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A NOTE ON THE UNITS OF MANTACI-REUTENAUER ALGEBRA

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Abstract. In this paper, we have first presented the construction of the linear characters of a finite Coxeter group G_n of type B_n by lifting all linear characters of the quotient group $G_n/[G_n, G_n]$ of the commutator subgroup $[G_n, G_n]$. Also we show that the sets of distinguished coset representatives D_A and $D_{A'}$ for any two signed compositions A, A' of n which are G_n-conjugate to each other and for each conjugate class $\mathcal{C}_\lambda$ of G_n, where $\lambda \in \mathcal{BP}(n)$, the equality $|\mathcal{C}_\lambda \cap D_A| = |\mathcal{C}_\lambda \cap D_{A'}|$ holds. Finally, we have given the general structure of units of Mantaci-Reutenauer algebra.

1. Introduction

As a convention, throughout this paper, we denote by $\mathcal{MR}(G_n)$, $\mathcal{SC}(n)$ and $\mathcal{BP}(n)$ the Mantaci-Reutenauer algebra, the set of all signed compositions of n and the set of all double partitions of n, respectively.

We assume that G_n is a Coxeter group of type B_n. First of all, we will strictly describe the structure of the commutator subgroup $[G_n, G_n]$ of G_n by using combinatorial properties of G_n. Then we will obtain all of the linear characters of G_n by using lifts of the irreducible characters of the quotient group $G_n/[G_n, G_n]$.

Mantaci-Reutenauer algebra $\mathcal{MR}(G_n)$, that is a subalgebra of the group algebra $\mathbb{Q}G_n$ and contained the classical Solomon's descent algebra of type A_n and B_n, was firstly constructed in [7]. In [3], Bonnafé and Hohlweg have reconstructed this algebra by the methods which depend more on the structure of G_n as a Coxeter group. It is well-known by [3, Proposition 2.9] that if ${}^{w}A = A'$ for $A, A' \in \mathcal{SC}(n)$ and $w \in G_n$, then D_A and $D_{A'} = D_A w^{-1}$ are in general not G_n-conjugate as sets. In [5, Theorem 1.1], Fleischmann has proved that the sets of distinguished coset representatives of any two conjugate standard parabolic subgroups of a given Coxeter group are pointwise conjugate to each other and also he has given an example not verifying the pointwise conjugate statement

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for non-standard parabolic subgroups of a Coxeter group. Although the collection of the reflection subgroups of G_n corresponding to signed compositions of n also contains all standard parabolic subgroups and some non-parabolic subgroups of G_n, the sets of distinguished coset representatives of conjugate reflection subgroups are pointwise conjugate to each other.

In Theorem 3.3, we will also give an effective formula to determine how the structure of units of Mantaci-Reutenauer algebra $\mathcal{MR}(G_n)$. As a result of this, for any signed composition A of n containing both positive and negative components, the corresponding basis element y_A is not invertible in $\mathcal{MR}(G_n)$. Then we shall give an example to illustrate the method developed in Section 3.

2. The commutator subgroup of the Coxeter group G_n of type B_n

Let (G_n, R_n) denote a Coxeter system of type B_n and write its generating set as $R_n = \{t, s_1, \ldots, s_{n-1}\}$. The Coxeter group G_n acts by the permutation on the set $I_n = \{-n, \ldots, -1, 1, \ldots, n\}$ such that for every $i \in I_n$, $w(-i) = -w(i)$. So we have,

$$G_n = \{w \in \mathrm{Perm}(I_n) : \forall i \in I_n,\ w(-i) = -w(i)\}.$$

The Dynkin diagram of (G_n, R_n) is as follows:

$$B_n : \overset{t}{\circ} \Leftarrow \overset{s_1}{\circ} - \overset{s_2}{\circ} - \cdots - \overset{s_{n-1}}{\circ}.$$

For $I \subset R_n$, if G_I is generated by I, then G_I is called a *standard parabolic subgroup* of G_n. If H is a subgroup of G_n conjugate to G_I for some $I \subset R_n$, then we call H a *parabolic subgroup* of G_n. Let $t_0 := t$ and $t_{i+1} := s_{i+1} t_i s_{i+1}$ for each i, $0 \le i \le n-2$. If we put $T_n := \{t_0, t_1, \ldots, t_{n-1}\}$, then the defining relations between the elements of R_n and T_n are stated in such a way that:

(1) $t_i^2 = 1, s_j^2 = 1$ for all $i, j,\ 0 \le i \le n-1,\ 1 \le j \le n-1$;
(2) $(s_1 t)^4 = 1$;
(3) $(s_i s_{i+1})^3 = 1$ for all $i,\ 1 \le i \le n-2$;
(4) $(s_i t)^2 = 1,$ for all $i,\ 1 < i \le n-1$;
(5) $(s_i s_j)^2 = 1$ for $|i - j| \ge 2$;
(6) $(t_i t_j)^2 = 1$ for $0 \le i, j \le n-1$.

We sometimes represent $w \in G_n$ as the word $w(1)w(2)\cdots w(n)$. Denote by $l : G_n \to \mathbb{N}$ the length function attached to R_n and let $l_t : G_n \to \mathbb{N}$ be the function, which assigns to each element w of G_n the number of t appearing in a reduced expression of w. If we denote by $\mathcal{T}_n$ the reflection subgroup of G_n generated by T_n, then $\mathcal{T}_n$ is a normal subgroup of G_n. Now let $R_{-n} = \{s_1, \ldots, s_{n-1}\}$. The reflection subgroup of G_n generated by R_{-n} is represented by G_{-n} and isomorphic to the symmetric group Ξ_n of degree n. Thus $G_n = G_{-n} \ltimes \mathcal{T}_n$. Therefore, we have $|G_n| = 2^n \cdot n!$.

