolutionary phenomena of various real-world issues. For fundamental concepts about this theory and on its applications, we suggest the reader to refer [6, 14, 16, 24, 29]. In the past few years, impulsive integro-differential equations have grown to be an important area of research simply because of their applications to diverse problems coming up in communications, control technology, impact mechanics and electrical engineering. On the other hand, the related theory of impulsive integro-differential systems in abstract spaces is even now in its developing phase and many factors of the concept stay to be resolved.

The concept of control theory is to create systems execute certain tasks by employing acceptable control behavior. Among the list of essential aspects in current mathematical control theory, controllability performs an crucial factor in deterministic control theory and engineering. The idea of controllability is centered on the mathematical account of the dynamical system. In accordance with control theory, a dynamical system is controllable if, with a proper option of inputs, it is usually influenced from any initial state to any preferred last state within specific time. In the mathematical perspective, the issues of exact and approximate controllability are to be distinguished. Exact controllability allows to steer the system to arbitrary final state while approximate controllability signifies that the system is usually steered to arbitrary small neighborhood of final state. Especially, approximate controllable systems are more common and frequently approximate controllability is fully acceptable in applications. There are actually a lot of papers on the exact and approximate controllability of the different kinds of nonlinear systems under various hypotheses (see for instance [5, 8–10, 13, 15, 18–21, 23, 27, 28, 30] and references cited therein). Second-order differential and integro-differential equations provide as an theoretical formulation of several integro-differential equations which occur in problems linked with the transverse motion of an extensible beam, the vibration of hinged bars and various other physical phenomena. So it is very huge to concentrate the controllability issue for such systems in Banach spaces.

The literary works relevant to existence and controllability of second-order systems with impulses continues to be restricted. Chang et al. [1] analyzed

a class of second order IDEs with state-dependent delay by employing a suitable fixed point theorem bundled with concepts of a strongly continuous cosine family of bounded linear operators.. Sakthivel et al. [22] studied the controllability of second-order impulsive systems in Banach spaces without imposing the compactness condition on the cosine family of operators under Banach contraction mapping principle. In [2–4], the authors discussed the different types of second-order impulsive differential systems with different conditions on the given functions. The results are obtained by using the classical fixed point theorems. Dimplekumar N. Chalishajar [7] analyzed the controllability of a partial neutral functional differential inclusion of second order with impulse effect and infinite delay without assuming the compactness conditions of the family of cosine operators and also author introduced a new phase space axioms to derive the results. Lately, Meili Li and Junling Ma [17] studied the approximate controllability of second order impulsive functional differential systems with infinite delay in Banach spaces. Sufficient conditions are formulated and proved for the approximate controllability of such system under the assumption that the associated linear part of system is approximately controllable. However, it needs to be pointed out, to the best of our knowledge, the existence and approximate controllability results for second-order impulsive functional integro-differential equations with nonlocal conditions of the form (1.1)-(1.3) has not been examined yet. According to fixed point techniques, the proposed work in this manuscript on the second-order functional integro-differential systems with nonlocal and impulsive conditions is new in the literature. This fact is the important objective of this work.

The structure of this manuscript is as per the following. In Section 2, some fundamental certainties are reviewed. Section 3 is dedicated to the existence of mild solutions to problem (1.1)-(1.3). The approximate controllability result is shown in Section 4. In Section 5, a case is given to delineate our outcomes.

## 2. Preliminaries

In this section, we review some basic concepts, notations and properties needed to establish our results.

**Definition 2.1** ([17])**.** A one parameter family $\{C(t) : t \in \mathbb{R}\}$, of bounded linear operators in the Banach space $\mathbb{X}$ is called a strongly continuous cosine family if and only if

(i) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s, t, \in \mathbb{R}$;
(ii) $C(0) = I$;
(iii) $C(t)u$ is strongly continuous in $t$ on $\mathbb{R}$ for each fixed $u \in \mathbb{X}$.

Throughout this work, $\mathscr{A}$ is the infinitesimal generator of a strongly continuous cosine family, $\{C(t) : t \in \mathbb{R}\}$, of bounded linear operators defined on a Banach space $\mathbb{X}$ endowed with a norm $\|\cdot\|$. We denote by $\{S(t) : t \in \mathbb{R}\}$ the

sine function associated to $\{C(t) : t \in \mathbb{R}\}$ which is defined by

$$S(t)u = \int_0^t C(s)uds \quad \text{for} \quad u \in \mathbb{X} \quad \text{and} \quad t \in \mathbb{R}.$$

Furthermore, $\widehat{M_1}$ and $\widehat{M_2}$ are positive constants such that $\|C(t)\| \leq \widehat{M_1}$ and $\|S(t)\| \leq \widehat{M_2}$ for every $t \in \mathscr{J}$.

The infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ is the operator $\mathscr{A} : \mathbb{X} \to \mathbb{X}$ defined by

$$\mathscr{A}u = \frac{d^2}{dt^2}C(t)u|_{t=0}, \quad u \in \mathscr{D}(\mathscr{A}),$$

where $\mathscr{D}(\mathscr{A}) = \{u \in \mathbb{X} : C(t)u$ is twice continuously differentiable in $t\}$, endowed with the norm

$$\|u\|_{\mathscr{A}} = \|u\| + \|\mathscr{A}u\|, \quad u \in \mathscr{D}(\mathscr{A}).$$

Define $E = \{u \in \mathbb{X} : C(t)u$ is once continuously differentiable in $t\}$, endowed with the norm

$$\|u\|_E = \|u\| + \sup_{0 \leq t \leq 1} \|\mathscr{A}S(t)u\|, \quad u \in E,$$

then $E$ is a Banach space. The operator valued function $G(t) = \left[\begin{smallmatrix} C(t) & S(t) \\ \mathscr{A}S(t) & C(t) \end{smallmatrix}\right]$ is a strongly continuous group of bounded linear operators on the space $E \times \mathbb{X}$ generated by the operator $\mathcal{A} = \left[\begin{smallmatrix} 0 & I \\ \mathscr{A} & 0 \end{smallmatrix}\right]$ defined on $\mathscr{D}(\mathscr{A}) \times E$. It follows from this that $\mathscr{A}S(t) : E \to \mathbb{X}$ is a bounded linear operator and that $\mathscr{A}S(t)u \to 0,\ t \to 0,$ for each $u \in E$. Furthermore, if $u : [0, \infty) \to \mathbb{X}$ is a locally integrable function, then $z(t) = \int_0^t S(t-s)u(s)ds$ defines an $E$-valued continuous function. This is a consequence of the fact that

$$\int_0^t G(t-s) \begin{bmatrix} 0 \\ u(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)u(s)\, ds, & \int_0^t C(t-s)u(s)\, ds \end{bmatrix}^T$$

defines an $E \times \mathbb{X}$-valued continuous function.

The existence of solutions for the second order abstract Cauchy problem

$$\begin{cases} u''(t) &= \mathscr{A}u(t) + h(t), \ \ t \in \mathscr{J} = [0, b], \\ u(0) &= z, \ \ u'(0) = w, \end{cases} \tag{2.1}$$

where $h : \mathscr{J} \to \mathbb{X}$ is an integrable function has been discussed in [25]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem has been treated in [26]. We only mention here that the function $u(\cdot)$ given by

$$u(t) = C(t)z + S(t)w + \int_0^t S(t-s)h(s)ds, \quad t \in \mathscr{J},$$

is called mild solution of (2.1) and that when $z \in E$, $u(\cdot)$ is continuously differentiable and

$$u'(t) = \mathscr{A}S(t)z + C(t)w + \int_0^t C(t-s)h(s)ds, \quad t \in \mathscr{J}.$$

For additional details about cosine function theory, we refer to [12, 25, 26].

To take into account the impulsive conditions (1.1)-(1.3), it is easy to present some more aspects and notations.

A function $u : [\sigma, \tau] \to \mathbb{X}$ is considered to be a normalized piecewise continuous function (NPCF) on $[\sigma, \tau]$ if $u$ is piecewise continuous and left continuous on $(\sigma, \tau]$. We represent by $\mathcal{PC}([\sigma, \tau], \mathbb{X})$ the space of NPCF from $[\sigma, \tau]$ into $\mathbb{X}$. Especially, we present the space $\mathcal{PC}$ formed by all NPCF $u : [0, b] \to \mathbb{X}$ in ways that $u$ is continuous at $t \neq t_k$, $k = 1, \ldots, m$. It is obvious that $\mathcal{PC}$ rendered with the norm $\| u \|_{\mathcal{PC}} = \sup_{s \in \mathscr{J}} \| u(s) \|$ is a Banach space. Likewise, $\mathcal{PC}^1$ will be the space of the functions $u(\cdot) \in \mathcal{PC}$ such that $u(\cdot)$ is continuously differentiable on $\mathscr{J} - \{t_k : k = 1, 2, \ldots, m\}$ and the lateral derivatives $u'_R(t) = \lim_{s \to 0^+} \frac{u(t+s)-u(t^+)}{s}, u'_L(t) = \lim_{s \to 0^-} \frac{u(t+s)-u(t^-)}{s}$ are continuous functions on $[t_k, t_{k+1})$ and $(t_k, t_{k+1}]$, respectively. Next, for $u \in \mathcal{PC}^1$ we represent by $u'(t)$ the left derivative at $t \in (0, b]$ and by $u'(0)$ the right derivative at zero.

In what follows, we set $t_0 = 0$, $t_{m+1} = b$, and for $u \in \mathcal{PC}$ we signify by $\tilde{u}_k$, for $k = 0, 1, \ldots, m$, the function $\tilde{u}_k \in C([t_k, t_{k+1}]; \mathbb{X})$ given by $\widetilde{u}_k(t) = u(t)$ for $t \in (t_k, t_{k+1}]$ and $\widetilde{u}_k(t_k) = \lim_{t \to t_k^+} u(t)$. Moreover, for a set $B \subseteq \mathcal{PC}$, we represent by $\widetilde{B}_k$, for $k = 0, 1, \ldots, m$, the set $\widetilde{B}_k = \{\tilde{u}_k : u \in B\}$.

**Lemma 2.1.** *A set $B \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if, and only if, each set $\widetilde{B}_k$, $k = 0, 1, \ldots, m$, is relatively compact in $C([t_i, t_{i+1}], \mathbb{X})$.*

Now, we are in a position to present the mild solution for the system (1.1)-(1.3).

**Definition 2.2.** A function $u(\cdot) \in \mathcal{PC}(\mathscr{J}, \mathbb{X})$ is said to be a mild solution to the problem (1.1)-(1.3) if it satisfies the following integral equation

$$\begin{aligned} u(t) &= C(t)[u_0 + q(u)] + S(t)[\widetilde{u}_0 + \widetilde{q}(u)] \\ &\quad + \int_0^t S(t-s)\Bigg[\mathscr{F}\Bigg(s, u(\xi_1(s)), \ldots, u(\xi_n(s)), \int_0^s k_1(s, \tau, u(\xi_{n+1}(\tau)))d\tau\Bigg) \\ &\quad + \mathscr{G}\Bigg(s, u(\zeta_1(s)), \ldots, u(\zeta_p(s)), \int_0^s k_2(s, \tau, u(\zeta_{p+1}(\tau)))d\tau\Bigg)\Bigg]ds \\ &\quad + \sum_{0 < t_k < t} C(t - t_k)I_k(u(t_k)) + \sum_{0 < t_k < t} S(t - t_k)\overline{I}_k(u(t_k)), \quad t \in \mathscr{J}. \end{aligned}$$

The key tool in our approach is the following fixed point theorem.

