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REAL POLYHEDRAL PRODUCTS, MOORE'S CONJECTURE, AND SIMPLICIAL ACTIONS ON REAL TORIC SPACES

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Abstract. The real moment-angle complex (or, more generally, real polyhedral product) and its real toric space have recently attracted much attention in toric topology. The aim of this paper is to give two interesting remarks regarding real polyhedral products and real toric spaces. That is, we first show that Moore's conjecture holds to be true for certain real polyhedral products. In general, real polyhedral products show some drastic difference between the rational and torsion homotopy groups. Our result shows that at least in terms of the homotopy exponent at a prime this is not the case for real polyhedral products associated to a simplicial complex whose minimal missing faces are all k-simplices with $k \geq 2$. Moreover, we also show a structural theorem for a finite group G acting simplicially on the real toric space. In other words, we show that G always contains an element of order 2, and so the order of G should be even.

1. Introduction

Our main concern in this paper is the real moment-angle complex (or, more generally, real polyhedral product) and its real toric space which have recently attracted much attention in toric topology (see, e.g., [3], [4], and [8]). The real toric space is a notion which is a generalization of a real toric variety in algebraic geometry, and is given by a certain quotient space of the real moment-angle complex.

A space X is said to have a *homotopy exponent* n of X at a prime p if n is the least positive integer such that p^n annihilates the p-torsion in the homotopy groups $\pi_*(X)$. A conjecture of Moore claims in [7] and [15] that a finite simply-connected CW-complex X has a homotopy exponent at p if and only if the dimension of $\pi_*(X) \otimes \mathbb{Q}$ is finite, i.e., X is *elliptic*. On the other hand, a finite simply-connected CW-complex is said to be *hyperbolic* if it has infinitely many rational homotopy groups. In this paper, however, we do not restrict the CW-complex to be simply-connected, and even in this case the

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above conjecture will be said to be a conjecture of Moore, following the paper [11].

In order to explain our first result more precisely, we need to set up some notation and definitions. To do so, let K be a simplicial complex on the vertex set $[m] := \{1, 2, \dots, m\}$. For each $1 \le i \le m$, let (X_i, A_i) be a pair of pointed CW-complexes such that A_i is a pointed subspace of X_i. Let us denote by $(\underline{X}, \underline{A})$ the sequence $\{(X_i, A_i)\}_{i=1}^m$ of CW-pairs (X_i, A_i). For each face $\sigma \in K$, let $(\underline{X}, \underline{A})^\sigma$ be the subspace of $\prod_{i=1}^m X_i$ defined by

$$(\underline{X}, \underline{A})^\sigma = \prod_{i=1}^m Y_i,$$

where $Y_i = X_i$, if $i \in \sigma$ and $Y_i = A_i$, otherwise.

With these understood, the *polyhedral product* $(\underline{X}, \underline{A})$ determined by $(\underline{X}, \underline{A})$ and K is defined by

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \subset \prod_{i=1}^m X_i.$$

In particular, when each pair (X_i, A_i) is the same as (D^2, S^1) (resp. (D^{n_i}, S^{n_i-1}) with $n_i \ge 2$), the polyhedral product $(\underline{X}, \underline{A})^K$ is called a *moment-angle complex* $\mathcal{Z}_K(D^2, S^1)$ (resp. *generalized moment-angle complex* $\mathcal{Z}_K(D^{n_i}, S^{n_i-1})$). On the other hand, when each pair (X_i, A_i) is the same as (D^1, S^0), $(\underline{X}, \underline{A})^K$ is called a *real moment-angle complex* (or *real polyhedral product*) $\mathbb{R}\mathcal{Z}_K(D^1, S^0)$. Refer to [1], [2], [3], and [4] for more details on polyhedral products.

Recently, Hao, Sun, and Theriault proved in [11] that Moore's conjecture holds to be true for polyhedral products $(\underline{X}, \underline{A})^K$ with $(X_i, A_i) = (D^{n_i}, S^{n_i-1})$, $n_i \ge 2$. The first aim of this paper is to point out that the conjecture still holds to be true for certain real polyhedral products $(\underline{X}, \underline{A})^K$ with $(X_i, A_i) = (D^1, S^0)$. To be precise, our first main result is:

Theorem 1.1. *Let K be a simplicial complex on the vertex set $[m]$ such that all minimal missing faces are at least k-simplices with $k \ge 2$. Let $(\underline{X}, \underline{A})$ be a sequence of pairs (D^1, S^0). Then Moore's conjecture is true for the real polyhedral product $(\underline{X}, \underline{A})^K$, i.e., $(\underline{X}, \underline{A})^K$ is elliptic if and only if it has a finite homotopy exponent at every prime. Furthermore, $(\underline{X}, \underline{A})^K$ is elliptic if and only if all missing faces of K are mutually disjoint.*

Remark 1.2. The proof of Theorem 1.1 given in Section 3 actually shows that for a simplicial complex K having two minimal missing faces that intersect and are k-simplices for $k \ge 2$ the real polyhedral product $(\underline{X}, \underline{A})^K$ has no homotopy exponent at any prime.