Let $\{e_1, \ldots, e_n\}$ be the standard basis of the real inner product space $\mathbb{R}^n$ over $\mathbb{R}$. Let

$$\Gamma_n^+ = \{e_i : 1 \le i \le n\} \cup \{e_j + \alpha e_i : \alpha \in \{-1, 1\} \text{ and } 1 \le i < j \le n\}.$$

Then $\Gamma_n = \Gamma_n^+ \uplus \Gamma_n^-$ is a root system of G_n. The set $\Pi_n = \{e_1, e_2 - e_1, \ldots, e_n - e_{n-1}\}$ is a simple system of Γ_n. As a set of simple reflections, the generating set R_n of G_n is denoted by $\{s_\alpha : \alpha \in \Pi_n\}$. For further information on the Coxeter groups of type B_n, one may apply [6].

Taking into account the defining relations of G_n, it is well-known by [6] that there is a unique group homomorphism $\varepsilon' : G_n \to \{\pm 1\}$ such that $\varepsilon'(t) = -1$ and $\varepsilon'(s_i) = 1$ for $i = 1, \ldots, n-1$. Note that the function ε' is just one of the linear characters of G_n. If the kernel of ε' is denoted by $G_n^{'}$, then $\ker\varepsilon'$ is a normal subgroup of G_n of index two. From the definition of ε', it follows that an element $w \in G_n$ is contained in $G_n^{'}$ if and only if the number of the reflection t occurring as a factor in a reduced expression of w is even. Let $s_0 := ts_1t \in G_n^{'}$ and let $\mathcal{T}_n^{'} = \mathcal{T}_n \cap G_n^{'}$. Then $\mathcal{T}_n^{'} \lhd G_n^{'}$. We set $v_1 = s_0 s_1$ and $v_i = s_i v_{i-1} s_i$ for any i, $2 \le i \le n-1$. Thus for each $1 \le i \le n-1$ the element v_i is equal to tt_i and so v_i is an element of $\mathcal{T}_n^{'}$. We note here that every element $v_i \in \mathcal{T}_n^{'}$ is a commutator of G_n. It follows that the group $\mathcal{T}_n^{'}$ is a normal subgroup of $\mathcal{T}_n$ of index two and it is generated by the set $\{v_1, \ldots, v_{n-1}\}$. Therefore, the group $G_n^{'}$ is a semidirect product $G_n^{'} = \Xi_n \ltimes \mathcal{T}_n^{'}$. Since $s_0^2 = 1$, the generating set of $G_n^{'}$ is $R^{'} = \{s_0, s_1, \ldots, s_{n-1}\}$. The reflection group $G_n^{'}$ is a Coxeter group of type D_n. Note that the Coxeter relations

$$(s_0 s_1)^2 = 1;\ (s_0 s_2)^3 = 1;\ (s_0 s_i)^2 = 1 \text{ for } i \ge 3$$

constitute a presentation of $G_n^{'}$. The symmetric group Ξ_n is a standard parabolic subgroup of $G_n^{'}$, but the Coxeter group $G_n^{'}$ of type D_n is not a standard parabolic subgroup of G_n. Now let $l^{'}$ denote the length function on $G_n^{'}$. From Lemma 1.4.12(b) of [6], there exists the equality

$$l(w) = l^{'}(w) + l_t(w) \tag{1}$$

for every $w \in G_n^{'}$.

For $x, y \in G_n$, the number of the factor t in a reduced expression of $xyx^{-1}y^{-1}$ in terms of the generating set R_n is even. Therefore, it is easily seen that the commutator subgroup $[G_n, G_n]$ of G_n is also a subgroup of $G_n^{'}$. Moreover, $[G_n, G_n] \lhd G_n^{'}$. In particular, for $\Xi_n \le G_n$ we have $[\Xi_n, \Xi_n] = Alt_n$ and so $Alt_n \le [G_n, G_n]$, where Alt_n stands for alternating subgroup of Ξ_n.

Proposition 2.1. *The commutator subgroup $[G_n, G_n]$ of G_n can be expressed as a semidirect product $[G_n, G_n] = Alt_n \ltimes \mathcal{T}_n^{'}$.*

Proof. Let $sgn : G_n^{'} \to \{\pm 1\}$ be the sign character of $G_n^{'}$. Any element $w \in \ker(sgn)$ can be uniquely written as $w = w_S w_{T'}$ such that $w_S \in \Xi_n$ and $w_{T'} \in \mathcal{T}_n^{'}$. Since the number of the multipliers belong to R_n in a reduced expression of $w_{T'}$ is even, then $\varepsilon_n(w_{T'}) = 1$, where ε_n is the sign character of G_n. Hence by (1), the following equation holds:

$$sgn(w) = (-1)^{l^{'}(w)} = (-1)^{l(w)} = \varepsilon_n(w) = \varepsilon_n(w_S)\varepsilon_n(w_{T'}) = \varepsilon_n(w_S).$$

Thus we obtain that $w \in \ker(sgn)$ if and only if $\varepsilon_n(w_S) = 1$, or equivalently $w_S \in Alt_n$. From this point of view, we have $\ker(sgn) \subset Alt_n \ltimes \mathcal{T}_n^{'}$. It is clear that the reverse inclusion holds. Hence $\ker(sgn) = Alt_n \ltimes \mathcal{T}_n^{'}$. Since every generator of $\mathcal{T}_n^{'}$ is a commutator of G_n and $\mathcal{T}_n^{'}$ is a normal subgroup of $G_n^{'}$, we then get $\mathcal{T}_n^{'} \lhd [G_n, G_n]$. If we consider the fact that the alternating group Alt_n is a subgroup of $[G_n, G_n]$, then we have $Alt_n \ltimes \mathcal{T}_n^{'} \leq [G_n, G_n]$. At the same time, it can be easily seen that the commutator subgroup $[G_n, G_n]$ is a subgroup of $\ker(sgn)$. Hence, the commutator subgroup of G_n is $[G_n, G_n] = Alt_n \ltimes \mathcal{T}_n^{'}$, as required. □

The commutator subgroup $Alt_n \ltimes \mathcal{T}_n^{'}$ of G_n is extremely useful to obtain all the linear characters of the Coxeter group G_n. Since the factor group $G_n/[G_n, G_n]$ is commutative, then all the characters of the factor group are linear, and so irreducible. Likewise, the commutator subgroup $[G_n, G_n]$ is the intersection the kernels of all the linear characters of G_n. Thus, the commutator subgroup of G_n can be obtained by using the character table of G_n. Hence, we get G_n has four linear characters since $|G_n/Alt_n \ltimes \mathcal{T}_n^{'}| = 4$.