**Lemma 2.2** (Leray-Schauder Nonlinear Alternative [11]). *Let $\mathbb{X}$ be a Banach space with $Z \subset \mathbb{X}$ closed and convex. Assume that $U$ is a relatively open subset of $Z$ with $0 \in U$ and $\Upsilon : \overline{U} \to Z$ is a compact map. Then either*

(i) *$\Upsilon$ has a fixed point in $\overline{U}$, or*
(ii) *there is a point $v \in \partial U$ such that $v \in \lambda\Upsilon(v)$ for some $\lambda \in (0,1)$.*

## 3. Existence results

In this section, we present and prove the existence results for the problem (1.1)-(1.3). In order to utilize Lemma 2.2, we need to list the subsequent hypotheses:

(H1) The functions $\mathscr{F} : \mathscr{J} \times \mathbb{X}^{n+1} \to \mathbb{X}$ and $\mathscr{G} : \mathscr{J} \times \mathbb{X}^{p+1} \to \mathbb{X}$ are continuous and there exist constants $\mathscr{L} > 0$, $\widetilde{\mathscr{L}} > 0$, $\mathscr{L}_1 \geq 0$, $\widetilde{\mathscr{L}_1} \geq 0$ such that for all $x_i, y_i \in \mathbb{X}$, $i = 1, \ldots, n+1$ and $x_l, y_l \in \mathbb{X}$, $l = 1, \ldots, p+1$, we have

$$\| \mathscr{F}(t, x_1, x_2, \ldots, x_{n+1}) - \mathscr{F}(t, y_1, y_2, ldots, y_{n+1}) \| \leq \mathscr{L}\Big( \sum_{i=1}^{n+1} \|x_i - y_i\| \Big)$$

and

$$\| \mathscr{G}(t, x_1, x_2, \ldots, x_{p+1}) - \mathscr{G}(t, y_1, y_2, \ldots, y_{p+1}) \| \leq \widetilde{\mathscr{L}}\Big( \sum_{l=1}^{p+1} \|x_l - y_l\| \Big)$$

with $\mathscr{L}_1 = \max_{t \in \mathscr{J}} \|\mathscr{F}(t, 0, \ldots, 0)\|$ and $\widetilde{\mathscr{L}_1} = \max_{t \in \mathscr{J}} \|\mathscr{G}(t, 0, \ldots, 0)\|$.

(H2) The functions $k_1, k_2 : \mathscr{J} \times \mathscr{J} \times \mathbb{X} \to \mathbb{X}$ are continuous and we can find constants $\mathscr{N} > 0, \widetilde{\mathscr{N}} > 0, \mathscr{N}_1 \geq 0, \widetilde{\mathscr{N}_1} \geq 0$ such that for all $x, y \in \mathbb{X}$,

$$\|k_1(t, s, x) - k_1(t, s, y)\| \leq \mathscr{N}\|x - y\|$$

and

$$\|k_2(t, s, x) - k_2(t, s, y)\| \leq \widetilde{\mathscr{N}}\|x - y\|$$

with $\mathscr{N}_1 = \max_{0 \leq s \leq t \leq b} \|k_1(t, s, 0)\|$ and $\widetilde{\mathscr{N}_1} = \max_{0 \leq s \leq t \leq b} \|k_2(t, s, 0)\|$.

(H3) The functions $\xi_i : \mathscr{J} \to \mathscr{J}$, $i = 1, \ldots, n+1$ and $\zeta_l : \mathscr{J} \to \mathscr{J}$, $l = 1, \ldots, p+1$ are continuous functions such that $\xi_i(t) \leq t$, $i = 1, \ldots, n+1$ and $\zeta_l(t) \leq t$, $l = 1, \ldots, p+1$.

(H4) (i) $I_k \in C(\mathbb{X}, \mathbb{X})$, $k = 1, \ldots, m$ are all compact operators, and there exist continuous nondecreasing functions $\Psi_k : [0, \infty) \to (0, \infty)$, $k = 1, \ldots, m$, such that

$$\|I_k(u)\| \leq \Psi_k(\|u\|) \quad \text{for each } u \in \mathbb{X}.$$

(ii) $\overline{I}_k \in C(\mathbb{X}, \mathbb{X})$, $k = 1, \ldots, m$ are all compact operators, and there exist continuous nondecreasing functions $\widetilde{\Psi}_k : [0, \infty) \to (0, \infty)$,

$k = 1, \ldots, m$, such that

$$\|\overline{I}_k(u)\| \leq \widetilde{\Psi}_k(\|u\|) \text{ for every } u \in \mathbb{X}.$$

(H5) (i) The function $q(\cdot) : \mathcal{PC}(\mathscr{J}, \mathbb{X}) \to \mathbb{X}$ is continuous and we can find a $\delta \in (0, t_1)$ in a way that $q(\phi) = q(\widetilde{\phi})$ for any $\phi, \widetilde{\phi} \in \mathcal{PC}(\mathscr{J}, \mathbb{X})$ with $\phi = \widetilde{\phi}$ on $[\delta, b]$.

(ii) The function $\widetilde{q}(\cdot) : \mathcal{PC}(\mathscr{J}, \mathbb{X}) \to \mathbb{X}$ is continuous and we can find a $\delta \in (0, t_1)$ in ways that $\widetilde{q}(\phi) = \widetilde{q}(\widetilde{\phi})$ for any $\phi, \widetilde{\phi} \in \mathcal{PC}(\mathscr{J}, \mathbb{X})$ with $\phi = \widetilde{\phi}$ on $[\delta, b]$.

(iii) There is a continuous nondecreasing function $\Lambda : [0, \infty) \to (0, \infty)$ in a way that

$$\|q(\phi)\| \leq \Lambda(\|\phi\|_{\mathcal{PC}}),\ \phi \in \mathcal{PC}(\mathscr{J}, \mathbb{X}).$$

(iv) There is a continuous nondecreasing function $\widetilde{\Lambda} : [0, \infty) \to (0, \infty)$ in a way that

$$\|\widetilde{q}(\phi)\| \leq \widetilde{\Lambda}(\|\phi\|_{\mathcal{PC}}),\ \phi \in \mathcal{PC}(\mathscr{J}, \mathbb{X}).$$

(H6) We can find a constant $\widehat{M}^* > 0$ in a way that

(3.1)
$$\frac{\widehat{M}^*}{e^{\eta}\left[\widehat{M}_* + \widehat{M}_1\left[\Lambda(\widehat{M}^*) + \sum_{k=1}^{m} \Psi_k(\widehat{M}^*)\right] + \widehat{M}_2\left[\widetilde{\Lambda}(\widehat{M}^*) + \sum_{k=1}^{m} \widetilde{\Psi}_k(\widehat{M}^*)\right]\right]} > 1,$$

where $\eta = \widehat{M}_2 b\big[(\mathscr{L}n + \widetilde{\mathscr{L}}p) + b(\mathscr{L}\mathscr{N} + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}})\big], \widehat{M}_* = \widehat{M}_1\|u_0\| + \widehat{M}_2\|\widetilde{u}_0\| + \widehat{M}_2 b\left[b(\mathscr{L}\mathscr{N}_1 + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}_1}) + (\mathscr{L}_1 + \widetilde{\mathscr{L}_1})\right]$.

**Theorem 3.1.** *Let $u(0), u'(0) \in \mathbb{X}$. If assumptions* (H1)-(H6) *are fulfilled, then the impulsive nonlocal Cauchy problem* (1.1)-(1.3) *has at least one mild solution on $\mathscr{J}$.*

*Proof.* Let $\mathscr{L}_0 = 2\widehat{M}_2\left[(\mathscr{L}n + \widetilde{\mathscr{L}}p) + (\mathscr{L}\mathscr{N} + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}})b\right]$ and we introduce in the space $\mathcal{PC}(\mathscr{J}, \mathbb{X})$ the equivalent norm defined as

$$\|\phi\|_{\mathcal{V}} := \sup_{t \in \mathscr{J}} e^{-\mathscr{L}_0 t}\|\phi(t)\|.$$

Then, it is easy to see that $\mathcal{V} := (\mathcal{PC}(\mathscr{J}, \mathbb{X}), \|\cdot\|_{\mathcal{V}})$ is a Banach space. Fix $v \in \mathcal{PC}(\mathscr{J}, \mathbb{X})$ and for $t \in \mathscr{J}, \phi \in \mathcal{V}$, we now define an operator

(3.2)
$$\begin{aligned}
&(\Upsilon_v\phi)(t)\\
&= C(t)[u_0 + q(v)] + S(t)[\widetilde{u}_0 + \widetilde{q}(v)]\\
&\quad + \int_0^t S(t-s)\Big[\mathscr{F}\Big(s, \phi(\xi_1(s)), \ldots, \phi(\xi_n(s)), \int_0^s k_1(s, \tau, \phi(\xi_{n+1}(\tau)))d\tau\Big)
\end{aligned}$$

$$+\mathscr{G}\bigg(s,\phi(\zeta_1(s)),\dots,\phi(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi(\zeta_{p+1}(\tau)))d\tau\bigg)\bigg]ds$$
$$\sum_{0<t_k<t} C(t-t_k)I_k(v(t_k))+\sum_{0<t_k<t} S(t-t_k)\overline{I}_k(v(t_k)).$$

Since $C(\cdot)(u_0+q(v))$ and $S(\cdot)(\widetilde{u}_0+\widetilde{q}(v))$ are belongs to $\mathcal{PC}(\mathscr{J},\mathbb{X})$, it allows from (H1)-(H3) that $(\Upsilon_v\phi)(t)\in\mathcal{V}$ for all $\phi\in\mathcal{V}$. Allow $\phi,\widetilde{\phi}\in\mathcal{V}$, we obtain

$$\begin{aligned}
&e^{-\mathscr{L}_0 t}\|(\Upsilon_v\phi)(t)-(\Upsilon_v\widetilde{\phi})(t)\|\\
&\le e^{-\mathscr{L}_0 t}\int_0^t\bigg\|S(t-s)\bigg[\mathscr{F}\bigg(s,\phi(\xi_1(s)),\dots,\phi(\xi_n(s)),\int_0^s k_1(s,\tau,\phi(\xi_{n+1}(\tau)))d\tau\bigg)\\
&\quad-\mathscr{F}\bigg(s,\widetilde{\phi}(\xi_1(s)),\dots,\widetilde{\phi}(\xi_n(s)),\int_0^s k_1(s,\tau,\widetilde{\phi}(\xi_{n+1}(\tau)))d\tau\bigg)\bigg]\bigg\|ds\\
&\quad+e^{-\mathscr{L}_0 t}\int_0^t\bigg\|S(t-s)\bigg[\mathscr{G}\bigg(s,\phi(\zeta_1(s)),\dots,\phi(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi(\zeta_{p+1}(\tau)))d\tau\bigg)\\
&\quad-\mathscr{G}\bigg(s,\widetilde{\phi}(\zeta_1(s)),\dots,\widetilde{\phi}(\zeta_p(s)),\int_0^s k_2(s,\tau,\widetilde{\phi}(\zeta_{p+1}(\tau)))d\tau\bigg)\bigg]\bigg\|ds\\
&\le \widehat{M}_2\mathscr{L}\int_0^t e^{-\mathscr{L}_0 t}\bigg[\|\phi(\xi_1(s))-\widetilde{\phi}(\xi_1(s))\|+\cdots+\|\phi(\xi_n(s))-\widetilde{\phi}(\xi_n(s))\|\\
&\quad+\bigg\|\int_0^s k_1(s,\tau,\phi(\xi_{n+1}(\tau)))d\tau-\int_0^s k_1(s,\tau,\widetilde{\phi}(\xi_{n+1}(\tau)))d\tau\bigg\|\bigg]ds\\
&\quad+\widehat{M}_2\widetilde{\mathscr{L}}\int_0^t e^{-\mathscr{L}_0 t}\bigg[\|\phi(\zeta_1(s))-\widetilde{\phi}(\zeta_1(s))\|+\cdots+\|\phi(\zeta_p(s))-\widetilde{\phi}(\zeta_p(s))\|\\
&\quad+\bigg\|\int_0^s k_2(s,\tau,\phi(\zeta_{p+1}(\tau)))d\tau-\int_0^s k_2(s,\tau,\widetilde{\phi}(\zeta_{p+1}(\tau)))d\tau\bigg\|\bigg]ds\\
&\le \widehat{M}_2\mathscr{L}\int_0^t e^{-\mathscr{L}_0 t}\bigg[e^{\mathscr{L}_0\xi_1(s)}\sup_{s\in\mathscr{J}}e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|+\cdots\\
&\quad+e^{\mathscr{L}_0\xi_n(s)}\sup_{s\in\mathscr{J}}e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|+\mathscr{N}\int_0^s\|\phi(\xi_{n+1}(\tau))-\widetilde{\phi}(\xi_{n+1}(\tau))\|d\tau\bigg]ds\\
&\quad+\widehat{M}_2\widetilde{\mathscr{L}}\int_0^t e^{-\mathscr{L}_0 t}\bigg[e^{\mathscr{L}_0\zeta_1(s)}\sup_{s\in\mathscr{J}}e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|+\cdots\\
&\quad+e^{\mathscr{L}_0\zeta_p(s)}\sup_{s\in\mathscr{J}}e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|+\widetilde{\mathscr{N}}\int_0^s\|\phi(\zeta_{p+1}(\tau))-\widetilde{\phi}(\zeta_{p+1}(\tau))\|d\tau\bigg]ds\\
&\le \widehat{M}_2\mathscr{L}\int_0^t e^{-\mathscr{L}_0 t}\bigg[ne^{\mathscr{L}_0 s}\sup_{s\in\mathscr{J}}e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|\\
&\qquad+\mathscr{N}be^{\mathscr{L}_0\xi_{n+1}(s)}\sup_{s\in\mathscr{J}}e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|\bigg]ds
\end{aligned}$$