In general, it is believed that real polyhedral products have very complicated torsion in homotopy and homology groups. To the contrary, Theorem 1.1 shows that at least real polyhedral products whose minimal missing faces are k-simplices with $k \ge 2$ do not have so much complicated torsion in their homotopy groups. Even though the proof of Theorem 1.1 is an extension of the arguments in [11], in this sense our main result (Theorem 1.1) seems to be interesting and remarkable. In order to make this paper concise as much as we can, however, we want to emphasize only the necessary points different from [11], so that much of the details will be left (refer to the papers [11] and [9] for more details).

Recall that the real toric variety is given by the fixed point set of a toric variety under the involution defined by the complex conjugation. By definition, any finite subgroup G of the permutation group acting simplicially on a simplicial complex K naturally acts on the real moment-angle complex associated to K, and it further induces an action on the real toric space under a certain invariance condition of G. The second aim of this paper is to show a structure theorem of such a finite group G acting simplicially on the real toric space (see Section 4 for more details).

In order to explain our second main result more precisely, as above let K be a simplicial complex on the vertex set $[m]$, and let D^1 and S^0 denote the closed interval $[-1,1]$ and its boundary $\partial D^1 = \{-1,1\}$, respectively. Then the real moment-angle complex $\mathbb{R}\mathcal{Z}_K(S^1,S^0)$ (or, simply denoted $\mathbb{R}\mathcal{Z}_K$) of K can be also given by

$$\begin{aligned}\mathbb{R}\mathcal{Z}_K &:= (\underline{D}^1, \underline{S}^0)^K \\ &= \bigcup_{\sigma\in K} \{(x_1,x_2,\dots,x_m)\in (D^1)^m \mid x_i \in S^0 \text{ for } i\notin\sigma\}.\end{aligned}$$

Let $\mathbb{Z}_2$ denote the vector space $\{0,1\}$ over $\mathbb{Z}_2$ under the natural multiplication. Then $\mathbb{Z}_2^m$ as a group acts on $(D^1)^m$ diagonally by the sign, and in turn it induces the action on the real moment-angle complex $\mathbb{R}\mathcal{Z}_K$. Let $\Lambda : \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ be a linear map with $m > n$. Then the *real toric space* associated to the pair (K,Λ) is defined by the quotient space $\mathbb{R}\mathcal{Z}_K/\ker\Lambda$. It turns out that the action of $\ker\Lambda$ on $\mathbb{R}\mathcal{Z}_K$ is free if and only if Λ satisfies a certain regularity condition.

Now, let G be a finite subgroup of the permutation group S_m on m letters. We say that G *acts simplicially* on K if G acts on the vertex set $[m]$ of K as a subgroup of S_m in such a way that it preserves the simplices of K. Clearly any simplicial action of G induces an action on the real moment-angle complex $\mathbb{R}\mathcal{Z}_K$. Moreover, such a G-action on $\mathbb{R}\mathcal{Z}_K$ induces an action on the quotient space $\mathbb{R}\mathcal{Z}_K/\ker\Lambda$, whenever the kernel $\ker\Lambda$ of Λ is *invariant under* G in that for any $z = (z_1,z_2,\dots,z_m)\in\ker\Lambda$ and $g\in G$, we have

$$g\cdot z := (z_{g(1)}, z_{g(2)}, \dots, z_{g(m)}) \in \ker\Lambda.$$

With these understood, our second main result is to show the following theorem for the structure of the finite group G acting simplicially on the real toric space.

Theorem 1.3. *Let K be a simplicial complex of dimension $n-1$ on the vertex set $[m]$ with $m > n$ and let G act simplicially on K as a subgroup of the permutation group on m letters. Assume that the kernel $\ker\Lambda$ of a linear map*

$\Lambda : \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ is invariant under G. Then the action of G on K induces an action on the real toric space. Moreover, G always contains an element of order 2, and thus the order of G should be even.

The first statement of Theorem 1.3 is a result given in [5, Theorem 2.3], while the second statement is a new observation.

One wide class of examples supporting the validity of Theorem 1.3 can be provided with the real toric varieties associated to the Weyl chambers of classical groups. In these cases, the simplicial complexes are the Coxeter complexes of type R and the Weyl groups play the role of the finite groups G which preserve the kernel of a characteristic map (see [5, Section 3]). Note that all of the Weyl groups of classical groups always have an even order (see [12]).