We write H for the commutator subgroup $[G_n, G_n]$ of G_n. The factor group G_n/H, the collection of the elements H, s_1H, tH, s_1tH, is Klein 4-group and it is generated by the set $\{s_1H, tH\}$. Therefore, all the characters of G_n/H are as follows:

(1) $\tilde{f}_1(s_1H) = 1,\ \tilde{f}_1(tH) = 1$ (the trivial character of G_n/H);
(2) $\tilde{f}_2(s_1H) = -1,\ \tilde{f}_2(tH) = -1$ (the sign character of G_n/H);
(3) $\tilde{f}_3(s_1H) = 1,\ \tilde{f}_3(tH) = -1$;
(4) $\tilde{f}_4(s_1H) = -1,\ \tilde{f}_4(tH) = 1$.

Hence by lifting to G_n the characters $\tilde{f}_i$, $1 \leq i \leq 4$, we obtain the all linear characters of G_n in the following way:

(1) $f_1(s_i) = 1,\ 1 \leq i \leq n-1,\ f_1(t) = 1$ (the trivial character of G_n);
(2) $f_2(s_i) = -1,\ 1 \leq i \leq n-1,\ f_2(t) = -1$ (the sign character of G_n);
(3) $f_3(s_i) = 1,\ 1 \leq i \leq n-1,\ f_3(t) = -1$;
(4) $f_4(s_i) = -1,\ 1 \leq i \leq n-1,\ f_4(t) = 1$.

There is the relation $f_4 = f_3 \cdot f_2$ between the characters $f_2,\ f_3,\ f_4$. The character f_3 is nothing else but the function $\varepsilon^{'}$ given in the beginning of this section.

3. About the some units of Mantaci-Reutenauer algebra

Now we recall the structure of Mantaci-Reutenauer algebra due to [3]:

For a positive integer n, a *signed composition* of n is an expression of n as a finite sequence $A = (a_1, \ldots, a_k)$ whose each part consists of non-zero integers such that the summation of the absolute values of all parts equals n. We set $|A| = \sum_{i=1}^{k} |a_i|$. In order to denote the set of all signed compositions of n, we use the notation $\mathcal{SC}(n)$. Note that the size of $\mathcal{SC}(n)$ is $2 \cdot 3^{n-1}$. Let

$A = (a_1, \ldots, a_k) \in \mathcal{SC}(n)$. If $a_i > 0$ (resp. $a_i < 0$) for every $i \geq 1$, then A is said to be a *positive* (resp. *negative*) signed composition of n . If $a_i < 0$ for every $i \geq 2$, in this case A is called *parabolic* signed composition of n. Let define $A^+ = (|a_1|, \ldots, |a_r|)$. Then A^+ is a positive signed composition of n. We will denote by $\mathcal{SC}^+(n)$ and $\mathcal{SC}_p(n)$ the set of positive and parabolic signed compositions of n, respectively. A *double partition* $\lambda = (\lambda^+; \lambda^-)$ of n consists of a pair of partitions λ^+ and λ^- such that $|\lambda| = |\lambda^+| + |\lambda^-| = n$. If the length of λ^+ (resp. the length of λ^-) is equal to zero, then we write $\emptyset$ instead of λ^+ (resp. λ^-). We denote the set of all double partitions of n by $\mathcal{BP}(n)$. For a $\lambda = (\lambda^+; \lambda^-)$ double partition of n, $\hat{\lambda}$ denotes the signed composition of n obtained by concatenating λ^+ and $-\lambda^-$, that is, $\hat{\lambda} = \lambda^+ \sqcup -\lambda^-$ is the signed composition obtained by appending the sequence of components of λ^+ to that of $-\lambda^-$ and let $R_n^{'}$ be the set $\{s_1 \cdots s_{n-1}, t_0, t_1, \ldots, t_{n-1}\}$ [3].

In [3], the authors have introduced some reflection subgroup of G_n for any signed composition of n as follows: For $A = (a_1, \ldots, a_k) \in \mathcal{SC}(n)$, the set R_A is defined as

$$\begin{aligned} R_A =& \{s_p \in R_{-n} : |a_1| + \cdots + |a_{i-1}| + 1 \leq p \leq |a_1| + \cdots + |a_i| - 1\} \\ & \cup \{t_{|a_1|+\cdots+|a_{j-1}|+1} \in T_n \mid a_j > 0\} \subset R_n^{'}. \end{aligned}$$

The reflection subgroup G_A of G_n, which is generated by R_A, is a Coxeter group [3]. Let $R_A^{'} = R_n^{'} \cap G_A$, $\Gamma_A = \{\alpha \in \Gamma_n : s_\alpha \in G_A\}$ and $\Gamma_A^+ = \Gamma_A \cap \Gamma_n^+$. Thus the set Γ_A^+ is a positive root system of Γ_A and Π_A is a fundamental system of Γ_A contained in Γ_A^+. Hence we write $R_A = \{s_\alpha : \alpha \in \Pi_A\}$. Moreover, $G_A = \Xi_{A^+} \ltimes \langle T_A \rangle$, where $T_A = G_A \cap \mathcal{T}_n$. We use $A \subset B$ if $G_A \subset G_B$. Denote cox_A the Coxeter element of G_A attached to R_A. For any $A \in \mathcal{SC}(n)$, the set of distinguished coset representatives of G_A in G_n is defined in the following way:

$$D_A = \{x \in G_n : \forall\, s \in R_A,\ l(xs) > l(x)\}.$$

In other words, the set D_A can also be expressed as $\{x \in G_n : \forall\, \alpha \in \Pi_A,\ x(\alpha) \in \Gamma_n^+\}$. For $A, B \in \mathcal{SC}(n)$ such that $B \subset A$, the set $D_B^A = D_B \cap G_A$ is also the set of distinguished coset representatives of G_B in G_A. Setting

$$d_A = \sum_{w \in D_A} w \in \mathbb{Q}G_n,$$

then by [7] Mantaci-Reutenauer algebra, a subalgebra of group algebra $\mathbb{Q}G_n$, is described explicitly as follows:

$$\mathcal{MR}(G_n) = \bigoplus_{A \in \mathcal{SC}(n)} \mathbb{Q}d_A.$$

In [3], for $A, B \in \mathcal{SC}(n)$, the set of distinguished representatives of double cosets $G_A \backslash G_n / G_B$ is defined as $D_{AB} = D_A^{-1} \cap D_B$. Let the map Φ_n : $\mathcal{MR}(G_n) \to \mathbb{Q}\mathrm{Irr}G_n$ be the unique $\mathbb{Q}$-linear map such that $\Phi_n(d_A) = \mathrm{Ind}_{G_A}^{G_n} 1_A$

for every $A \in \mathcal{SC}(n)$, where $\mathbb{Q}\mathrm{Irr}G_n$ and 1_A stand for the algebra of the irreducible characters of G_n and the trivial character of G_A, respectively. Furthermore, it is well-known from [3] that the radical of $\mathcal{MR}(G_n)$ is $\mathrm{Ker}\Phi_n = \sum_{A \equiv_n A'} \mathbb{Q}(d_A - d_{A'})$.

Now we define $\phi_\lambda = \mathrm{Ind}_{G_{\hat{\lambda}}}^{G_n} 1_{\hat{\lambda}}$ for each $\lambda \in \mathcal{BP}(n)$. Let cox_δ be a Coxeter element of $G_{\hat{\delta}}$ for a double partition δ of n. Since the matrix $(\phi_\lambda(\mathrm{cox}_\delta))_{\lambda,\delta \in \mathcal{BP}(n)}$ is upper triangular and has positive diagonal entries, then $(\phi_\lambda(\mathrm{cox}_\delta))_{\lambda,\delta \in \mathcal{BP}(n)}$ is invertible in $\mathbb{Q}$. In what follows, the inverse of $(\phi_\lambda(\mathrm{cox}_\delta))_{\lambda,\delta \in \mathcal{BP}(n)}$ will be denoted by $(v_{\lambda\delta})_{\lambda,\delta \in \mathcal{BP}(n)}$.

We have obtained in [1] that for each $\lambda \in \mathcal{BP}(n)$ the orthogonal primitive idempotent $\zeta_\lambda = \sum_{\delta \in \mathcal{BP}(n)} v_{\lambda\delta}\phi_\delta$ of $\mathbb{Q}\mathrm{Irr}G_n$ is also the characteristic class function of G_n corresponding to the conjugate class $\mathcal{C}_\lambda$. When we extend linearly the class function ζ_λ to the group algebra $\mathbb{Q}G_n$, we have the following proposition.

Proposition 3.1. *Let $A, A' \in \mathcal{SC}(n)$ such that G_A is G_n-conjugate to $G_{A'}$. Then for each $\lambda \in \mathcal{BP}(n)$*

$$|\mathcal{C}_\lambda \cap D_A| = |\mathcal{C}_\lambda \cap D_{A'}|. \tag{2}$$

Proof. Because of the nilpotency of $d_A - d_{A'}$, we immediately see that $\zeta_\lambda(d_A) = \zeta_\lambda(d_{A'})$. Since ζ_λ is the characteristic class function, then the sizes of two sets $\mathcal{C}_\lambda \cap D_A$ and $\mathcal{C}_\lambda \cap D_{A'}$ are the same. □

As a result of the proposition given above, we say that if G_A is conjugate to $G_{A'}$ under the action of G_n, then the sets of distinguished coset representatives D_A and $D_{A'}$ are pointwise conjugate in the sense of [5, Theorem 1.1].

For a signed composition of $B = (b_1, \ldots, b_r)$, in [3], the authors have defined the sets $A_B = \{s_{|b_1|+\cdots+|b_i|} : i \in [1, r] \text{ and } b_i < 0 \text{ and } b_{i+1} > 0\}$ and $\mathcal{A}_B = R'_B \uplus A_B$. Also they have assigned to each element $x \in G_n$ a signed composition $\mathbf{C}(x) \in \mathcal{SC}(n)$ with a surjective map $\mathbf{C} : G_n \to \mathcal{SC}(n),\ x \mapsto \mathbf{C}(x)$. For instance, the element $(7. -3 -1. -6.245) \in G_7$ corresponds to the signed composition $\mathbf{C}(7. -3 -1. -6.245) = (1, -2, -1, 3) \in \mathcal{SC}(7)$. For a signed composition A of n, let $Y_A = \{x \in G_n : \mathbf{C}(x) = A\}$. Thus, there is a decomposition $G_n = \biguplus_{A \in \mathcal{SC}(n)} Y_A$. Furthermore, setting

$$y_A = \sum_{w \in Y_A} w,$$

it is well-known that the collection $\{y_A : A \in \mathcal{SC}(n)\}$ is another basis of the algebra $\mathcal{MR}(G_n)$ in [3].

Lemma 3.2 ([3, Lemma 2.21]). *For $A, B \in \mathcal{SC}(n)$. Then*

1. *when $Y_A \cap D_B \neq \emptyset$, the set Y_A is a subset of D_B,*
2. *the longest element η_A of D_A is contained in Y_A and so $Y_A \subset D_A$.*

The relation $\to$ on $\mathcal{SC}(n)$ is defined in [3] as follows: for $A, B \in \mathcal{SC}(n)$, we write $B \to A$ if $Y_B \subset D_A$ or equivalently $R_A \subset \mathcal{A}_B$. Transitive closure of $\to$ is denoted by $\ll$. Thus by [3], the relation $\ll$ is an partial order on $\mathcal{SC}(n)$, and moreover, there is a decomposition $D_A = \biguplus_{B \to A} Y_B$.