$$
\begin{aligned}
&+ \widehat{M}_2\widetilde{\mathscr{L}}\int_0^t e^{-\mathscr{L}_0 t}\Big[pe^{\mathscr{L}_0 s}\sup_{s\in\mathscr{J}} e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\| \\
&\qquad + \widetilde{\mathscr{N}}be^{\mathscr{L}_0\zeta_{p+1}(s)}\sup_{s\in\mathscr{J}} e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|\Big]ds \\
&\leq \widehat{M}_2\mathscr{L}\int_0^t e^{-\mathscr{L}_0(t-s)}\Big[n\sup_{s\in\mathscr{J}} e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\| \\
&\qquad + \mathscr{N}b\sup_{s\in\mathscr{J}} e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|\Big]ds \\
&+ \widehat{M}_2\widetilde{\mathscr{L}}\int_0^t e^{-\mathscr{L}_0(t-s)}\Big[p\sup_{s\in\mathscr{J}} e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\| \\
&\qquad + \widetilde{\mathscr{N}}b\sup_{s\in\mathscr{J}} e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|\Big]ds \\
&\leq \widehat{M}_2\mathscr{L}(n+\mathscr{N}b)\int_0^t e^{-\mathscr{L}_0(t-s)}ds\|\phi-\widetilde{\phi}\|_{\mathcal{V}} \\
&+ \widehat{M}_2\widetilde{\mathscr{L}}(p+\widetilde{\mathscr{N}}b)\int_0^t e^{-\mathscr{L}_0(t-s)}ds\|\phi-\widetilde{\phi}\|_{\mathcal{V}} \\
&\leq \widehat{M}_2\Big[(\mathscr{L}n+\widetilde{\mathscr{L}}p)+(\mathscr{L}\mathscr{N}+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}})b\Big]\int_0^t e^{-\mathscr{L}_0(t-s)}ds\|\phi-\widetilde{\phi}\|_{\mathcal{V}} \\
&\leq \frac{\widehat{M}_2\Big[(\mathscr{L}n+\widetilde{\mathscr{L}}p)+(\mathscr{L}\mathscr{N}+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}})b\Big]}{\mathscr{L}_0}\|\phi-\widetilde{\phi}\|_{\mathcal{V}},\ t\in\mathscr{J},
\end{aligned}
$$

which indicates that

$$e^{-\mathscr{L}_0 t}\|(\Upsilon_v\phi)(t)-(\Upsilon_v\widetilde{\phi})(t)\|\leq\frac{1}{2}\|\phi-\widetilde{\phi}\|_{\mathcal{V}},\ t\in\mathscr{J}.$$

Hence

$$\|\Upsilon_v\phi-\Upsilon_v\widetilde{\phi}\|_{\mathcal{V}}\leq\frac{1}{2}\|\phi-\widetilde{\phi}\|_{\mathcal{V}},\ \phi,\widetilde{\phi}\in\mathcal{V}.$$

Thus, the operator $\Upsilon_v$ is a strict contraction. By the Banach contraction principle, we observe that $\Upsilon_v$ has a unique fixed point $\phi_v\in\mathcal{V}$ and the equation (3.2) has a unique mild solution on $[0,b]$.

Fix

$$\widetilde{v}(t):=\begin{cases}v(t) & \text{if } t\in(\delta,b],\\ v(\delta) & \text{if } t\in[0,\delta].\end{cases}$$

From (3.2), we have

$$
\begin{aligned}
\phi_{\widetilde{v}}(t) &= C(t)[u_0+q(\widetilde{v})]+S(t)[\widetilde{u}_0+\widetilde{q}(\widetilde{v})] \\
&\quad+\int_0^t S(t-s)\Big[\mathscr{F}\Big(s,\phi_{\widetilde{v}}(\xi_1(s)),\ldots,\phi_{\widetilde{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)
\end{aligned}
$$

$$+\mathscr{G}\Big(s,\phi_{\widetilde{v}}(\zeta_1(s)),\dots,\phi_{\widetilde{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds$$

$$+\sum_{0<t_k<t} C(t-t_k)I_k(v(t_k))+\sum_{0<t_k<t} S(t-t_k)\overline{I}_k(v(t_k)). \tag{3.3}$$

Consider the map $\Gamma:\mathcal{PC}_\delta=\mathcal{PC}([\delta,b],\mathbb{X})\to\mathcal{PC}_\delta$ defined by

$$(\Gamma v)(t)=\phi_{\widetilde{v}}(t),\ t\in[\delta,b].$$

We should demonstrate that $\Gamma$ fulfills every one of the states of Lemma 2.2. For better understandability, the proof is going to be presented in a few stages.

**Step 1**. $\Gamma$ maps bounded sets into bounded sets in $\mathcal{PC}_\delta$.

In fact, it is sufficient to demonstrate that we can find a positive constant $\Lambda_2$ in ways that for every $v\in B_r(\delta):=\Big\{\phi\in\mathcal{PC}_\delta;\ \sup_{\delta\le t\le b}\|\phi(t)\|\le r\Big\}$ one has $\|\Gamma v\|_{\mathcal{PC}}\le\Lambda_2$.

Let $v\in B_r(\delta)$, then for $t\in(0,b]$, we have

$$\begin{aligned}
&\|\phi_{\widetilde{v}}(t)\|\\
&\le\|C(t)[u_0+q(\widetilde{v})]\|+\|S(t)[\widetilde{u}_0+\widetilde{q}(\widetilde{v})]\|\\
&\quad+\int_0^t\Big\|S(t-s)\mathscr{F}\Big(s,\phi_{\widetilde{v}}(\xi_1(s)),\dots,\phi_{\widetilde{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\Big\|ds\\
&\quad+\int_0^t\Big\|S(t-s)\mathscr{G}\Big(s,\phi_{\widetilde{v}}(\zeta_1(s)),\dots,\phi_{\widetilde{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big\|ds\\
&\quad+\Big\|\sum_{0<t_k<t}C(t-t_k)I_k(v(t_k))\Big\|+\Big\|\sum_{0<t_k<t}S(t-t_k)\overline{I}_k(v(t_k))\Big\|\\
&\le\widehat{M}_1[\|u_0+q(\widetilde{v})\|]+\widehat{M}_2[\|\widetilde{u}_0+\widetilde{q}(\widetilde{v})\|]\\
&\quad+\widehat{M}_1\sum_{k=1}^m\|I_k(v(t_k))\|+\widehat{M}_2\sum_{k=1}^m\|\overline{I}_k(v(t_k))\|\\
&\quad+\widehat{M}_2\int_0^t\Big[\Big\|\mathscr{F}\Big(s,\phi_{\widetilde{v}}(\xi_1(s)),\dots,\phi_{\widetilde{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\\
&\quad-\mathscr{F}(s,0,\dots,0)\Big\|+\|\mathscr{F}(s,0,\dots,0)\|\Big]ds\\
&\quad+\widehat{M}_2\int_0^t\Big[\Big\|\mathscr{G}\Big(s,\phi_{\widetilde{v}}(\zeta_1(s)),\dots,\phi_{\widetilde{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\\
&\quad-\mathscr{G}(s,0,\dots,0)\Big\|+\|\mathscr{G}(s,0,\dots,0)\|\Big]ds\\
&\le\widehat{M}_1[\|u_0\|+\|q(\widetilde{v})\|]+\widehat{M}_2[\|\widetilde{u}_0\|+\|\widetilde{q}(\widetilde{v})\|]\\
&\quad+\widehat{M}_1\sum_{k=1}^m\Psi_k(\|v(t_k)\|)+\widehat{M}_2\sum_{k=1}^m\widetilde{\Psi}_k(\|v(t_k)\|)
\end{aligned}$$

$$
\begin{aligned}
&+\widehat{M}_2\int_0^t\Big\{\mathscr{L}\Big[\sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(s)\|+\cdots+\sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(s)\|\\
&+\int_0^s[\|k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))-k_1(s,\tau,0)\|+\|k_1(s,\tau,0)\|]d\tau\Big]+\mathscr{L}_1\Big\}ds\\
&+\widehat{M}_2\int_0^t\Big\{\widetilde{\mathscr{L}}\Big[\sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(s)\|+\cdots+\sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(s)\|\\
&+\int_0^s[\|k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))-k_2(s,\tau,0)\|+\|k_2(s,\tau,0)\|]d\tau\Big]+\widetilde{\mathscr{L}_1}\Big\}ds\\
\le\ &\widehat{M}_1[\|u_0\|+\Lambda(\|\widetilde{v}\|_{\mathcal{PC}})]+\widehat{M}_2[\|\widetilde{u}_0\|+\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}})]\\
&+\widehat{M}_1\sum_{k=1}^m\Psi_k(\|v(t_k)\|)+\widehat{M}_2\sum_{k=1}^m\widetilde{\Psi}_k(\|v(t_k)\|)\\
&+\widehat{M}_2\int_0^t\left\{\mathscr{L}\left[n\sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(s)\|+b\left(\mathscr{N}\sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(s)\|+\mathscr{N}_1\right)\right]+\mathscr{L}_1\right\}ds\\
&+\widehat{M}_2\int_0^t\left\{\widetilde{\mathscr{L}}\left[p\sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(s)\|+b\left(\widetilde{\mathscr{N}}\sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(s)\|+\widetilde{\mathscr{N}_1}\right)\right]+\widetilde{\mathscr{L}_1}\right\}ds\\
\le\ &\widehat{M}_*+\widehat{M}_1\Lambda(r)+\widehat{M}_2\widetilde{\Lambda}(r)+\widehat{M}_1\sum_{k=1}^m\Psi_k(r)+\widehat{M}_2\sum_{k=1}^m\widetilde{\Psi}_k(r)\\
&+\widehat{M}_2\Big[(\mathscr{L}n+\widetilde{\mathscr{L}}p)+b(\mathscr{L}\mathscr{N}+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}})\Big]\int_0^t\sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(s)\|ds,
\end{aligned}
$$

where $\widehat{M}_*=\widehat{M}_1\|u_0\|+\widehat{M}_2\|\widetilde{u}_0\|+\widehat{M}_2b\Big[b(\mathscr{L}\mathscr{N}_1+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}_1})+(\mathscr{L}_1+\widetilde{\mathscr{L}_1})\Big]$.