As an immediate consequence of Theorem 1.3, we have the following corollary.

Corollary 1.4. *Let K be a simplicial complex of dimension $n-1$ on the vertex set $[m]$ with $m > n$, and let G act simplicially on K as a subgroup of the permutation group on m letters. If the order of G is odd, then G cannot act simplicially on K in such a way that the kernel $\ker\Lambda$ of a linear map $\Lambda : \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ is invariant under G.*

We organize this paper, as follows. In Section 2, we set up some notation and basic facts necessary for the proof of Theorem 1.1. Section 3 is devoted to giving a proof of Theorem 1.1. In Section 4, we briefly summarize some notation and basic facts necessary for the proof of Theorem 1.3 given in Section 5. Section 5 is devoted to giving a proof of Theorem 1.3. To do so, we first establish a nontrivial homomorphism Ψ from the finite group G to the endomorphism group $\mathrm{End}(\ker\Lambda, \ker\Gamma)$. With this homomorphism Ψ in place, the proof of Theorem 1.3 immediately follows, as we can see in Section 5.

2. Simplicial complexes and polyhedral products

The aim of this section is to collect some basic and important facts necessary for the proof of Theorem 1.1.

2.1. Simplicial complexes and minimal missing faces

As before, let K be a simplicial complex on the vertex set $[m]$. An element σ of K is called a *simplex* and $\partial\sigma$ denotes the boundary of σ. In this paper, we use the notation Δ^{m-1} to denote the standard $(m-1)$-simplex on the vertex set $[m]$. Clearly K is always a subcomplex of Δ^{m-1}. A face $\sigma \in \Delta^{m-1}$ is called a *missing face* of K if $\sigma \notin K$, while a missing face σ is called a *minimal missing face* of K if any proper face of σ is a face of K. Refer to [11] for more details.

2.2. Some fuctoriality properties of polyhedral products

This subsection is largely taken from [9, Subsection 2.2].

To begin with, we need the notion of a full subcomplex of a given simplicial complex. To be precise, as before let K be a simplicial complex on the vertex set $[m]$, and let I be a subset of $[m]$. A *full subcomplex* K_I of K is a simplicial complex such that every simplex of K on the vertices in I is also a simplex of K_I. It follows from the definition that the inclusion of K_I into K is a simplicial map, and this induces a map of polyhedral products from $(\underline{X},\underline{A})^{K_I}$ into $(\underline{X},\underline{A})^K$. Moreover, the following lemma ([9, Lemma 2.2.3] or [11, Proposition 2.2]) holds.

Lemma 2.1. *Let K be a simplicial complex on the vertex set $[m]$, and let $(\underline{X},\underline{A})$ be any sequence of pointed, not necessarily path-connected, CW-pairs (X_i,A_i) with a base point $*\in A_i$. Let j denote the inclusion of a full subcomplex K_I into K for $I\subset[m]$. Then the following statements hold:*

(1) *The canonical projection $\prod_{i\in[m]}X_i\to\prod_{k\in I}X_k$ restricts to a surjective map*
$$\mathcal{Z}^j:(\underline{X},\underline{A})^K\to(\underline{X},\underline{A})^{K_I}.$$
(2) *The map $\hat{j}:\prod_{k\in I}X_k\to\prod_{i\in[m]}X_i$ defined by $\hat{j}(x)_k=x_i$, if $j(i)=k$, and $\hat{j}(x)_k=*$, otherwise, restricts to an injective map*
$$\mathcal{Z}_{j,*}:(\underline{X},\underline{A})^{K_I}\to(\underline{X},\underline{A})^K.$$
Here $\hat{j}(x)_k$ denotes the j-th component of $\hat{j}(x)$ in $\prod_{i\in[m]}X_i$.

(3) *We have the identity*
$$\mathcal{Z}^j\circ\mathcal{Z}_{j,*}=\mathrm{Id}.$$
That is, the inclusion map $\mathcal{Z}_{j,*}$ has a left inverse.

It is important to note that we do need the path-connectedness of pointed CW-pairs in Lemma 2.1, contrary to [11, Proposition 2.2].

For a pointed space Y with a base point $*$, the *reduced cone* CY of Y is defined by the quotient space
$$CY=(Y\times[0,1])/\sim,$$
where $(y,1)\sim(y',1)$ for all $y,y'\in Y$, and $(*,t)\sim(*,0)$ for all $t\in[0,1]$. Then Y can be regarded as a subspace of CY by mapping $y\in Y$ to $(y,0)\in CY$. The following theorem ([11, Theorem 2.3]) holds.