Since the Mantaci-Reutenauer algebra is an algebra with unity element 1, it is possible to mention about the units of this algebra. Considering Theorem 3.3 and the structure of $\mathcal{MR}(G_n)$, we shall investigate whether the basis elements $y_A, A \in \mathcal{SC}(n)$ of the algebra $\mathcal{MR}(G_n)$ are invertible.

For any $\lambda \in \mathcal{BP}(n)$, the algebra homomorphism $\tau_\lambda : \mathcal{MR}(G_n) \to \mathbb{Q}$, $x \mapsto \Phi_n(x)(\mathrm{cox}_\lambda)$, which is defined in [3], is an irreducible character of $\mathcal{MR}(G_n)$. Let $x \in \mathcal{MR}(G_n)$. Since the inner product of the characters $\Phi_n(x)$ and ζ_λ is $\langle \Phi_n(x), \zeta_\lambda \rangle = \frac{|\mathcal{C}_\lambda|}{|G_n|} \Phi_n(x)(\mathrm{cox}_\lambda)$, then we can express $\Phi_n(x)$ by

$$\Phi_n(x) = \sum_{\lambda \in \mathcal{BP}(n)} \tau_\lambda(x) \zeta_\lambda \tag{3}$$

in terms of the basis $\{\zeta_\lambda : \lambda \in \mathcal{BP}(n)\}$ of $\mathbb{Q}\mathrm{Irr} G_n$.

Since the map Φ is a surjective algebra morphism, by [2] there is a collection of orthogonal primitive idempotents $(e_\lambda)_{\lambda \in \mathcal{BP}(n)}$ of $\mathcal{MR}(G_n)$ which satisfies the following conditions:

(1) $\forall \lambda \in \mathcal{BP}(n),\ \Phi_n(e_\lambda) = \zeta_\lambda$,
(2) $\forall \lambda, \mu \in \mathcal{BP}(n)$, $e_\lambda e_\mu = \delta_{\lambda,\mu} e_\lambda$,
(3) $\sum_{\lambda \in \mathcal{BP}(n)} e_\lambda = 1$.

We denote $\sum_{\mathcal{BP}(n)}(G_n)$ the subspace of $\mathcal{MR}(G_n)$ spanned by the set $(e_\lambda)_{\lambda \in \mathcal{BP}(n)}$, then we have a decomposition

$$\mathcal{MR}(G_n) = \mathrm{Ker}\Phi_n \bigoplus \sum_{\mathcal{BP}(n)} (G_n).$$

Since each e_λ for $\lambda \in \mathcal{BP}(n)$ is an orthogonal primitive idempotent, then the subspace $\sum_{\mathcal{BP}(n)}(G_n)$ is also a subalgebra of $\mathcal{MR}(G_n)$. It is not difficult to see that the set $(e_\lambda)_{\lambda \in \mathcal{BP}(n)}$ is a basis for $\sum_{\mathcal{BP}(n)}(G_n)$. Since every element of $\mathrm{Ker}\Phi_n$ is nilpotent, neither element of $\mathrm{Ker}\Phi_n$ is unit in $\mathcal{MR}(G_n)$.

Theorem 3.3. *Let $x = a + b \in \mathcal{MR}(G_n)$ such that $a \in Ker\Phi_n$ and $b \in \sum_{\mathcal{BP}(n)}(G_n)$. The element x is a unit in $\mathcal{MR}(G_n)$ if and only if b is a unit in $\sum_{\mathcal{BP}(n)}(G_n)$ if and only if $\tau_\lambda(b) \neq 0$ for every $\lambda \in \mathcal{BP}(n)$.*

Proof. Suppose $x = a + b$ is a unit in $\mathcal{MR}(G_n)$. Then $\Phi_n(x) = \Phi_n(b)$ is a unit in $\mathbb{Q}\mathrm{Irr} G_n$. Taking into account (3), we have $\Phi_n(b) = \sum_{\lambda \in \mathcal{BP}(n)} \tau_\lambda(b) \zeta_\lambda$, where $\tau_\lambda(b) \neq 0$ for each $\lambda \in \mathcal{BP}(n)$. Assume to the contrary that $\tau_\gamma(b)$ equals to zero for some $\gamma \in \mathcal{BP}(n)$. Then since the set $\{\zeta_\lambda, \lambda \in \mathcal{BP}(n)\}$ consists of orthogonal primitive idempotents, we have $\Phi_n(b) \cdot \zeta_\gamma = \tau_\gamma(b) \zeta_\gamma = 0$. Therefore, $\Phi_n(b)$ is a non-zero zero divisor element and so it is not unit in $\mathbb{Q}\mathrm{Irr} G_n$. This is a contradiction by assumption. Accordingly, $\tau_\lambda(b) \neq 0$ for each $\lambda \in \mathcal{BP}(n)$. This also shows that $\Phi_n(x)(w) = \Phi_n(b)(w) \neq 0$ for all $w \in G_n$.