Utilizing the Gronwall's inequality, we obtain

$$
\begin{aligned}
&\sup_{(0,b]}\|\phi_{\widetilde{v}}(s)\|\le e^{\widehat{M}_2b\left[(\mathscr{L}n+\widetilde{\mathscr{L}}p)+b(\mathscr{L}\mathscr{N}+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}})\right]}\\
&\qquad\cdot\left[\widehat{M}_*+\widehat{M}_1\left[\Lambda(r)+\sum_{k=1}^m\Psi_k(r)\right]+\widehat{M}_2\left[\widetilde{\Lambda}(r)+\sum_{k=1}^m\widetilde{\Psi}_k(r)\right]\right].
\end{aligned}
$$

Thus

$$
\|\Gamma v\|_{\mathcal{PC}}\le e^{\eta}\left[\widehat{M}_*+\widehat{M}_1\left[\Lambda(r)+\sum_{k=1}^m\Psi_k(r)\right]+\widehat{M}_2\left[\widetilde{\Lambda}(r)+\sum_{k=1}^m\widetilde{\Psi}_k(r)\right]\right].:=\Lambda_2.
$$

**Step 2**. $\Gamma$ is continuous on $B_r(\delta)$.

From (3.2) and (H1)-(H5), we consider that for $v_1,v_2\in B_r(\delta)$, $t\in(0,b]$,

$$
\begin{aligned}
&\|\phi_{\widetilde{v}_1}(t)-\phi_{\widetilde{v}_2}(t)\|\\
\le\ &\|C(t)[q(\widetilde{v_1})-q(\widetilde{v_2})]\|+\|S(t)[\widetilde{q}(\widetilde{v_1})-\widetilde{q}(\widetilde{v_2})]\|
\end{aligned}
$$

$$
\begin{aligned}
&+\left\| \sum_{0<t_k<t} C(t-t_k)I_k(v_1(t_k)) - \sum_{0<t_k<t} C(t-t_k)I_k(v_2(t_k)) \right\| \\
&+\left\| \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(v_1(t_k)) - \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(v_2(t_k)) \right\| \\
&+\int_0^t \Bigg\| S(t-s)\Bigg[\mathscr{F}\Bigg(s,\phi_{\widetilde{v}_1}(\xi_1(s)),\dots,\phi_{\widetilde{v}_1}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}_1}(\xi_{n+1}(\tau)))d\tau\Bigg) \\
&-\mathscr{F}\Bigg(s,\phi_{\widetilde{v}_2}(\xi_1(s)),\dots,\phi_{\widetilde{v}_2}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}_2}(\xi_{n+1}(\tau)))d\tau\Bigg)\Bigg]\Bigg\| ds \\
&+\int_0^t \Bigg\| S(t-s)\Bigg[\mathscr{G}\Bigg(s,\phi_{\widetilde{v}_1}(\zeta_1(s)),\dots,\phi_{\widetilde{v}_1}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}_1}(\zeta_{p+1}(\tau)))d\tau\Bigg) \\
&-\mathscr{G}\Bigg(s,\phi_{\widetilde{v}_2}(\zeta_1(s)),\dots,\phi_{\widetilde{v}_2}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}_2}(\zeta_{p+1}(\tau)))d\tau\Bigg)\Bigg]\Bigg\| ds \\
\le\ & \widehat{M}_1\|q(\widetilde{v_1})-q(\widetilde{v_2})\| + \widehat{M}_2\|\widetilde{q}(\widetilde{v_1})-\widetilde{q}(\widetilde{v_2})\| + \widehat{M}_1\sum_{k=1}^m \|I_k(v_1(t_k))-I_k(v_2(t_k))\| \\
&+\widehat{M}_2\sum_{k=1}^m \|\overline{I}_k(v_1(t_k))-\overline{I}_k(v_2(t_k))\| \\
&+\widehat{M}_2\mathscr{L}\int_0^t \Bigg[\|\phi_{\widetilde{v}_1}(\xi_1(s))-\phi_{\widetilde{v}_2}(\xi_1(s))\|+\cdots+\|\phi_{\widetilde{v}_1}(\xi_n(s))-\phi_{\widetilde{v}_2}(\xi_n(s))\| \\
&+\int_0^s [\|k_1(s,\tau,\phi_{\widetilde{v}_1}(\xi_{n+1}(\tau)))-k_1(s,\tau,\phi_{\widetilde{v}_2}(\xi_{n+1}(\tau)))\|]d\tau\Bigg] ds \\
&+\widehat{M}_2\widetilde{\mathscr{L}}\int_0^t \Bigg[\|\phi_{\widetilde{v}_1}(\zeta_1(s))-\phi_{\widetilde{v}_2}(\zeta_1(s))\|+\cdots+\|\phi_{\widetilde{v}_1}(\zeta_p(s))-\phi_{\widetilde{v}_2}(\zeta_p(s))\| \\
&+\int_0^s [\|k_2(s,\tau,\phi_{\widetilde{v}_1}(\zeta_{p+1}(\tau)))-k_2(s,\tau,\phi_{\widetilde{v}_2}(\zeta_{p+1}(\tau)))\|]d\tau\Bigg] ds \\
\le\ & \widehat{M}_1\|q(\widetilde{v_1})-q(\widetilde{v_2})\| + \widehat{M}_2\|\widetilde{q}(\widetilde{v_1})-\widetilde{q}(\widetilde{v_2})\| + \widehat{M}_1\sum_{k=1}^m \|I_k(v_1(t_k))-I_k(v_2(t_k))\| \\
&+\widehat{M}_2\sum_{k=1}^m \|\overline{I}_k(v_1(t_k))-\overline{I}_k(v_2(t_k))\| + \widehat{M}_2\mathscr{L}\int_0^t \Bigg[\sup_{s\in[0,b]}\|\phi_{\widetilde{v}_1}(s)-\phi_{\widetilde{v}_2}(s)\|+\cdots \\
&+\sup_{s\in[0,b]}\|\phi_{\widetilde{v}_1}(s)-\phi_{\widetilde{v}_2}(s)\| + \mathscr{N}\int_0^s \|\phi_{\widetilde{v}_1}(\xi_{n+1}(\tau))-\phi_{\widetilde{v}_2}(\xi_{n+1}(\tau))\|d\tau\Bigg] ds \\
&+\widehat{M}_2\widetilde{\mathscr{L}}\int_0^t \Bigg[\sup_{s\in[0,b]}\|\phi_{\widetilde{v}_1}(s)-\phi_{\widetilde{v}_2}(s)\|+\cdots+\sup_{s\in[0,b]}\|\phi_{\widetilde{v}_1}(s)-\phi_{\widetilde{v}_2}(s)\| \\
&+\widetilde{\mathscr{N}}\int_0^s \|\phi_{\widetilde{v}_1}(\zeta_{p+1}(\tau))-\phi_{\widetilde{v}_2}(\zeta_{p+1}(\tau))\|d\tau\Bigg] ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \widehat{M}_1\|q(\widetilde{v_1}) - q(\widetilde{v_2})\| + \widehat{M}_2\|\widetilde{q}(\widetilde{v_1}) - \widetilde{q}(\widetilde{v_2})\| + \widehat{M}_1\sum_{k=1}^{m}\|I_k(v_1(t_k)) - I_k(v_2(t_k))\| \\
&\quad + \widehat{M}_2\sum_{k=1}^{m}\|\overline{I}_k(v_1(t_k)) - \overline{I}_k(v_2(t_k))\| + \widehat{M}_2\mathscr{L}\int_0^t\Big[n\sup_{s\in[0,b]}\|\phi_{\widetilde{v}_1}(s) - \phi_{\widetilde{v}_2}(s)\| \\
&\quad + \mathscr{N}b\sup_{s\in[0,b]}\|\phi_{\widetilde{v}_1}(s) - \phi_{\widetilde{v}_2}(s)\|\Big]ds \\
&\quad + \widehat{M}_2\widetilde{\mathscr{L}}\int_0^t\Big[p\sup_{s\in[0,b]}\|\phi_{\widetilde{v}_1}(s) - \phi_{\widetilde{v}_2}(s)\| + \widetilde{\mathscr{N}}b\sup_{s\in[0,b]}\|\phi_{\widetilde{v}_1}(s) - \phi_{\widetilde{v}_2}(s)\|\Big]ds \\
&\leq \widehat{M}_1\Big[\|q(\widetilde{v_1}) - q(\widetilde{v_2})\| + \sum_{k=1}^{m}\|I_k(v_1(t_k)) - I_k(v_2(t_k))\|\Big] + \widehat{M}_2\Big[\|\widetilde{q}(\widetilde{v_1}) - \widetilde{q}(\widetilde{v_2})\| \\
&\quad + \sum_{k=1}^{m}\|\overline{I}_k(v_1(t_k)) - \overline{I}_k(v_2(t_k))\|\Big] \\
&\quad + \widehat{M}_2\Big[(\mathscr{L}n + \widetilde{\mathscr{L}}p) + b(\mathscr{L}\mathscr{N} + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}})\Big]\int_0^t\sup_{s\in[0,b]}\|\phi_{\widetilde{v}_1}(s) - \phi_{\widetilde{v}_2}(s)\|ds.
\end{aligned}
$$

Making use of Gronwall's inequality just as before, for $t, v_1, v_2$ as above

$$
\begin{aligned}
&\sup_{s\in[0,b]}\|\phi_{\widetilde{v}_1}(t) - \phi_{\widetilde{v}_2}(t)\| \\
&\leq \Bigg[\widehat{M}_1\Big[\|q(\widetilde{v_1}) - q(\widetilde{v_2})\| + \sum_{k=1}^{m}\|I_k(v_1(t_k)) - I_k(v_2(t_k))\|\Big] + \widehat{M}_2\Big[\|\widetilde{q}(\widetilde{v_1}) - \widetilde{q}(\widetilde{v_2})\| \\
&\quad + \sum_{k=1}^{m}\|\overline{I}_k(v_1(t_k)) - \overline{I}_k(v_2(t_k))\|\Big]\Bigg]e^{\eta}
\end{aligned}
$$

for all $t \in [0, b]$, which implies that

$$
\begin{aligned}
&\|\Gamma v_1 - \Gamma v_2\|_{\mathcal{PC}} \\
&\leq \Bigg[\widehat{M}_1\Big[\|q(\widetilde{v_1}) - q(\widetilde{v_2})\| + \sum_{k=1}^{m}\|I_k(v_1(t_k)) - I_k(v_2(t_k))\|\Big] + \widehat{M}_2\Big[\|\widetilde{q}(\widetilde{v_1}) - \widetilde{q}(\widetilde{v_2})\| \\
&\quad + \sum_{k=1}^{m}\|\overline{I}_k(v_1(t_k)) - \overline{I}_k(v_2(t_k))\|\Big]\Bigg]e^{\eta}
\end{aligned}
$$

for all $t \in [\delta, b], v_1, v_2 \in B_r(\delta)$. Thus, the operator $\Gamma$ is continuous.

**Step 3**. $\Gamma$ is a compact operator.

To this end, we consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$, where $\Gamma_1$ and $\Gamma_2$ are the operators on $B_r(\delta)$ defined respectively by

$$
\begin{aligned}
(\Gamma_1 v)(t) &= C(t)[u_0 + q(\widetilde{v})] + S(t)[\widetilde{u}_0 + \widetilde{q}(\widetilde{v})] \\
&\quad + \int_0^t S(t-s)\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \ldots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s, \tau, \phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)ds
\end{aligned}
$$

$$+\int_0^t S(t-s)\mathscr{G}\Big(s,\phi_{\widetilde{v}}(\zeta_1(s)),\dots,\phi_{\widetilde{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)ds,$$
$$t\in[\delta,b],$$
$$(\Gamma_2 v)(t)=\sum_{0<t_k<t}C(t-t_k)I_k(v(t_k))+\sum_{0<t_k<t}S(t-t_k)\overline{I}_k(v(t_k)),\ t\in[\delta,b].$$

We first show that $\Gamma_1$ is a compact operator.

(i) $\Gamma_1(B_r(\delta))$ is equicontinuous.