Theorem 2.2. *Let K be a simplicial complex on the vertex set $[m]$, and let $(\underline{X},\underline{A})$ be any sequence of pointed, not necessarily path-connected, CW-pairs. For each $1\le i\le m$, let Y_i denote the homotopy fiber of the inclusion of A_i into X_i, and let CY_i denote the reduced cone of Y_i. Then there is a homotopy fibration*
$$(\underline{CY},\underline{Y})^K\to(\underline{X},\underline{A})^K\to\prod_{i\in[m]}X_i.$$
Moreover, there is a homotopy equivalence
$$\Omega(\underline{X},\underline{A})^K\simeq\prod_{i\in[m]}\Omega X_i\times\Omega(\underline{CY},\underline{Y})^K.$$

Proof. The proof follows immediately from that of [11, Theorem 2.3]. Indeed, since by Lemma 2.1 there is a homotopy left inverse of

$$X_i = (\underline{X}, \underline{A})^{K_{\{i\}}} \to (\underline{X}, \underline{A})^K, \quad 1 \le i \le m,$$

after the looping ΩX_i and their multiplication $\prod_{i\in[m]} \Omega X_i$ we can obtain a homotopy right inverse

$$\prod_{i=1}^{m} \Omega X_i \to \Omega(\underline{X}, \underline{A})^K$$

of the map

$$\Omega(\underline{X}, \underline{A})^K \to \prod_{i=1}^{m} \Omega X_i.$$

Hence, by using the loop multiplication of $\Omega(\underline{X}, \underline{A})^K$ and the fact that the homotopy fiber of $(\underline{X}, \underline{A})^K \to \prod_{i\in[m]} X_i$ is $(\underline{CY}, \underline{Y})^K$ we can see that the composite map

$$\prod_{i=1}^{m} \Omega X_i \times \Omega(\underline{CY}, \underline{Y})^K \to \Omega(\underline{X}, \underline{A})^K \times \Omega(\underline{X}, \underline{A})^K \to \Omega(\underline{X}, \underline{A})^K,$$

is a homotopy equivalence, as desired. □

The following lemma ([11, Lemma 2.5]) originally from [10, Proposition 2.3]) was stated under the condition that each Y_i was path-connected. But, in fact, it is obvious to see that there is no need to assume the path-connectedness of each Y_i (see [10, Proposition 2.3] for more details).

Lemma 2.3. *Let Δ^k denote the k-simplex and $\partial\Delta^k$ its boundary complex. Let Y_i be a, not necessarily path-connected, space for $1 \le i \le k+1$. Then the following identities hold*:

(1) $(\underline{CY}, \underline{Y})^{\Delta^k} = \prod_{i\in[k+1]} CY_i$.
(2) $(\underline{CY}, \underline{Y})^{\partial\Delta^k}$ *is homotopy equivalent to* $\Sigma^k Y_1 \wedge \cdots \wedge Y_{k+1}$.

3. Proof of Theorem 1.1

For the proof, we assume that each pair (X_i, A_i) is (D^1, S^0). As in the case of $(X_i, A_i) = (D^n, S^{n-1})$ with $n \ge 2$, the homotopy fiber Y_i of the inclusion of S^0 into D^1 is S^0. So the pair (CY_i, Y_i) is just (D^1, S^0). Note also that, since X_i is equal to D^1, the product $\prod_{i\in[m]} \Omega X_i$ is contractible. Moreover, it follows from Lemma 2.3 that for a minimal missing k-face σ of K $(\underline{CY}, \underline{Y})^{\partial\sigma} = (\underline{D^1}, \underline{S^0})^{\partial\sigma}$ is homotopy equivalent to

$$\Sigma^k Y_1 \wedge Y_2 \wedge \cdots \wedge Y_{k+1} = \Sigma^k (S^0)^{\wedge k+1}. \tag{3.1}$$

The following lemma plays an important role in the proof of Theorem 1.1.

Lemma 3.1. *For a k-simplex σ of K,*
$$(\underline{CY}, \underline{Y})^{\partial\sigma} = (\underline{D^1}, \underline{S^0})^{\partial\sigma}$$
is homotopy equivalent to a sphere S^k.

Proof. Observe that $\Sigma S^0 \wedge S^0 \simeq S^1 \simeq \Sigma S^0$. Iterating this gives
$$\Sigma (S^0)^{\wedge k+1} \simeq \Sigma S^0.$$
As already noticed in (3.1), we have
$$(\underline{D^1}, \underline{S^0})^{\partial\sigma} \simeq \Sigma^k (S^0)^{\wedge k+1}.$$
Thus we can obtain
$$(\underline{D^1}, \underline{S^0})^{\partial\sigma} \simeq \Sigma^k S^0 \simeq S^k,$$
as desired. □

As a consequence of Lemma 3.1, if k is greater than or equal to 2, then $(\underline{CY}, \underline{Y})^{\partial\sigma}$ is homotopy equivalent to a simply-connected sphere S^k.