On the other hand, in order to prove sufficient condition we assume that $\tau_\lambda(b) \neq 0$ for every $\lambda \in \mathcal{BP}(n)$. Since the isomorphism

$$\sum_{\mathcal{BP}(n)} (G_n) \cong \mathbb{Q}\mathrm{Irr}G_n, \tag{4}$$

then we may write $\Phi_n(b) = \sum_{\lambda\in\mathcal{BP}(n)} \tau_\lambda(b)\zeta_\lambda$. If we consider the definition of $\mathrm{Ker}\Phi_n$, then we obtain $\Phi_n(a) = 0$. From these facts, we get $\Phi_n(x) = \sum_{\lambda\in\mathcal{BP}(n)} \tau_\lambda(x)\zeta_\lambda$. As $\tau_\lambda(x) \neq 0$ for every $\lambda \in \mathcal{BP}(n)$, we conclude that

$$\Phi_n(x) \cdot \Big(\sum_{\lambda\in\mathcal{BP}(n)} \frac{1}{\tau_\lambda(x)}\zeta_\lambda\Big) = \sum_{\lambda\in\mathcal{BP}(n)} {\zeta_\lambda}^2 = \sum_{\lambda\in\mathcal{BP}(n)} \zeta_\lambda = 1. \tag{5}$$

As a consequence of (5), both $\Phi_n(x)$ and $\Phi_n(b)$ are units. From (4), the element b is a unit in $\mathcal{MR}(G_n)$. Since a is a nilpotent element and b is a unit, then the element x is invertible in $\mathcal{MR}(G_n)$. □

Example 3.4. In [3], Bonnafé and Hohlweg have obtained a collection of orthogonal primitive idempotents of $\mathcal{MR}(G_2)$ as follows:

$$\begin{aligned}
e_{(2;\emptyset)} &= d_{(2)} - \frac{1}{2}d_{(1,1)} - \frac{1}{2}d_{(1,-1)} + \frac{1}{2}d_{(-1,1)} - \frac{1}{2}d_{(-2)} + \frac{1}{4}d_{(-1,-1)},\\
e_{(1,1;\emptyset)} &= \frac{1}{2}d_{(1,1)} - \frac{1}{4}d_{(1,-1)} - \frac{1}{4}d_{(-1,1)} + \frac{1}{8}d_{(-1,-1)},\\
e_{(1;1)} &= \frac{1}{2}d_{(1,-1)} - \frac{1}{4}d_{(-1,-1)},\\
e_{(\emptyset;2)} &= \frac{1}{2}d_{(-2)} - \frac{1}{4}d_{(-1,-1)},\\
e_{(\emptyset;1,1)} &= \frac{1}{8}d_{(-1,-1)}.
\end{aligned}$$

Also $\mathrm{Ker}\Phi_2 = \mathbb{Q}(d_{(1,-1)} - d_{(-1,1)})$. If we consider the element $y_{(1,1)}$ of $\mathcal{MR}(G_2)$, then it may be written as

$$y_{(1,1)} = \frac{1}{2}(d_{(1,-1)} - d_{(-1,1)}) - e_{(2;\emptyset)} + e_{(1,1;\emptyset)} - e_{(1;1)} + e_{(\emptyset;2)} + e_{(\emptyset;1,1)}.$$

Here, we see that the coefficient of each idempotent e_λ, $\lambda \in \mathcal{BP}(2)$ in the expression above is different from zero. From Theorem 3.3, $y_{(1,1)}$ is invertible in $\mathcal{MR}(G_2)$. In fact, the inverse of $y_{(1,1)}$ is itself, for if $\Phi_2(y_{(1,1)}) = f_4$ and f_4 is a linear character of the Coxeter group G_2.

Let $A \in \mathcal{SC}^+(n)$. Then $A^+ = A$ and $\mathcal{T}_n \leq W_A$ and so we have obtained that the longest element $w_n = t_0 \cdots t_{n-1}$ belongs to W_A and the inclusion $Y_A \subset D_A \subset \Xi_n$ holds. Moreover, the set $\{Y_A : A \in \mathcal{SC}^+(n)\}$ is a partition of the symmetric group Ξ_n. Thus Ξ_n can be expressed as

$$\Xi_n = D_{(1,1,\ldots,1)} = \coprod_{A\in\mathcal{SC}^+(n)} Y_A.$$

Since the reflection subgroup G_A is a semidirect product of Ξ_A and $\mathcal{T}_n$, then we may write

$$G_n = D_A G_A = D_A(\Xi_A \ltimes \mathcal{T}_n).$$

Thus, each element $w \in G_n$ is uniquely expressible in the form $w = d_A w_A = d_A w_S w_T$. Since $G_n = \Xi_n \ltimes \mathcal{T}_n$, then $\Xi_n = D_A \Xi_A$. As a consequence, D_A is the set of distinguished coset representatives of Ξ_A in Ξ_n, since Ξ_A is a standard parabolic subgroup of Ξ_n.

Example 3.5. For $A = (3,1) \in \mathcal{SC}(4)$, we have the generating set $R_{(3,1)} = \{s_1, s_2, t_1, t_4\}$. Since $G_{(3,1)} = \Xi_{(3,1)} \ltimes \mathcal{T}_4 = \langle s_1, s_2 \rangle \ltimes \mathcal{T}_4$, then we get $G_4 = D_{(3,1)} G_{(3,1)} = (D_{(3,1)}\langle s_1, s_2 \rangle) \ltimes \mathcal{T}_4$. Thus $D_{(3,1)}$ is the collection of distinguished coset representatives of the reflection subgroup $\Xi_{(3,1)} = \langle s_1, s_2 \rangle$ in Ξ_4. Furthermore, we have $D_{(3,1)} = \{1, s_3, s_2 s_3, s_1 s_2 s_3\}$ and so $D_{(3,1)} \subset \Xi_4$. We also note that $Y_{(3,1)} = \{s_3, s_2 s_3, s_1 s_2 s_3\} \subset D_{(3,1)}$.

Now for $A \in \mathcal{SC}^+(n)$, we define the set

$$\widetilde{Y}_A = \{x \in \Xi_n : x \in D_A \text{ and } l(x s_\alpha) < l(x) \text{ for all } s_\alpha \in R_{-n} \backslash R_A\}.$$

The following lemma gives us the relation between the sets $\widetilde{Y}_A$ and Y_A.