Let $\delta\le\tau_1<\tau_2\le b$, and $\epsilon>0$ be small, note that

$$\begin{aligned}
&\Big\|\mathscr{F}\Big(s,\phi_{\widetilde{v}}(\xi_1(s)),\dots,\phi_{\widetilde{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\Big\|\\
&\le\Big\|\mathscr{F}\Big(s,\phi_{\widetilde{v}}(\xi_1(s)),\dots,\phi_{\widetilde{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)-\mathscr{F}(s,0,\dots,0)\Big\|\\
&\quad+\|\mathscr{F}(s,0,\dots,0)\|\\
&\le\mathscr{L}\Big[\|\phi_{\widetilde{v}}(\xi_1(s))\|\cdots+\|\phi_{\widetilde{v}}(\xi_n(s))\|+\Big\|\int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big\|\Big]+\mathscr{L}_1\\
&\le\mathscr{L}\Big[\sup_{s\in[\delta,b]}\|\phi_{\widetilde{v}}(s)\|+\cdots+\sup_{s\in[\delta,b]}\|\phi_{\widetilde{v}}(s)\|\\
&\quad+\int_0^s[\|k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))-k_1(s,\tau,0)\|+\|k_1(s,\tau,0)\|]d\tau\Big]+\mathscr{L}_1\\
&\le\mathscr{L}\Big\{n\sup_{s\in[\delta,b]}\|\phi_{\widetilde{v}}(s)\|+b\Big[\mathscr{N}\sup_{s\in[\delta,b]}\|\phi_{\widetilde{v}}(s)\|+\mathscr{N}_1\Big]\Big\}+\mathscr{L}_1\\
&\le\mathscr{L}\Big[(n+\mathscr{N}b)\sup_{s\in[\delta,b]}\|\phi_{\widetilde{v}}(s)\|+b\mathscr{N}_1\Big]+\mathscr{L}_1\\
&\le\mathscr{L}[(n+\mathscr{N}b)r+b\mathscr{N}_1]+\mathscr{L}_1:=\widehat{M}^{**}
\end{aligned}$$

and

$$\begin{aligned}
&\Big\|\mathscr{G}\Big(s,\phi_{\widetilde{v}}(\zeta_1(s)),\dots,\phi_{\widetilde{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big\|\\
&\le\Big\|\mathscr{G}\Big(s,\phi_{\widetilde{v}}(\zeta_1(s)),\dots,\phi_{\widetilde{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)-\mathscr{G}(s,0,\dots,0)\Big\|\\
&\quad+\|\mathscr{G}(s,0,\dots,0)\|\\
&\le\widetilde{\mathscr{L}}\Big[\|\phi_{\widetilde{v}}(\zeta_1(s))\|+\cdots+\|\phi_{\widetilde{v}}(\zeta_p(s))\|+\Big\|\int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\xi_{p+1}(\tau)))d\tau\Big\|\Big]+\widetilde{\mathscr{L}_1}\\
&\le\widetilde{\mathscr{L}}\Big[\sup_{s\in[\delta,b]}\|\phi_{\widetilde{v}}(s)\|+\cdots+\sup_{s\in[\delta,b]}\|\phi_{\widetilde{v}}(s)\|\\
&\quad+\int_0^s[\|k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))-k_2(s,\tau,0)\|+\|k_2(s,\tau,0)\|]d\tau\Big]+\widetilde{\mathscr{L}_1}
\end{aligned}$$

$$\leq \widetilde{\mathscr{L}}\left\{p \sup_{s\in[\delta,b]}\|\phi_{\widetilde{v}}(s)\| + b\left[\widetilde{\mathscr{N}} \sup_{s\in[\delta,b]}\|\phi_{\widetilde{v}}(s)\| + \widetilde{\mathscr{N}_1}\right]\right\} + \widetilde{\mathscr{L}_1}$$
$$\leq \widetilde{\mathscr{L}}\left[(p+\widetilde{\mathscr{N}}b) \sup_{s\in[\delta,b]}\|\phi_{\widetilde{v}}(s)\| + b\widetilde{\mathscr{N}_1}\right] + \widetilde{\mathscr{L}_1}$$
$$\leq \widetilde{\mathscr{L}}[(p+\widetilde{\mathscr{N}}b)r + b\widetilde{\mathscr{N}_1}] + \widetilde{\mathscr{L}_1} := \widehat{M}_1^{**}.$$

From the above estimations, we have

$$\begin{aligned}
&\|\Gamma_1 v(\tau_2) - \Gamma_1 v(\tau_1)\| \\
&\leq \|[C(\tau_2) - C(\tau_1)][u_0 + q(\widetilde{v})]\| + \|[S(\tau_2) - S(\tau_1)][\widetilde{u}_0 + \widetilde{q}(\widetilde{v})]\| \\
&\quad + \Big\| \int_0^{\tau_2} S(\tau_2 - s)\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \ldots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)ds \\
&\quad - \int_0^{\tau_1} S(\tau_1 - s)\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \ldots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)ds\Big\| \\
&\quad + \Big\| \int_0^{\tau_2} S(\tau_2 - s)\mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \ldots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)ds \\
&\quad - \int_0^{\tau_1} S(\tau_1 - s)\mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \ldots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)ds\Big\| \\
&\leq \|[C(\tau_2) - C(\tau_1)][u_0 + q(\widetilde{v})]\| + \|[S(\tau_2) - S(\tau_1)][\widetilde{u}_0 + \widetilde{q}(\widetilde{v})]\| \\
&\quad + \int_0^{\tau_1-\epsilon} \Big\|[S(\tau_2 - s) - S(\tau_1 - s)]\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \ldots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\Big\|ds \\
&\quad + \int_{\tau_1-\epsilon}^{\tau_1} \Big\|[S(\tau_2 - s) - S(\tau_1 - s)]\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \ldots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\Big\|ds \\
&\quad + \int_{\tau_1}^{\tau_2} \Big\|S(\tau_2 - s)\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \ldots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\Big\|ds \\
&\quad + \int_0^{\tau_1-\epsilon} \Big\|[S(\tau_2 - s) - S(\tau_1 - s)]\mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \ldots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big\|ds \\
&\quad + \int_{\tau_1-\epsilon}^{\tau_1} \Big\|[S(\tau_2 - s) - S(\tau_1 - s)]\mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \ldots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big\|ds \\
&\quad + \int_{\tau_1}^{\tau_2} \Big\|S(\tau_2 - s)\mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \ldots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big\|ds \\
&\leq \|[C(\tau_2) - C(\tau_1)][u_0 + q(\widetilde{v})]\| + \|[S(\tau_2) - S(\tau_1)][\widetilde{u}_0 + \widetilde{q}(\widetilde{v})]\| \\
&\quad + \widehat{M}^{**}\int_0^{\tau_1-\epsilon} \|S(\tau_2 - s) - S(\tau_1 - s)\|ds + \widehat{M}^{**}\int_{\tau_1-\epsilon}^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)]\|ds \\
&\quad + \widehat{M}^{**}\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|ds + \widehat{M}_1^{**}\int_0^{\tau_1-\epsilon} \|S(\tau_2 - s) - S(\tau_1 - s)\|ds \\
&\quad + \widehat{M}_1^{**}\int_{\tau_1-\epsilon}^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|ds + \widehat{M}_1^{**}\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|ds.
\end{aligned}$$

We see that $\|\Gamma_1 v(\tau_2) - \Gamma v(\tau_1)\|$ tends to zero independently of $v \in B_r(\delta)$ as $\tau_2 - \tau_1 \to 0$, since the compactness of the operator $S(t)$ for $t > 0$, implies

the continuity in the uniform operator topology. Thus, $\Gamma_1$ maps $B_r(\delta)$ into an equicontinuous family of the functions.

(ii) The set $\Gamma_1(B_r(\delta))(t)$ is precompact in $\mathbb{X}$.

Let $\delta < t \leq s \leq b$ be fixed and $\epsilon$ a real number satisfying $0 < \epsilon < t$. For $v \in B_r(\delta)$, we define

$$\begin{aligned}
&(\Gamma_{1,\epsilon}v)(t)\\
&= C(t)[u_0 + q(\widetilde{v})] + S(t)[\widetilde{u}_0 + \widetilde{q}(\widetilde{v})]\\
&\quad + \int_0^{t-\epsilon} S(t-s)\Big[\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \ldots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\\
&\quad + \mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \ldots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds.
\end{aligned}$$

Working with the compactness of $C(t)$ for $t > 0$, we deduce that the set $\{(\Gamma_{1,\epsilon}v)(t) : v \in B_r(\delta)\}$ is precompact $v \in B_r(\delta)$ for $\epsilon, 0 < \epsilon < t$. Furthermore, for each $v \in B_r(\delta)$ we sustain

$$\begin{aligned}
&\|(\Gamma_1 v)(t) - (\Gamma_{1,\epsilon}v)(t)\|\\
&\leq \int_{t-\epsilon}^{t} \Big\| S(t-s)\Big[\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \ldots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\\
&\qquad + \mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \ldots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]\Big\| ds\\
&\leq \widehat{M}_2 \int_{t-\epsilon}^{t} \big(\widehat{M}^{**} + \widehat{M}_1^{**}\big) ds\\
&\leq \widehat{M}_2\big(\widehat{M}^{**} + \widehat{M}_1^{**}\big)\epsilon.
\end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $\{(\Gamma_1 v) : v \in B_r(\delta)\}$. Hence the set $\{(\Gamma_1 v) : v \in B_r(\delta)\}$ is a precompact in $\mathbb{X}$. It is easy to see that $\Gamma_1(B_r(\delta))$ is uniformly bounded. Since we have proven that $\Gamma_1(B_r(\delta))$ is an equicontinuous collection, by the Arzela-Ascoli theorem it suffices to demonstrate that $\Gamma_1$ maps $B_r(\delta)$ into a precompact set in $\mathbb{X}$.

Next, it stays to check that $\Gamma_2$ is also a compact operator. From [1, Theorem 3.2], we observe that $\Gamma_2$ is compact operator and hence $\Gamma$ is a compact operator.

**Step 4**. We now reveal that we can find an open set $U \subseteq \mathcal{PC}_\delta$ with $v \notin \lambda\Gamma v$ for $\lambda \in (0,1)$ and $v \in \partial U$. Let $\lambda \in (0,1)$ and allow $v \in \mathcal{PC}_\delta$ be a possible solution of $v = \lambda\Gamma(v)$ for some $0 < \lambda < 1$. Therefore, for every $t \in (0, b]$,

$$\begin{aligned}
v(t) = \lambda\phi_{\widetilde{v}}(t) &= \lambda C(t)[u_0 + q(\widetilde{v})] + \lambda S(t)[\widetilde{u}_0 + \widetilde{q}(\widetilde{v})]\\
&\quad + \lambda\int_0^t S(t-s)\Big[\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \ldots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\\
&\quad + \mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \ldots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds
\end{aligned}$$

$$+\lambda\sum_{0<t_k<t}C(t-t_k)I_k(v(t_k))+\lambda\sum_{0<t_k<t}S(t-t_k)\overline{I}_k(v(t_k)).$$

This suggests by (H1)-(H5) and for each $t\in(0,b]$, we have $\|v(t)\|\leq\|\phi_{\widetilde{v}}(t)\|$ and

$$\begin{aligned}
&\|\phi_{\widetilde{v}}(t)\|\\
&\leq\|C(t)[u_0+q(\widetilde{v})]\|+\|S(t)[\widetilde{u}_0+\widetilde{q}(\widetilde{v})]\|\\
&\quad+\int_0^t\left\|S(t-s)\mathscr{F}\left(s,\phi_{\widetilde{v}}(\xi_1(s)),\ldots,\phi_{\widetilde{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\right)\right\|ds\\
&\quad+\int_0^t\left\|S(t-s)\mathscr{G}\left(s,\phi_{\widetilde{v}}(\zeta_1(s)),\ldots,\phi_{\widetilde{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\right)\right\|ds\\
&\quad+\left\|\sum_{0<t_k<t}C(t-t_k)I_k(v(t_k))\right\|+\left\|\sum_{0<t_k<t}S(t-t_k)\overline{I}_k(v(t_k))\right\|\\
&\leq\widehat{M}_*+\widehat{M}_1\left[\Lambda(\|\widetilde{v}\|_{\mathcal{PC}})+\sum_{k=1}^m\Psi_k(\|v(t_k)\|)\right]+\widehat{M}_2\left[\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}})+\sum_{k=1}^m\widetilde{\Psi}_k(\|v(t_k)\|)\right]\\
&\quad+\widehat{M}_2\left[(\mathscr{L}n+\widetilde{\mathscr{L}}p)+b(\mathscr{L}\mathscr{N}+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}})\right]\int_0^t\sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(s)\|ds,
\end{aligned}$$

where $\widehat{M}_*=\widehat{M}_1\|u_0\|+\widehat{M}_2\|\widetilde{u}_0\|+\widehat{M}_2b\left[b(\mathscr{L}\mathscr{N}_1+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}_1})+(\mathscr{L}_1+\widetilde{\mathscr{L}_1})\right]$.