We also need the following Theorem 4.2 in [11] originally stated under the assumption that each pair (X_i, A_i) is a pointed, path-connected, CW-pair. We remark that Theorem 4.2 in [11] continues to hold without the extra assumption that the CW-pair is path-connected.

Theorem 3.2. *Let K be a simplicial complex on the vertex set $[m]$, and let $(\underline{X}, \underline{A})$ be any sequence of pointed, not necessarily path-connected, CW-pairs. The following statements hold*:

(1) *Assume that the collection of all minimal missing faces consists of σ_1, σ_2, …, σ_n and that all of σ_i's are mutually disjoint. Then we have the following homotopy equivalence*
$$\Omega(\underline{X}, \underline{A})^K \simeq \prod_{i\in[m]} \Omega X_i \times \prod_{j=1}^{n} \Omega(\underline{CY}, \underline{Y})^{\partial\sigma_j}.$$

(2) *Assume that two minimal missing faces σ_1 and σ_2 have a non-trivial intersection. Then $\Omega((\underline{CY}, \underline{Y})^{\partial\sigma_1} \vee (\underline{CY}, \underline{Y})^{\partial\sigma_2})$ is a retract of $\Omega(\underline{CY}, \underline{Y})^K$.*

Finally, we are ready to prove Theorem 1.1 which is closely adapted from the proof of [11, Theorem 1.1]. For the sake of reader's convenience, we provide a detailed proof.

Proof of Theorem 1.1. For the proof, we first recall that all minimal missing faces σ_i are assumed to be k-simplices with $k \geq 2$. So it follows from Lemma 3.1 that $(\underline{CY}, \underline{Y})^{\partial\sigma_i}$ is homotopy equivalent to a simply-connected sphere S^k.

Assume now that two minimal missing faces σ_1 and σ_2 have a non-trivial intersection. Then it follows from Lemma 3.1 that $(\underline{CY}, \underline{Y})^{\partial\sigma_1} \vee (\underline{CY}, \underline{Y})^{\partial\sigma_2}$ is homotopy equivalent to a wedge of two simply-connected spheres. Note also that by Theorem 3.2 (2) its loop space
$$\Omega((\underline{CY}, \underline{Y})^{\partial\sigma_1} \vee (\underline{CY}, \underline{Y})^{\partial\sigma_2})$$

is a retract of $\Omega(\underline{X}, \underline{A})^K$, so that a wedge of two simply-connected spheres is homotopy equivalent to $(\underline{X}, \underline{A})^K$. But a wedge of two simply-connected spheres is known to be hyperbolic by a theorem of Hilton and Milnor. Moreover, such a wedge is shown to have no homotopy exponent at any prime by Neisendorfer and Selick in [14].

Next, we assume that all the minimal missing faces σ_j, $1 \leq j \leq k$, of K are mutually disjoint. By Lemma 2.3 (2), $(\underline{CY}, \underline{Y})^{\partial\sigma_i}$ is homotopy equivalent to a simply-connected sphere. Thus

$$\prod_{j=1}^{k} (\underline{CY}, \underline{Y})^{\partial\sigma_j}$$

is homotopy equivalent to a finite product of simply-connected spheres which is known to be elliptic. Since each $\Omega X_i = \Omega D^1$ is contractible for $1 \leq i \leq m$, by Theorem 3.2(1) $\Omega(\underline{X}, \underline{A})^K$ is now homotopy equivalent to

$$\prod_{j=1}^{k} \Omega(\underline{CY}, \underline{Y})^{\partial\sigma_j}.$$

Hence $(\underline{X}, \underline{A})^K$ is homotopy equivalent to a finite product of simply-connected spheres. Note that every simply-connected sphere has a homotopy exponent at every prime, so that a finite product of simply-connected spheres also has a homotopy exponent at every prime.

Therefore, we can conclude that the real polyhedral product $(\underline{X}, \underline{A})^K$ for a simplicial complex K as in Theorem 1.1 is elliptic if and only if it has a homotopy exponent at every prime. This completes the proof of Theorem 1.1. □

4. Real moment-angle complexes and real toric spaces

The aim of this section is to quickly review some basic material regarding moment-angle complexes and real toric spaces. Refer to [1] and [2] for more details on real moment-angle complexes.

Recall that the real moment-angle complex $\mathbb{R}\mathcal{Z}_K$ of K is given by

$$\begin{aligned} \mathbb{R}\mathcal{Z}_K := (\underline{D}^1, \underline{S}^0)^K &= \bigcup_{\sigma \in K} (\underline{D}^1, \underline{S}^0)^\sigma \\ &= \bigcup_{\sigma \in K} \{(x_1, x_2, \ldots, x_m) \in (D^1)^m \mid x_i \in S^0 \text{ for } i \notin \sigma\}. \end{aligned}$$

It can be shown as in [3, Lemma 6.13] that $\mathbb{R}\mathcal{Z}_K$ is a topological manifold if K is a simplicial sphere.