Lemma 3.6. *If A is a positive signed composition of n, then we have*

$$Y_A = \widetilde{Y}_A. \tag{6}$$

Proof. Let $A = (a_1, a_2, \ldots, a_r) \in \mathcal{SC}^+(n)$. Then two subsets Y_A and $\widetilde{Y}_A$ of G_n are contained in Ξ_n. We first assume that w is any element of Y_A. Since $R_{-n} \backslash R_A$ equals the set $\{s_{a_1}, s_{a_1+a_2}, \ldots, s_{a_1+a_2+\cdots+a_{r-1}}\}$ and $\mathbf{C}(w) = A$, so we have

$$w(e_{a_1+a_2+\cdots+a_i+1} - e_{a_1+a_2+\cdots+a_i}) = w(e_{a_1+a_2+\cdots+a_i+1}) - w(e_{a_1+a_2+\cdots+a_i}) < 0$$

for all $1 \leq i \leq r-1$. For this reason, we obtain $l(w s_{a_1+a_2+\cdots+a_i}) < l(w)$ and so $w \in \widetilde{Y}_A$. The reverse inclusion follows immediately from the definitions of the sets $\widetilde{Y}_A$ and Y_A. This completes the proof. $\square$

Example 3.7. For the positive signed composition $A = (3,1)$ of $n = 4$, from Lemma 3.6 the set $Y_{(3,1)} = \{s_3, s_2 s_3, s_1 s_2 s_3\}$ coincides with the set $\widetilde{Y}_{(3,1)} = \{w \in \Xi_4 : w \in D_A \text{ and } l(w s_\alpha) < l(w) \text{ for all } s_\alpha \in R_{-4} \backslash R_A\}$. In fact, the set $R_{-4} \backslash R_A$ is $\{s_3\}$ and any element w in $Y_{(3,1)}$ satisfies the condition $w(e_4 - e_3) < 0$, thus $l(w s_3) < l(w)$. On the other hand, for every $w \in \widetilde{Y}_{(3,1)}$ the relations $w(e_2 - e_1) > 0$, $w(e_3 - e_2) > 0$ and $w(e_4 - e_3) < 0$ are hold, so we may write $w(e_1) < w(e_2) < w(e_3) > w(e_4) > 0$. Therefore, we have $\mathbf{C}(w) = (3,1)$. This means that $w \in Y_{(3,1)}$.

It follows from part (2) of Lemma 3.2 that when $A \subset B$ for $A, B \in \mathcal{SC}(n)$ there exists the relation $B \to A$.

Proposition 3.8. *If $A \in \mathcal{SC}^+(n)$ and $B \to A$, then we have $A \subset B$.*

Proof. Since the signed composition B can be obtained by means of refinement from A in the sense of [3], then B is a positive signed composition of n. The set A_B is empty because of $B \in \mathcal{SC}^+(n)$. From the definition of $\to$, we get $R_A \subset \mathcal{A}_B = R_B^{'} \uplus A_B$. We deduce from this that $R_A \subset R_B^{'}$ and so $G_A \subset G_B$. Therefore, we conclude that $A \subset B$, as desired. □

As a result of Proposition 3.8, for $A \in \mathcal{SC}^+(n)$, we have $D_A = \biguplus_{B \to A} Y_B = \biguplus_{A \subset B} Y_B$. Thus, we may write $d_A = \sum_{A \subset B} y_B$ as in [8]. From Möbius Inversion formula, we obtain that

$$y_A = \sum_{A \subset B} (-1)^{|R_B^{'}| - |R_A^{'}|} d_B. \tag{7}$$

Since A is a positive signed composition of n, then $|R_A^{'}| = n + |R_{-n} \cap R_A|$. Hence the equation (7) can be rewritten as

$$y_A = \sum_{A \subset B} (-1)^{(|R_{-n} \cap R_B| - |R_{-n} \cap R_A|)} d_B.$$

As the longest element $\sigma_n \in \Xi_n$ is not central, by [4, Proposition 2.1] some elements y_A, $A \in \mathcal{SC}^+(n)$ are not invertible, i.e., one can find an element $w \in \Xi_n \leq G_n$ such that $\Phi_n(y_A)(w) = 0$. For each $A \in \mathcal{SC}^+(n)$, there is the unique $B \in \mathcal{SC}^-(n)$(all components of B are negative) such that

$$Y_B = Y_A w_n, \tag{8}$$

where w_n is the longest element of G_n. If we take the image of the both sides of equation (8) under the map Φ_n and take into account the fact $\Phi_n(w_n) = f_2$ in [8], then we have reached that

$$\Phi_n(y_A) = f_2 \Phi_n(y_B) \tag{9}$$

and so we say from the equation (9) that y_A is invertible if and only if y_B is invertible. It is clear from the equation (9) that $\Phi_n(y_{(n)}) f_2 = \Phi_n(y_{(-1,\dots,-1)})$ and $\Phi_n(y_{(1,\dots,1)}) f_2 = \Phi_n(y_{(-n)})$. For $A = (1, \dots, 1), (n) \in \mathcal{SC}^+(n)$, there is not any element w of G_n such that $\Phi_n(y_A)(w) = 0$. Therefore, not only the elements $y_{(n)}$ and $y_{(1,\dots,1)}$ but the elements $y_{(-n)}$ and $y_{(-1,\dots,-1)}$ are invertible as well.

Example 3.9. Let $n \geq 2$. Since

$Y_{(n-1,1)} = \{s_{n-1}, s_{n-2}s_{n-1}, \dots, s_1 s_2 \cdots s_{n-2} s_{n-1}\}$, $D_{(n-1,1)} = Y_{(n-1,1)} \uplus Y_{(n)}$ and $|D_{(n-1,1)}| = n$, then we have

$$D_{(n-1,1)} = \{1, s_{n-1},\ s_{n-2}s_{n-1}, \dots, s_1 s_2 \cdots s_{n-2} s_{n-1}\}.$$

Accordingly, for every $n \geq 3$,

$$D_{(n-1,1)(n-1,1)} = \{1, s_{n-1}\},$$

from this $\Phi_n(d_{(n-1,1)})(\mathrm{cox}_{(n-1;1)}) = 1$. Hence

$$\Phi_n(y_{(n-1,1)})(\mathrm{cox}_{(n-1;1)}) = \Phi_n(d_{(n-1,1)})(\mathrm{cox}_{(n-1;1)}) - \Phi_n(d_{(n)})(\mathrm{cox}_{(n-1;1)}) = 0$$

and so the element $y_{(n-1,1)}$ is not invertible in $\mathcal{MR}(G_n)$.