Utilizing the Gronwall's inequality, we obtain

$$\begin{aligned}
&\sup_{(0,b]}\|\phi_{\widetilde{v}}(s)\|\\
&\leq e^{\eta}\left[\widehat{M}_*+\widehat{M}_1\left[\Lambda(\|\widetilde{v}\|_{\mathcal{PC}})+\sum_{k=1}^m\Psi_k(\|v\|_{\mathcal{PC}})\right]+\widehat{M}_2\left[\widetilde{\Lambda}(\|v\|_{\mathcal{PC}})+\sum_{k=1}^m\widetilde{\Psi}_k(\|\widetilde{v}\|_{\mathcal{PC}})\right]\right].
\end{aligned}$$

and the previous inequality holds. Consequently,

$$\|v\|_{\mathcal{PC}}\leq e^{\eta}\left[\widehat{M}_*+\widehat{M}_1\left[\Lambda(\|\widetilde{v}\|_{\mathcal{PC}})+\sum_{k=1}^m\Psi_k(\|v\|_{\mathcal{PC}})\right]+\widehat{M}_2\left[\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}})+\sum_{k=1}^m\widetilde{\Psi}_k(\|v\|_{\mathcal{PC}})\right]\right],$$

and therefore

$$\frac{\|v\|_{\mathcal{PC}}}{e^{\eta}\left[\widehat{M}_*+\widehat{M}_1\left[\Lambda(\|\widetilde{v}\|_{\mathcal{PC}})+\sum\limits_{k=1}^m\Psi_k(\|v\|_{\mathcal{PC}})\right]+\widehat{M}_2\left[\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}})+\sum\limits_{k=1}^m\widetilde{\Psi}_k(\|v\|_{\mathcal{PC}})\right]\right]}\leq 1.$$

From the hypothesis (H6), we can find a constant $\widehat{M}^*$ such that $\|v\|_{\mathcal{PC}}\neq\widehat{M}^*$. Set

$$U=\left\{v\in\mathcal{PC}([\delta,b],\mathbb{X});\ \sup_{\delta\leq t\leq b}\|v(t)\|<\widehat{M}^*\right\}.$$

As an outcome of Steps 1-3 in Theorem 3.1, it suffices to demonstrate that $\Gamma:\overline{U}\to\mathcal{PC}_\delta$ is a compact map.

With the selection of $U$, there is no $u \in \partial U$ in a way that $v \in \lambda \Gamma v$ for $\lambda \in (0,1)$. As being a end result of Lemma 2.2, we consider that the operator $\Gamma$ has a fixed point $\widetilde{v}_* \in \overline{U}$. Thus, we obtain

$$\begin{aligned} u(t) &= C(t)[u_0 + q(\widetilde{v}_*)] + S(t)[\widetilde{u}_0 + \widetilde{q}(\widetilde{v}_*)] \\ &\quad + \int_0^t S(t-s)\Bigg[\mathscr{F}\Bigg(s, u(\xi_1(s)), \ldots, u(\xi_n(s)), \int_0^s k_1(s,\tau,u(\xi_{n+1}(\tau)))d\tau\Bigg) \\ &\quad + \mathscr{G}\Bigg(s, u(\zeta_1(s)), \ldots, u(\zeta_p(s)), \int_0^s k_2(s,\tau,u(\zeta_{p+1}(\tau)))d\tau\Bigg)\Bigg]ds \\ &\quad + \sum_{0<t_k<t} C(t-t_k)I_k(v(t_k)) + \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(v(t_k)). \end{aligned} \tag{3.4}$$

Noting that $u = \phi_{\widetilde{v}_*} = (\Gamma\widetilde{v}_*)(t) = \widetilde{v}_*$, $t \in [\delta, b]$. By (H5)(i)(ii), we obtain $q(u) = q(\widetilde{v}_*)$ and $\widetilde{q}(u) = \widetilde{q}(\widetilde{v}_*)$. This suggests, joined with (3.4), that $u(t)$ is a mild solution of problem (1.1)-(1.3). This completes the proof of this theorem. □

## 4. Approximate results

As an application of Theorem 3.1, we shall consider the system (1.1) with control parameters such as:

$$\begin{aligned} u''(t) &= \mathscr{A}u(t) + \mathscr{F}\left(t, u(\xi_1(t)), \ldots, u(\xi_n(t)), \int_0^t k_1(t,s,u(\xi_{n+1}(s)))ds\right) \\ &\quad + \mathscr{G}\left(t, u(\zeta_1(t)), \ldots, u(\zeta_p(t)), \int_0^t k_2(t,s,u(\zeta_{p+1}(s)))ds\right) + B\widehat{u}(t), \\ &\quad t \in \mathscr{J} = [0,b],\ t \neq t_k,\ k = 1,2,\ldots,m, \end{aligned} \tag{4.1}$$

with the conditions (1.2) and (1.3). The functions $\mathscr{F}(\cdot)$, $\mathscr{G}(\cdot)$, $k_1(\cdot)$, $k_2(\cdot)$, $q(\cdot)$, $\widetilde{q}(\cdot)$, $I_k(\cdot)$, $\overline{I}_k(\cdot)$, $\xi_i$, $i = 1,2,\ldots,n+1$ and $\zeta_l$, $l = 1,\ldots,p+1$, are same as defined in (1.1)-(1.3). The control function $\widehat{u}(\cdot) \in \mathscr{L}^2(\mathscr{J}, U)$, a Banach space of admissible control function with $U$ as a Banach space and $B$ is a bounded linear operator from $U$ to $\mathbb{X}$.

**Definition 4.1.** A function $u(\cdot) \in \mathcal{PC}(\mathscr{J}, \mathbb{X})$ is said to be a mild solution of problem (4.1) with the conditions (1.2) and (1.3) if it satisfies the following integral equation

$$\begin{aligned} u(t) &= C(t)[u_0 + q(u)] + S(t)[\widetilde{u}_0 + \widetilde{q}(u)] \\ &\quad + \int_0^t S(t-s)\Bigg[\mathscr{F}\Bigg(s, u(\xi_1(s)), \ldots, u(\xi_n(s)), \int_0^s k_1(s,\tau,u(\xi_{n+1}(\tau)))d\tau\Bigg) \\ &\quad + \mathscr{G}\Bigg(s, u(\zeta_1(s)), \ldots, u(\zeta_p(s)), \int_0^s k_2(s,\tau,u(\zeta_{p+1}(\tau)))d\tau\Bigg) + B\widehat{u}(s)\Bigg]ds \\ &\quad + \sum_{0<t_k<t} C(t-t_k)I_k(u(t_k)) + \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(u(t_k)), \quad t \in \mathscr{J}. \end{aligned}$$

**Definition 4.2.** The control system (4.1) with the conditions (1.2) and (1.3) is said to be approximately controllable on $\mathscr{J}$ if for all $u_0 \in \mathbb{X}_\alpha$, there is some control $\widehat{u} \in \mathscr{L}^2(\mathscr{J}, U)$, the closure of the reachable set, $\overline{R(b, u_0)}$ is dense in $\mathbb{X}$, i.e., $\overline{R(b, u_0)} = \mathbb{X}$, where $R(b, u_0) = \{u(b, \widehat{u}) : \widehat{u} \in \mathscr{L}^2(\mathscr{J}, U), u(0, \widehat{u}) = u_0\}$ is a mild solution of the system (4.1) with the conditions (1.2) and (1.3).

In order to address the problem, it is helpful now to present two significant operators and essential presumptions on these operators:

$$\overline{\Upsilon}_0^b = \int_0^b S(b-s)BB^*S^*(b-s)ds : \mathbb{X} \to \mathbb{X},$$
$$R(\gamma, \overline{\Upsilon}_0^b) = (\gamma I + \overline{\Upsilon}_0^b)^{-1} : \mathbb{X} \to \mathbb{X}, \quad 0 < \gamma < 1,$$

where $B^*$ denotes the adjoint of $B$ and $S^*(t)$ is the adjoint of $S(t)$. It is straightforward that the operator $\overline{\Upsilon}_0^b$ is a linear bounded operator.

To investigate the approximate controllability of system (4.1) with the conditions (1.2) and (1.3), we impose the following condition:

(H0) $\gamma R(\gamma, \overline{\Upsilon}_0^b) \to 0$ as $\gamma \to 0^+$ in the strong operator topology.

In view of [19], hypothesis (H0) holds if and only if the linear system

$$u'(t) = \mathscr{A}u(t) + B\widehat{u}(t), \quad t \in [0, b], \tag{4.2}$$
$$u(0) = u_0 \tag{4.3}$$

is approximate controllability on $\mathscr{J}$.

It will be shown that the system (1.4) with the conditions (1.2) and (1.3) is approximately controllable, if for all $\gamma > 0$, there exists a function $u(\cdot) \in \mathcal{PC}$ and $u_b \in \mathbb{X}$ such that

$$\begin{aligned} u(t) &= C(t)[u_0 + q(u)] + S(t)[\widetilde{u}_0 + \widetilde{q}(u)] \\ &\quad + \int_0^t S(t-s)\Bigg[\mathscr{F}\bigg(s, u(\xi_1(s)), \dots, u(\xi_n(s)), \int_0^s k_1(s, \tau, u(\xi_{n+1}(\tau)))d\tau\bigg) \\ &\quad + \mathscr{G}\bigg(s, u(\zeta_1(s)), \dots, u(\zeta_p(s)), \int_0^s k_2(s, \tau, u(\zeta_{p+1}(\tau)))d\tau\bigg) + B\widehat{u}(s, u)\Bigg]ds \\ &\quad + \sum_{0<t_k<t} C(t-t_k)I_k(u(t_k)) + \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(u(t_k)), \end{aligned}$$

$$\widehat{u}(t, u) = B^*S^*(b-t)R(\gamma, \overline{\Upsilon}_0^b)\widehat{p}(u(\cdot)),$$

where

$$\begin{aligned} \widehat{p}(u(\cdot)) &= u_b - C(b)[u_0 + q(u)] - S(b)[\widetilde{u}_0 + \widetilde{q}(u)] \\ &\quad - \int_0^b S(b-s)\Bigg[\mathscr{F}\bigg(s, u(\xi_1(s)), \dots, u(\xi_n(s)), \int_0^s k_1(s, \tau, u(\xi_{n+1}(\tau)))d\tau\bigg) \\ &\quad + \mathscr{G}\bigg(s, u(\zeta_1(s)), \dots, u(\zeta_p(s)), \int_0^s k_2(s, \tau, u(\zeta_{p+1}(\tau)))d\tau\bigg)\Bigg]ds \end{aligned}$$

$$-\sum_{k=1}^{m} C(b-t_k)I_k(u(t_k)) - \sum_{k=1}^{m} S(b-t_k)\overline{I}_k(u(t_k)).$$

*Remark* 4.1. In view of equations (3.2), (3.3) and Step 1 of Theorem 3.1, if $v \in \mathcal{PC}_\delta$, we calculate the following estimate:

$$\begin{aligned}
&\left\| \int_0^t S(t-s)B\widehat{u}(s,v)ds \right\| \\
\leq\ & \int_0^t S(t-s)BB^*S^*(b-t)R(\gamma,\overline{\Upsilon}_0^b)\Bigg[u_b - C(b)[u_0+q(\widetilde{v})] - S(b)[\widetilde{u}_0+\widetilde{q}(\widetilde{v})] \\
&- \int_0^b S(b-s)\Bigg[\mathscr{F}\left(s,\phi_{\widetilde{v}}(\xi_1(s)),\dots,\phi_{\widetilde{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\right) \\
&+ \mathscr{G}\left(s,\phi_{\widetilde{v}}(\zeta_1(s)),\dots,\phi_{\widetilde{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\right)\Bigg]ds \\
&- \sum_{k=1}^{m} C(b-t_k)I_k(v(t_k)) - \sum_{k=1}^{m} S(b-t_k)\overline{I}_k(v(t_k))\Bigg](s)ds \\
\leq\ & \left(\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\left[\|u_b\| + \widehat{M}_1\left[\|u_0\| + \Lambda(r) + \sum_{k=1}^{m}\Psi_k(r)\right]\right. \\
&+ \widehat{M}_2\left[\|\widetilde{u}_0\| + \widetilde{\Lambda}(r) + \sum_{k=1}^{m}\widetilde{\Psi}_k(r)\right] \\
&\left.+ \widehat{M}_2 b\left[b(\mathscr{L}\mathscr{N}_1 + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}_1}) + (\mathscr{L}_1 + \widetilde{\mathscr{L}_1})\right]\right] \\
&+ \left(\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\widehat{M}_2\left[(\mathscr{L}n + \widetilde{\mathscr{L}}p) + b(\mathscr{L}\mathscr{N} + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}})\right]\int_0^b \sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(s)\|ds.
\end{aligned}$$