Let $\mathbb{Z}_2$ denote the vector space $\{0, 1\}$ over $\mathbb{Z}_2$ under the natural multiplication, and let $\Lambda : \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ be a linear map. Note that $\mathbb{Z}_2^m$ acts on $(D^1)^m$

diagonally, as follows.

$$\begin{aligned} &\mathbb{Z}_2^m \times (D^1)^m \to (D^1)^m \\ &((z_1,\ldots,z_m),(x_1,\ldots,x_m)) \mapsto ((-1)^{z_1}x_1,\ldots,(-1)^{z_m}x_m), \end{aligned} \tag{4.1}$$

so that in turn it induces an action on the real moment-angle complex $\mathbb{R}\mathcal{Z}_K$. Clearly any subgroup of $\mathbb{Z}_2^m$ can be realized as the kernel $\ker\Lambda$ of a linear map $\Lambda : \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ for $m > n$, and the quotient space $\mathbb{R}\mathcal{Z}_K/\ker\Lambda$ is called the *real toric space* associated to the pair (K,Λ). It turns out that the action of $\ker\Lambda$ on $\mathbb{R}\mathcal{Z}_K$ is free if and only if Λ satisfies the regularity condition as below (see, e.g., [6, Lemma 1.1]).

A real toric space is a generalization of a more specific space which is defined as the quotient space $\mathbb{R}\mathcal{Z}_K/\ker\Lambda$ for a characteristic map Λ. Recall that a linear map $\Lambda : \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ is called a *characteristic map* if it satisfies a certain *regularity condition*. That is, when we write the linear map Λ as an $n \times m$-matrix with the same notation

$$\Lambda = (\lambda(1)\lambda(2)\cdots\lambda(m-1)\lambda(m))_{n\times m}, \tag{4.2}$$

$\lambda(i_1)$, $\lambda(i_2)$, …, $\lambda(i_k)$ are linearly independent over $\mathbb{Z}_2$ for any simplex $\{i_1, i_2, \ldots, i_k\}$ in K. Here each $\lambda(i)$ is regarded as a column vector of size n, and the matrix in (4.2) is called the *characteristic matrix*.

For a characteristic map Λ, the quotient space $\mathbb{R}\mathcal{Z}_K/\ker\Lambda$ is called a *small cover* (resp. *real topological toric manifold*) if K is a polytopal sphere (resp. star-shaped sphere). Refer to [8] and [13] for more details. Note also that any small cover is in turn a generalization of a real toric variety which is given by the fixed point set of a toric variety under the natural involution defined by the complex conjugation.

Now, as before let G be a finite subgroup of the permutation group S_m on m letters. Recall that G acts simplicially on K if G acts on the vertex set $[m]$ of K as a subgroup of S_m in such a way that it preserves the simplices of K. It is easy to see that any simplicial action of G induces an action on the real moment-angle complex $\mathbb{R}\mathcal{Z}_K$. Moreover, it induces an action on the homology group

$$H_*(\mathbb{R}\mathcal{Z}_K;\mathbb{Q}) \cong \bigoplus_{S\in \mathrm{Row}(\Lambda)} \tilde{H}_{*-1}(K_S;\mathbb{Q}),$$

where $\mathrm{Row}(\Lambda) \subset \mathbb{Z}_2^m$ denotes the row space of Λ and K_S denote the full subcomplex of K with its vertex set $S \subseteq [m]$. Here, we identify any element S of $\mathrm{Row}(\Lambda)$ with an element I_S of $[m]$ in the natural way. That is, for $S = (s_1, s_2, \ldots, s_m) \in \mathrm{Row}(\Lambda)$, we set

$$I_S = \{i \in [m] \mid s_i \neq 0\}.$$

The following lemma and its proof plays an important role in the proof of our main Theorem 1.3 (see [5, Theorem 2.3]).

Lemma 4.1. *Assume that* $\ker\Lambda$ *is invariant under* G *in the sense that for any* $z = (z_1, z_2, \ldots, z_m) \in \ker\Lambda$ *and* $g \in G$*, we have*

$$g \cdot z = (z_{g(1)}, z_{g(2)}, \ldots, z_{g(m)}) \in \ker\Lambda.$$

Then the action of G *on* $\mathbb{R}\mathcal{Z}_K$ *induces one on the real toric space* $\mathbb{R}\mathcal{Z}_K/\ker\Lambda$.