Lemma 3.10. *Let $A \in \mathcal{SC}(n)$ such that it is different from both positive and negative signed composition of n. Then*

$$(-n) \not\twoheadrightarrow A. \tag{10}$$

Proof. By our assumption on A, the generating set R_A contains some element of T_n. We know that $R'_{-n} = R_{-n}$ and $A_{(-n)} = \emptyset$. Therefore, we may write $R_A \not\subset \mathcal{A}_{(-n)} = R'_{-n} = R_{-n}$. Thus $(-n) \not\twoheadrightarrow A$, as desired. □

When $A \in \mathcal{SC}(n)$ has both positive and negative parts, then for any $B \in \mathcal{SC}^-(n)$ the relation $B \not\twoheadrightarrow A$ already holds as a consequence of the proof of Lemma 3.10. On account of this reason, the basis elements $y_{(-n)}$ and y_B do not occur in the expression of the basis element d_A in terms of the basis set $\{y_C : C \in \mathcal{SC}(n)\}$. Since the relation $\ll$ is an order on $\mathcal{SC}(n)$ and $(-n) \not\twoheadrightarrow A$, by Möbius Inversion Formula, we obtain that the element $d_{(-n)}$ does not appear in the expression of y_A in terms of the basis $\{d_C : C \in \mathcal{SC}(n)\}$. Denote the first point by Y_E, which is corresponding to positive signed composition, in the expression of the set of distinguished coset representatives D_A in terms of $\{Y_C : C \in \mathcal{SC}(n)\}$ is obtained by means of broken operator, which is defined in [3]. Then, from Y_E on, it is obtained the decomposition of D_E into the sets Y_K corresponding to positive signed compositions K, which can be obtained by the refinement of E. Thus the term $d_{(n)}$ is not included in the expression of y_A in terms of the elements $d_C, C \in \mathcal{SC}(n)$. Consequently, we have $\Phi_n(y_A)(s_1 \cdots s_{n-1}) = 0$. Hence the basis element y_A is not invertible in $\mathcal{MR}(G_n)$.

Example 3.11. We consider the signed composition $A = (1,1,-1,1) \in \mathcal{SC}(4)$. Then, we have

$$\begin{aligned} &D_{(1,1,-1,1)} \\ &= Y_{(1,1,-1,1)} \uplus Y_{(1,1,1,1)} \uplus Y_{(2,1,1)} \uplus Y_{(1,2,1)} \uplus Y_{(1,1,2)} \uplus Y_{(2,2)} \uplus Y_{(1,3)} \uplus Y_{(3,1)} \uplus Y_{(4)} \end{aligned}$$

and it is easily seen from the above that the first term, which is corresponding to positive signed composition, in this decomposition is $E = (1,1,1,1) \in \mathcal{SC}^+(4)$. Thus, we have $D_{(1,1,1,1)} = Y_{(1,1,1,1)} \uplus Y_{(2,1,1)} \uplus Y_{(1,2,1)} \uplus Y_{(1,1,2)} \uplus Y_{(2,2)} \uplus Y_{(1,3)} \uplus Y_{(3,1)} \uplus Y_{(4)}$. Since $y_{(1,1,-1,1)} = d_{(1,1,-1,1)} - d_{(1,1,1,1)}$, then the element $d_{(4)}$ does not seem in this expression.

Now we give an example to illustrate the method in the preceding paragraph.

Example 3.12. Expression of the basis elements y_A in terms of another basis elements $\{d_A : A \in \mathcal{SC}(3)\}$ in $\mathcal{MR}(G_3)$ is as follows:

(1) $y_{(3)} = d_{(3)} = 1$,
(2) $y_{(2,1)} = d_{(2,1)} - d_{(3)}$,
(3) $y_{(1,2)} = d_{(1,2)} - d_{(3)}$,
(4) $y_{(1,1,1)} = d_{(1,1,1)} - d_{(2,1)} - d_{(1,2)} + d_{(3)}$,

(5) $y_{(2,-1)} = d_{(2,-1)} - d_{(2,1)}$,
(6) $y_{(-1,2)} = d_{(-1,2)} - d_{(1,2)}$,
(7) $y_{(1,-2)} = d_{(1,-2)} - d_{(1,2)} - d_{(1,-1,1)} + d_{(1,1,1)}$,
(8) $y_{(-2,1)} = d_{(-2,1)} - d_{(2,1)} - d_{(-1,1,1)} + d_{(1,1,1)}$,
(9) $y_{(1,-1,1)} = d_{(1,-1,1)} - d_{(1,1,1)}$,
(10) $y_{(1,1,-1)} = y_{(-2,1)} w_3$,
(11) $y_{(-1,1,1)} = y_{(1,-2)} w_3$,
(12) $y_{(1,-1,-1)} = y_{(-1,2)} w_3$,
(13) $y_{(-1,1,-1)} = y_{(1,-1,1)} w_3$,
(14) $y_{(-1,-1,1)} = y_{(2,-1)} w_3$,
(15) $y_{(-3)} = y_{(1,1,1)} w_3$,
(16) $y_{(-2,-1)} = y_{(1,2)} w_3$,
(17) $y_{(-1,-2)} = y_{(2,1)} w_3$,
(18) $y_{(-1,-1,-1)} = y_{(3)} w_3$,

where $w_3 = t_0 t_1 t_2$ is the longest element of G_3. By using the character table of $\mathcal{MR}(G_3)$ given in [2], for $A \in \mathcal{SC}(3) \backslash \{(3), (1,1,1), (-3), (-1,-1,-1)\}$, we see that there exists some $w \in G_3$ such that $\Phi_3(y_A)(w) = 0$. Thus the elements y_A are not invertible in $\mathcal{MR}(G_3)$.

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