**Theorem 4.1.** *Suppose that the hypotheses* (H0)-(H5) *are satisfied. Then the system* (4.1) *with the conditions* (1.2) *and* (1.3) *has at least one mild solution on* $\mathscr{J}$ *provided*

$$\frac{\widehat{M}_{***}}{e^{\widetilde{\eta}}\left[\widehat{M}_{**} + \left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\left[\widehat{M}_1\left(\Lambda(\widehat{M}_{***}) + \sum_{k=1}^{m}\Psi_k(\widehat{M}_{***})\right) + \widehat{M}_2\left(\widetilde{\Lambda}(\widehat{M}_{***}) + \sum_{k=1}^{m}\widetilde{\Psi}_k(\widehat{M}_{***})\right)\right]\right]} > 1,$$

*where* $\widehat{M}_B = \|B\|$, $\widehat{M}_{**} = \left(\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\|u_b\| + \left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\left[\widehat{M}_1\|u_0\| + \widehat{M}_2\|\widetilde{u}_0\| + \widehat{M}_2 b\left[b(\mathscr{L}\mathscr{N}_1 + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}_1}) + (\mathscr{L}_1 + \widetilde{\mathscr{L}_1})\right]\right]$ *and*

$\widetilde{\eta} = \left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\widehat{M}_2[(\mathscr{L}n + \widetilde{\mathscr{L}}p) + b(\mathscr{L}\mathscr{N} + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}})]b.$

*Proof.* By thinking of Theorem 3.1, we define

$\phi_{\widetilde{v}}(t) = C(t)[u_0 + q(\widetilde{v})] + S(t)[\widetilde{u}_0 + \widetilde{q}(\widetilde{v})]$

$$
\begin{aligned}
&+\int_0^t S(t-s)\Big[\mathscr{F}\Big(s,\phi_{\widetilde{v}}(\xi_1(s)),\dots,\phi_{\widetilde{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\\
&+\mathscr{G}\Big(s,\phi_{\widetilde{v}}(\zeta_1(s)),\dots,\phi_{\widetilde{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds\\
&+\int_0^t S(t-s)BB^*S^*(b-t)R(\gamma,\overline{\Upsilon}_0^b)\Big[u_b-C(b)[u_0+q(\widetilde{v})]-S(b)[\widetilde{u}_0+\widetilde{q}(\widetilde{v})]\\
&-\int_0^b S(b-s)\Big[\mathscr{F}\Big(s,\phi_{\widetilde{v}}(\xi_1(s)),\dots,\phi_{\widetilde{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\\
&+\mathscr{G}\Big(s,\phi_{\widetilde{v}}(\zeta_1(s)),\dots,\phi_{\widetilde{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds\\
&-\sum_{k=1}^m C(b-t_k)I_k(v(t_k))-\sum_{k=1}^m S(b-t_k)\overline{I}_k(v(t_k))\Big](s)ds\\
&+\sum_{0<t_k<t} C(t-t_k)I_k(v(t_k))+\sum_{0<t_k<t} S(t-t_k)\overline{I}_k(v(t_k)).
\end{aligned}
$$

Consider the map $\widetilde{\Gamma}:\mathcal{PC}_\delta=\mathcal{PC}([\delta,b],\mathbb{X})\to\mathcal{PC}_\delta$ defined by

$$(\widetilde{\Gamma}v)(t)=\phi_{\widetilde{v}}(t),\ t\in[\delta,b].$$

We might demonstrate that the operator $\widetilde{\Gamma}$ satisfies every one of the states of Lemma 2.2. For better understandability, the proof is going to be presented in a few stages.

**Step 1***. $\widetilde{\Gamma}$ maps bounded sets into bounded sets in $\mathcal{PC}_\delta$.

Without a doubt, it is sufficient to show that we can find a positive constant $\widetilde{\Lambda}_2$ in ways that for every $v\in B_r(\delta):=\Big\{\phi\in\mathcal{PC}_\delta;\ \sup_{\delta\le t\le b}\|\phi(t)\|\le r\Big\}$ one has $\|\widetilde{\Gamma}v\|_{\mathcal{PC}}\le\widetilde{\Lambda}_2$.

Let $v\in B_r(\delta)$, then for $t\in(0,b]$, we receive

$$
\begin{aligned}
&\|\phi_{\widetilde{v}}(t)\|\\
&\le\widehat{M}_{**}+\left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2b\right)\left[\widehat{M}_1\left(\Lambda(r)+\sum_{k=1}^m\Psi_k(r)\right)+\widehat{M}_2\left(\widetilde{\Lambda}(r)+\sum_{k=1}^m\widetilde{\Psi}_k(r)\right)\right]\\
&\quad+\left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2b\right)\widehat{M}_2\Big[(\mathscr{L}n+\widetilde{\mathscr{L}}p)+b(\mathscr{L}\mathscr{N}+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}})\Big]\int_0^b\sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(s)\|ds,
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{M}_{**}=&\left(\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2b\right)\|u_b\|+\left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2b\right)\Big[\widehat{M}_1\|u_0\|\\
&+\widehat{M}_2\|\widetilde{u}_0\|+\widehat{M}_2b\Big[b(\mathscr{L}\mathscr{N}_1+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}_1})+(\mathscr{L}_1+\widetilde{\mathscr{L}_1})\Big]\Big].
\end{aligned}
$$

Utilizing the Gronwall's inequality, we receive

$$\sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(t)\| \leq e^{\left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\widehat{M}_2[(\mathscr{L}n+\widetilde{\mathscr{L}}p)+b(\mathscr{L}\mathscr{N}+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}})]b}\Bigg[\widehat{M}_{**} + \left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\Bigg[\widehat{M}_1\left(\Lambda(r)+\sum_{k=1}^{m}\Psi_k(r)\right) + \widehat{M}_2\left(\widetilde{\Lambda}(r)+\sum_{k=1}^{m}\widetilde{\Psi}_k(r)\right)\Bigg].$$

Thus

$$\|\widetilde{\Gamma}v\|_{\mathcal{PC}} \leq e^{\overline{\eta}}\Bigg[\widehat{M}_{**} + \left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\Bigg[\widehat{M}_1\left(\Lambda(r)+\sum_{k=1}^{m}\Psi_k(r)\right) + \widehat{M}_2\left(\widetilde{\Lambda}(r)+\sum_{k=1}^{m}\widetilde{\Psi}_k(r)\right)\Bigg] := \widetilde{\Lambda}_2.$$

By implementing the techniques applied in Theorem 3.1 (Step 2 & Step 3), we deduce that the operator $\widetilde{\Gamma}$ is continuous and compact with simple modifications.

**Step 4***. We now present that we can find an open set $U \subseteq \mathcal{PC}_\delta$ with $v \notin \lambda\widetilde{\Gamma}v$ for $\lambda \in (0,1)$ and $v \in \partial U$. Let $\lambda \in (0,1)$ and let $v \in \mathcal{PC}_\delta$ be a possible solution of $v = \lambda\widetilde{\Gamma}(v)$ for some $0 < \lambda < 1$. Therefore, for every $t \in (0,b]$,

$$\begin{aligned}
v(t) = \lambda\phi_{\widetilde{v}}(t) &= \lambda C(t)[u_0+q(\widetilde{v})] + \lambda S(t)[\widetilde{u}_0+\widetilde{q}(\widetilde{v})] \\
&+ \lambda\int_0^t S(t-s)\Bigg[\mathscr{F}\Bigg(s,\phi_{\widetilde{v}}(\xi_1(s)),\dots,\phi_{\widetilde{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Bigg) \\
&+ \mathscr{G}\Bigg(s,\phi_{\widetilde{v}}(\zeta_1(s)),\dots,\phi_{\widetilde{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Bigg)\Bigg]ds \\
&+ \lambda\int_0^t S(t-s)BB^*S^*(b-t)R(\gamma,\overline{\Upsilon}_0^b)\Bigg[u_b - C(b)[u_0+q(\widetilde{v})] - S(b)[\widetilde{u}_0+\widetilde{q}(\widetilde{v})] \\
&- \int_0^b S(b-s)\Bigg[\mathscr{F}\Bigg(s,\phi_{\widetilde{v}}(\xi_1(s)),\dots,\phi_{\widetilde{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Bigg) \\
&+ \mathscr{G}\Bigg(s,\phi_{\widetilde{v}}(\zeta_1(s)),\dots,\phi_{\widetilde{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Bigg)\Bigg]ds \\
&- \sum_{k=1}^{m} C(b-t_k)I_k(v(t_k)) - \sum_{k=1}^{m} S(b-t_k)\overline{I}_k(v(t_k))\Bigg](s)ds \\
&+ \lambda\sum_{0<t_k<t} C(t-t_k)I_k(v(t_k)) + \lambda\sum_{0<t_k<t} S(t-t_k)\overline{I}_k(v(t_k)).
\end{aligned}$$

This implies by (H1)-(H5) and for each $t \in (0,b]$ we have $\|v(t)\| \leq \|\phi_{\widetilde{v}}(t)\|$ and

$$\begin{aligned}\|\phi_{\widetilde{v}}(t)\| \leq \widehat{M}_{**} + \left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\Bigg[\widehat{M}_1\left(\Lambda(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^{m}\Psi_k(\|v\|_{\mathcal{PC}})\right)\\ + \widehat{M}_2\left(\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^{m}\widetilde{\Psi}_k(\|v\|_{\mathcal{PC}})\right)\Bigg]\\ + \left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\widehat{M}_2\Big[(\mathscr{L}n + \widetilde{\mathscr{L}}p) + b(\mathscr{L}\mathscr{N} + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}})\Big]\\ \cdot\int_0^b \sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(s)\|ds.\end{aligned}$$

Utilizing the Gronwall's inequality, we receive

$$\begin{aligned}\sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(t)\| \leq e^{\widetilde{\eta}}\Bigg[\widehat{M}_{**} + \left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\Bigg[\widehat{M}_1\left(\Lambda(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^{m}\Psi_k(\|v\|_{\mathcal{PC}})\right)\\ + \widehat{M}_2\left(\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^{m}\widetilde{\Psi}_k(\|v\|_{\mathcal{PC}})\right)\Bigg]\Bigg],\end{aligned}$$

and the past inequality keeps. As a result,

$$\begin{aligned}\|v\|_{\mathcal{PC}} \leq e^{\widetilde{\eta}}\Bigg[\widehat{M}_{**} + \left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\Bigg[\widehat{M}_1\left(\Lambda(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^{m}\Psi_k(\|v\|_{\mathcal{PC}})\right)\\ + \widehat{M}_2\left(\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^{m}\widetilde{\Psi}_k(\|v\|_{\mathcal{PC}})\right)\Bigg]\Bigg],\end{aligned}$$

and therefore

$$\frac{\|v\|_{\mathcal{PC}}}{e^{\widetilde{\eta}}\left[\widehat{M}_{**} + \left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\left[\widehat{M}_1\left(\Lambda(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^{m}\Psi_k(\|v\|_{\mathcal{PC}})\right) + \widehat{M}_2\left(\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^{m}\widetilde{\Psi}_k(\|v\|_{\mathcal{PC}})\right)\right]\right]} \leq 1.$$

There exists a constant $\widehat{M}_{***} > 0$ in ways that $\|v\|_{\mathcal{PC}} \neq \widehat{M}_{***}$. Fix

$$U = \left\{v \in \mathcal{PC}([\delta,b],\mathbb{X}); \sup_{\delta\leq t\leq b}\|v(t)\| < \widehat{M}_{***}\right\}.$$

As a result of Step 1* and Step 4* in Theorem 4.1, it is enough to demonstrate that the operator $\widetilde{\Gamma} : \overline{U} \to \mathcal{PC}_\delta$ is a compact map.