Proof. Let x and y be any two elements of $\mathbb{R}\mathcal{Z}_K$ such that $x = z \cdot y$ for some $z = (z_1, z_2, \ldots, z_m) \in \ker\Lambda$. Then by (4.1) we have

$$\begin{aligned} g \cdot x = g \cdot (z \cdot y) &= g \cdot ((-1)^{z_1} y_1, \ldots, (-1)^{z_m} y_m) \\ &= ((-1)^{z_{g(1)}} y_{g(1)}, \ldots, (-1)^{z_{g(m)}} y_{g(m)}) \\ &= ((-1)^{z'_1} y_{g(1)}, \ldots, (-1)^{z'_m} y_{g(m)}), \end{aligned} \tag{4.3}$$

where $z'_i = z_{g(i)}$ for each $i \in [m]$. By assumption, note that

$$z' := (z'_1, z'_2, \ldots, z'_m)$$

is an element of $\ker\Lambda$.

On the other hand, it is also easy to obtain

$$\begin{aligned} z' \cdot (g \cdot y) &= z' \cdot (y_{g(1)}, y_{g(2)}, \ldots, y_{g(m)}) \\ &= ((-1)^{z'_1} y_{g(1)}, (-1)^{z'_2} y_{g(2)}, \ldots, (-1)^{z'_m} y_{g(m)}). \end{aligned} \tag{4.4}$$

Thus it follows from (4.3) and (4.4) that we have $g \cdot x = z' \cdot (g \cdot y)$. This implies that $g \cdot x$ and $g \cdot y$ represent the same element in $\mathbb{R}\mathcal{Z}_K/\ker\Lambda$.

Now, it is straightforward to show that there is a well-defined action of G on $\mathbb{R}\mathcal{Z}_K/\ker\Lambda$, which completes the proof of Lemma 4.1. □

5. Proof of Theorem 1.3

We first need to prove the following proposition.

Proposition 5.1. *Let* K *be a simplicial complex of dimension* $n-1$ *with the vertex set* $[m]$ *and* $m > n$*, and let* G *act simplicially on* K*. Assume that* $\ker\Lambda$ *is invariant under* G*. Then there is a group homomorphism*

$$\Psi : G \to \mathrm{End}(\ker\Lambda, \ker\Lambda).$$

In other words, the invariance of G *on* $\ker\Lambda$ *induces an action on* $\ker\Lambda$.

Proof. Since K is a simplicial complex on the vertex set $[m]$, it is important to note that by definition every singleton $\{i\}$ is an element of K for each $i \in [m]$. For example, if we let $\sigma = \{1\} \in K$, then we have

$$(\underline{D}^1, \underline{S}^0)^\sigma = D^1 \times S^0 \times \cdots \times S^0.$$

Thus we have an element

$$\mathbf{1} = (1, 1, \ldots, 1) \in (\underline{D}^1, \underline{S}^0)^\sigma \subset \mathbb{R}\mathcal{Z}_K.$$

Let G be the finite subgroup of the permutation group S_m on m letters, as before. Then clearly we have $g \cdot \mathbf{1} = \mathbf{1}$ for all $g \in G$. Hence the equivalence class $[\mathbf{1}]$ in $\mathbb{R}\mathcal{Z}_K/\ker\Lambda$ is fixed under the action of G on $\mathbb{R}\mathcal{Z}_K/\ker\Lambda$. One special and important feature of $\mathbf{1}$ we need in this paper is that no component of $\mathbf{1}$ is zero.

Next, we want to construct a homomorphism

$$\Psi : G \to \mathrm{End}(\ker\Lambda, \ker\Lambda),$$

as follows. To do so, note first that from the proof of Lemma 4.1 for any $z \in \ker\Lambda$ and $g \in G$ there is a unique element $z'_g \in \ker\Lambda$ such that

$$g \cdot (z \cdot \mathbf{1}) = z'_g \cdot (g \cdot \mathbf{1}) = z'_g \cdot \mathbf{1},$$

where $z'_g(i)$ is given by $z_{g(i)}$ for each $i \in [m]$, i.e., $z'_g = g \cdot z$. Since $\ker\Lambda$ is assumed to be invariant under G, for each $g \in G$ we can thus define

$$\varphi(g) : \ker\Lambda \to \ker\Lambda, \quad z \mapsto z'_g = g \cdot z.$$

Using the map φ, we now define a map

$$\Psi : G \to \mathrm{End}(\ker\Lambda, \ker\Lambda), \quad g \mapsto \varphi(g).$$

It is easy to see that Ψ is indeed a homomorphism we want. To be precise, for any two elements $g_1, g_2 \in G$ we have

$$(g_1 g_2) \cdot (z \cdot \mathbf{1}) = z'_{g_1 g_2}(g_1 g_2 \cdot \mathbf{1}) = z'_{g_1 g_2} \cdot \mathbf{1}. \tag{5.1}$$

On the other hand, it is also true that

$$\begin{aligned}(g_1 g_2) \cdot (z \cdot \mathbf{1}) &= g_1 \left(g_2 \cdot (z \cdot \mathbf{1})\right) \\ &= g_1 \left(z'_{g_2} \cdot (g_2 \cdot \mathbf{1})\right) = g_1 \left(z'_{g_2} \cdot \mathbf{1}\right) \\ &= (z'_{g_2})'_{g_1} \cdot (g_1 \cdot \mathbf{1}) = (z'_{g_2})'_{g_1} \cdot \mathbf{1}.\end{aligned} \tag{5.2}$$

Note that we have

$$(z'_{g_2})'_{g_1} = g_1 \cdot z'_{g_2} = g_1 \cdot (g_2 \cdot z). \tag{5.3}$$

Thus it follows from (5.1), (5.2), and (5.3) that we have

$$z'_{g_1 g_2} = (z'_{g_2})'_{g_1} = g_1 \cdot (g_2 \cdot z). \tag{5.4}$$

But this immediately implies that the map ψ is a homomorphism. Indeed, by (5.4) we have

$$\begin{aligned}\Psi(g_1 g_2)(z) = \varphi(g_1 g_2)(z) = z'_{g_1 g_2} &\overset{(5.4)}{=} g_1 \cdot (g_2 \cdot z) \\ &= g_1 \cdot (\varphi(g_2)(z)) = g_1 \cdot (\Psi(g_2)(z)) \\ &= \Psi(g_1) \circ \Psi(g_2)(z), \quad z \in \ker\Lambda.\end{aligned}$$

That is, we have

$$\Psi(g_1 g_2) = \Psi(g_1) \circ \Psi(g_2),$$

i.e., Ψ is a homomorphism. This completes the proof of Proposition 5.1. $\square$

We remark that by using the invariance of G on $\ker\Lambda$ it is also possible to directly show that there is a well-defined action of G on $\ker\Lambda$. It is a standard fact that this will then induce a group homomorphism Ψ as in the proof of Proposition 5.1. In view of the proof of Proposition 5.1, the action of G on $\ker\Lambda$ satisfies the property: for each $z \in \ker\Lambda$,

$$\begin{aligned} g_1 \cdot (g_2 \cdot z) &= (z_{g_2(g_1(1))}, z_{g_2(g_1(2))}, \dots z_{g_2(g_1(m))}) \\ &= (z_{g_2g_1(1)}, z_{g_2g_1(2)}, \dots z_{g_2g_1(m)}) \\ &= (g_2g_1) \cdot z, \quad g_1, g_2 \in G. \end{aligned}$$

Now, we are ready to prove Theorem 1.3, as follows.

Proof of Theorem 1.3. By Proposition 5.1, there is a group homomorphism

$$\Psi : G \to \mathrm{End}(\ker\Lambda, \ker\Lambda).$$

Thus, we have an isomorphism

$$G/\ker\Psi \cong \mathrm{Im}\Psi \subset \mathrm{End}(\ker\Lambda, \ker\Lambda).$$

Note that $\ker\Lambda$ contains a subspace of $\mathbb{Z}_2^m$ isomorphic to $\mathbb{Z}_2^{m-n}$ and so $\ker\Lambda$ is isomorphic to $\mathbb{Z}_2^l$ for some $l \geq m-n$. Hence $\mathrm{End}(\ker\Lambda, \ker\Lambda)$ is isomorphic to

$$\mathbb{Z}_2^l \otimes \mathbb{Z}_2^l \cong \mathbb{Z}_2^{l^2}.$$

Since the map $\varphi : G \to \ker\Lambda$ is non-trivial by definition, there should be an element of G whose image under Ψ is non-trivial. This together with the order of $\mathrm{End}(\ker\Lambda, \ker\Lambda)$ equal to 2^{l^2} immediately implies that there exists an element of G whose order is equal to two and that the order of G should be also even as well, as desired. □

Finally, we close this section with one simple example taken from [5, Example 2.4] and [8, Example 1.19], other than real toric varieties associated to the Weyl chambers of classical groups given in Section 1, which illustrates our main result.

Example 5.2. Let K_4 be the 4-gon on the vertex set [4], and let G be the cyclic group of order 4 acting on K_4 cyclically on four vertices. Let Λ be the characteristic map whose associated matrix is given by

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Then it is easy to see that the kernel $\ker\Lambda$ is invariant under the action of G. The real toric space associated to the pair (K_4, Λ) is actually the 2-dimensional torus $T^2 = S^1 \times S^1$ and the induced action of G on T^2 is generated by

$$g : T^2 \to T^2, \quad (x, y) \mapsto (-y, x).$$

At any rate, the order of G is four which is clearly even and contains an element g^2 of order two.

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