With the option of $U$, there is no $u \in \partial U$ in ways that $v \in \lambda\widetilde{\Gamma}v$ for $\lambda \in (0,1)$. As a result of Lemma 2.2, we infer that $\widetilde{\Gamma}$ has a fixed point $\widetilde{v}_* \in \overline{U}$. From the equation (3.4), we infer that $u(t)$ is a mild solution of the system (4.1) with the conditions (1.2). The proof is now completed. □

**Theorem 4.2.** *Assume that the conditions* (H0)-(H5) *hold and linear system* (4.2)-(4.3) *is approximately controllable on* $\mathscr{J}$. *The functions* $\mathscr{F} : J \times \mathbb{X}^{n+1} \to$

*$\mathbb{X}$ and $\mathscr{G} : J \times \mathbb{X}^{p+1} \to \mathbb{X}$ are continuous and uniformly bounded and there exist constants $\mathscr{N}^* > 0, \mathscr{N}^{**} > 0$ such that $\|\mathscr{F}(t, u_1, u_2, \ldots, u_{n+1})\|_\alpha \leq \mathscr{N}^*$ and $\|\mathscr{G}(t, u_1, u_2, \ldots, u_{p+1})\|_\alpha \leq \mathscr{N}^{**}$, then the system* (4.1) *with the conditions* (1.2) *and* (1.3) *is approximately controllable on $\mathscr{J}$.*

*Proof.* Let $u^\gamma(\cdot)$ be a fixed point of $\widetilde{\Gamma}$. By Theorem 4.1, any fixed point of $\widetilde{\Gamma}$ is a mild solution of (4.1) with the conditions (1.2) and (1.3) under the control

$$\hat{u}^\gamma(t) = B^* S^*(b-t) R(\gamma, \overline{\Upsilon}_0^b) \widehat{p}(u^\gamma)$$

and satisfies the inequality

$$u^\gamma(b) = u_b + \gamma R(\gamma, \overline{\Upsilon}_0^b) \widehat{p}(u^\gamma). \tag{4.2}$$

Moreover by assumptions on $\mathscr{F}$ and $\mathscr{G}$ with Dunford-Pettis theorem, we have that $\{f^\gamma(s)\}$ and $\{g^\gamma(s)\}$ are weakly compact in $\mathscr{L}^1(\mathscr{J}, \mathbb{X})$, so there is a subsequence, still denoted by $\{f^\gamma(s)\}$ and $\{g^\gamma(s)\}$, that converges weakly to say $f(s)$ and $g(s)$ in $\mathscr{L}^1(\mathscr{J}, \mathbb{X})$ respectively.

Define

$$w = u_b - C(b)[u_0 + q(u)] - S(b)[\widetilde{u}_0 + \widetilde{q}(u)] - \int_0^b S(b-s)[f(s) + g(s)]ds$$
$$- \sum_{k=1}^m C(b-t_k) I_k(u(t_k)) - \sum_{k=1}^m S(b-t_k) \overline{I}_k(u(t_k)).$$

Now, we have

$$\begin{aligned}
\|\widehat{p}(u^\gamma) - w\| &= \left\| \int_0^b S(b-s)[f(s, u_1^\gamma(s), u_2^\gamma(s), \ldots, u_{n+1}^\gamma(s)) - f(s)]ds \right\| \\
&\quad + \left\| \int_0^b S(b-s)[g(s, u_1^\gamma(s), u_2^\gamma(s), \ldots, u_{p+1}^\gamma(s)) - g(s)]ds \right\| \\
&\leq \sup_{t\in[0,b]} \left[ \left\| \int_0^t S(t-s)[f(s, u_1^\gamma(s), u_2^\gamma(s), \ldots, u_{n+1}^\gamma(s)) - f(s)]ds \right\| \right. \\
&\quad \left. + \left\| \int_0^t S(t-s)[g(s, u_1^\gamma(s), u_2^\gamma(s), \ldots, u_{p+1}^\gamma(s)) - g(s)]ds \right\| \right].
\end{aligned} \tag{4.3}$$

By using infinite-dimensional version of the Ascoli-Arzela theorem, one can show that an operator $l(\cdot) \to \int_0^\cdot S(\cdot - s) l(s) ds : \mathscr{L}^1(\mathscr{J}, \mathbb{X}) \to \mathscr{C}(\mathscr{J}, \mathbb{X})$ is compact. Consequently, we obtain that $\|\widehat{p}(u^\gamma) - w\| \to 0$ as $\gamma \to 0^+$. Moreover, from (4.2), we obtain

$$\begin{aligned}
\|u^\gamma(b) - u_b\| &\leq \|\gamma R(\gamma, \overline{\Upsilon}_0^b) \widehat{p}(u^\gamma)\| \\
&\leq \|\gamma R(\gamma, \overline{\Upsilon}_0^b)(\widehat{p}(u^\gamma) - w + w)\| \\
&\leq \|\gamma R(\gamma, \overline{\Upsilon}_0^b)(w)\| + \|\gamma R(\gamma, \overline{\Upsilon}_0^b)\| \|\widehat{p}(u^\gamma) - w\|
\end{aligned}$$

$$\leq \|\gamma R(\gamma, \overline{\Upsilon}_0^b)w\| + \|\widehat{p}(u^\gamma) - w\|.$$

It follows from assumption (H0) and the estimation (4.3) that $\|u^\gamma(b) - u_b\| \to 0$ as $\gamma \to 0^+$. This proves the approximate controllability of (4.1) with the conditions (1.2) and (1.3). □

## 5. Example

In this section, we shall give an example delineate our outcomes. In order to apply our abstract results, we need introduce some technical preliminaries. In the sequel, $\mathbb{X} = L^2([0,\pi])$, $\mathscr{D}(\mathscr{A}) = \{x \in \mathbb{X} : x'' \in \mathbb{X}, x(0) = x(\pi) = 0\}$ and $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subseteq \mathbb{X} \to \mathbb{X}$ is the linear operator defined by $\mathscr{A}x = x''$. It is well-known that $\mathscr{A}$ is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ on $\mathbb{X}$. Furthermore, $\mathscr{A}$ has a discrete spectrum, the eigenvalues are $-n^2$, for $n \in \mathbb{N}$, with corresponding eigenvectors $z_n(\tau) = \left(\frac{2}{\pi}\right)^{1/2}\sin(n\tau)$, and the properties (a)-(c) mentioned in [12] holds.

Consider the following impulsive partial functional integro-differential equation of the form:

$$\begin{aligned}\frac{\partial^2}{\partial t^2}z(t,x) &= \frac{\partial^2}{\partial x^2}z(t,x) + a_1(t)z(\sin t, x) + a_2(t)\sin z(t,x)\\ &\quad + \frac{1}{1+t^2}\int_0^t a_3(s)z(\sin s, x)ds\\ &\quad + \widetilde{a}_1(t)z(\sin t, x) + \widetilde{a}_2(t)\sin z(t,x) + \frac{1}{1+t^2}\int_0^t \widetilde{a}_3(s)z(\sin s, x)ds,\end{aligned} \tag{5.1}$$

$$\begin{aligned}\Delta z(t_k, x) &= \int_0^\pi p_k(x,y)z(t_k,y)dy \quad \text{and}\\ \Delta z'(t_k, x) &= \int_0^\pi \widetilde{p}_k(x,y)z(t_k,y)dy,\ k = 1,\dots,m,\end{aligned} \tag{5.2}$$

$$\begin{aligned}&z(t,0) = z(t,\pi) = 0; \quad z(0,x) = z_0(x);\\ &z_t(0,x) = z_1(x), \quad t \in \mathscr{J} = [0,1],\ 0 \leq x \leq \pi,\end{aligned} \tag{5.3}$$

$$\begin{aligned}z(0,x) &= z_0(x) + \sum_{k=1}^m \phi_k z(t_k,x), \text{ and}\\ z_t(0,x) &= z_1(x) + \sum_{k=1}^m \widetilde{\phi}_k z(t_k,x),\ 0 \leq x \leq \pi,\end{aligned} \tag{5.4}$$

where we assume the following conditions:

(a) the functions $a_i(\cdot)$ and $\widetilde{a}_i(\cdot)$, $i = 1,2,3$, are continuous on $[0,1]$, $n_i = \sup_{0\leq s\leq 1}|a_i(s)| < 1$, $i = 1,2,3$; and $\widetilde{n}_i = \sup_{0\leq s\leq 1}|\widetilde{a}_i(s)| < 1$, $i = 1,2,3$.

(b) the functions $p_k$, $\widetilde{p}_k : [0,\pi]\times[0,\pi] \to \mathbb{R}$, $k = 1, 1, \ldots, m$, are continuously differentiable and

$$\gamma_k = \left(\int_0^\pi \int_0^\pi \left(\frac{\partial}{\partial x} p_k(x,y)\right)^2 dxdy\right)^{\frac{1}{2}} < \infty, \quad \text{and}$$
$$\widetilde{\gamma}_k = \left(\int_0^\pi \int_0^\pi \left(\frac{\partial}{\partial x} \widetilde{p}_k(x,y)\right)^2 dxdy\right)^{\frac{1}{2}} < \infty$$

for every $k = 1, 2, \ldots, m$.

(c) The functions $\phi_k, \widetilde{\phi}_k \in \mathbb{R}$, $k = 1, 2, \ldots, m$.

To treat this system, we define the operators respectively $\mathscr{F} : \mathscr{J}\times\mathbb{X}\times\mathbb{X} \to \mathbb{X}, \mathscr{G} : \mathscr{J}\times\mathbb{X}\times\mathbb{X} \to \mathbb{X}, k_1 : \mathscr{J}\times\mathscr{J}\times\mathbb{X} \to \mathbb{X}, k_2 : \mathscr{J}\times\mathscr{J}\times\mathbb{X} \to \mathbb{X}$, $I_k, \overline{I}_k : \mathbb{X} \to \mathbb{X}$, $k = 1, 2, \ldots, m; q, \widetilde{q} : \mathcal{PC}(\mathscr{J}, \mathbb{X}) \to \mathbb{X}$ by

$$\mathscr{F}\left(t, z(\xi(t)), \int_0^t k_1(t,s,z(\xi(s)))ds\right)(x)$$
$$= a_1(t)z(\sin t, x) + a_2(t)\sin z(t,x) + \frac{1}{1+t^2}\int_0^t a_3(s)z(\sin s, x)ds,$$
$$\mathscr{G}\left(t, z(\zeta(t)), \int_0^t k_2(t,s,z(\zeta(s)))ds\right)(x)$$
$$= \widetilde{a}_1(t)z(\sin t, x) + \widetilde{a}_2(t)\sin z(t,x) + \frac{1}{1+t^2}\int_0^t \widetilde{a}_3(s)z(\sin s, x)ds,$$

$$\int_0^t k_1(t,s,\xi(z(s)))(x)ds = \frac{1}{1+t^2}\int_0^t a_3(s)z(\sin s, x)ds,$$
$$\int_0^t k_2(t,s,\zeta(z(s)))(x)ds = \frac{1}{1+t^2}\int_0^t \widetilde{a}_3(s)z(\sin s, x)ds,$$
$$I_k(z)(x) = \int_0^\pi p_k(x,y)z(t_k,y)dy, \ k = 1, \ldots, m,$$
$$\overline{I}_k(z)(x) = \int_0^\pi \widetilde{p}_k(x,y)z(t_k,y)dy, \ k = 1, \ldots, m,$$
$$q(z)(x) = \sum_{k=1}^m \phi_k z(t_k, x),$$

and

$$\widetilde{q}(z)(x) = \sum_{k=1}^m \widetilde{\phi}_k z(t_k, x).$$

Then equations (5.1)-(5.4) takes the abstract form (1.1)-(1.3). It is simple to view that with the selections of the above functions, conditions (H1)-(H6) of Theorem 3.1 are fulfilled. Hence by Theorem 3.1, we consider that nonlocal impulsive Cauchy problem (5.1)-(5.4) has a mild solution on $\mathscr{J}$.

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