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QUASI-ISOMETRIC AND WEAKLY QUASISYMMETRIC MAPS BETWEEN LOCALLY COMPACT NON-COMPLETE METRIC SPACES

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Abstract. The aim of this paper is to show that there exists a weakly quasisymmetric homeomorphism $f : (X, d) \to (Y, d')$ between two locally compact non-complete metric spaces such that $f : (X, d_h) \to (Y, d'_h)$ is not quasi-isometric, where d_h denotes the Gromov hyperbolic metric with respect to the metric d introduced by Ibragimov in 2011. This result shows that the answer to the related question asked by Ibragimov in 2013 is negative.

1. Introduction

In this paper, (X, d) and (Y, d') always denote metric spaces. Other notations and concepts appearing in this section will be introduced in the next section. In [2, Theorem 4.1], Ibragimov proved that if (X, d) and (Y, d') are locally compact non-complete and if $f : (X, d) \to (Y, d')$ is a power quasisymmetric map, then $f : (X, d_h) \to (Y, d'_h)$ is quasi-isometric, quantitatively, where d_h denotes the Gromov hyperbolic metric with respect to the metric d introduced by Ibragimov in [1]. Meanwhile, in [2], Ibragimov asked whether [2, Theorem 4.1] is true for weakly quasisymmetric maps or not. The aim of this paper is to discuss this question. Our result is as follows, which implies that the answer to the mentioned Ibragimov's question is negative.

Theorem 1.1. *There exists a weakly quasisymmetric homeomorphism*

$$f : \ (X, d) \to (Y, d')$$

between two locally compact non-complete metric spaces such that

$$f : \ (X, d_h) \to (Y, d'_h)$$

is not quasi-isometric.

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We prove the existence of such a homeomorphism in Theorem 1.1 in Section 2 (see Example 2.1 below).

2. An example

The aim of this section is to construct an example which shows the truth of Theorem 1.1. Before the construction of this example, we introduce several definitions.

Definition 2.1. Suppose $H \geq 1$ is a constant. For a homeomorphism $f : (X,d) \to (Y,d')$ between two metric spaces, it is called *weakly H-quasisymmetric* if for every triple (x,a,b) in X, $d(a,x) \leq d(x,b)$ implies

$$d'(a',x') \leq Hd'(x',b'),$$

where primes mean the images of points under f, for example, $a' = f(a)$ etc.

Definition 2.2. Suppose $\lambda \geq 1$ and $\mu \geq 0$ are constants. A map $f : (X,d) \to (Y,d')$ between two metric spaces is *(λ,μ)-quasi-isometric* if the following requirements are satisfied:

(i) $\mathrm{dist}(w, f(X)) \leq \nu$ for any $w \in Y$, where $\mathrm{dist}(w, f(X))$ denotes the distance from the point w to the set $f(X)$;

(ii) for all x, $y \in X$,

$$\lambda^{-1}d(x,y) - \nu \leq d'(x',y') \leq \lambda d(x,y) + \nu$$

(see [1] or [2]).

Definition 2.3. Suppose (X,d) is a metric space with $\partial X \neq \emptyset$, where ∂X is by definition the set $\overline{X} - X$ and $\overline{X}$ stands for the metric completion. Then the Gromov hyperbolic metric d_h on X is defined by

$$d_h(x,y) = 2\log \frac{d(x,y) + \max\{\mathrm{dist}(x,\partial X), \mathrm{dist}(y,\partial X)\}}{\sqrt{\mathrm{dist}(x,\partial X)\mathrm{dist}(y,\partial X)}}$$

for x and $y \in X$ (see [1]).

Now, we are ready to construct our needed example. Let

$$X = \bigcup_{n=1}^{\infty} A_n \subset \mathbb{R} \text{ and } Y = \bigcup_{n=1}^{\infty} B_n \subset \mathbb{R},$$

where $A_n = [n, n+\frac{1}{2})$ and $B_n = [n, n+\frac{1}{2^n})$ for each integer $n \geq 1$.

Example 2.1. Let $f : (X, |\cdot|) \to (Y, |\cdot|)$ be the homeomorphism such that for each $n \geq 1$, the restriction $f|_{A_n}$ of f on A_n is defined by

$$f|_{A_n}(t) = \Big(1 - \frac{1}{2^{n-1}}\Big)n + \frac{1}{2^{n-1}}t.$$

Then the following statements hold:

(1) $(X, |\cdot|)$ and $(Y, |\cdot|)$ are locally compact non-complete metric spaces, where $|\cdot|$ denotes the Euclidean metric;

(2) f is weakly 5-quasisymmetric;
(3) $f : (X, |\cdot|_h) \to (Y, |\cdot|_h)$ is not (λ, ν)-quasi-isometric for any $\lambda \geq 1$ and $\nu \geq 0$.

Proof. The statement (1) in the example is obviously true since $f(\partial X) = \partial Y = \{n + \frac{1}{2^n}\}_{n=1}^{\infty}$. To prove the statement (2) in the example, it suffices to show the following claim:

Claim 2.1. *For each triple (x, y, z) in X with $|y - x| \leq |x - z|$,*

$$|f(y) - f(x)| \leq 5|f(x) - f(z)|.$$

Obviously, if $\{x, y, z\} \subset A_n$ for some $n \geq 1$, then the claim is obvious. In the following, we always assume that there is no $n \geq 1$ such that $\{x, y, z\} \subset A_n$.

Further, if $x \leq \min\{y, z\}$ or $x \geq \max\{y, z\}$, then $|y - x| \leq |x - z|$ easily implies

$$|f(y) - f(x)| \leq |f(x) - f(z)|.$$

Hence there are two cases which we have to consider: $y \leq x \leq z$ and $z \leq x \leq y$. We only need to discuss the first case since the argument for the second case is very similar. Now, we assume that

$$y \leq x \leq z.$$

Under this assumption, we easily know that x and z cannot belong to a common A_n for some $n \geq 1$. Thus, to finish the proof, there are only two possibilities which we have to consider. For the first possibility when there are an n and some $k \geq 1$ such that $x,\, y \in A_n$ and $z \in A_{n+k}$, it follows that

$$f(x) - f(y) < \frac{1}{2} < f(z) - f(x).$$

For the remaining possibility, that is, there are integers $n,\, s_1,\, s_2 \geq 1$ and $t_i \in [0, \frac{1}{2})$ $(i = 1, 2, 3)$ such that $y = n - s_1 + t_2 \in A_{n-s_1}$, $x = n + t_1 \in A_n$ and $z = n + s_2 + t_3 \in A_{n+s_2}$, we infer from the assumption $x - y \leq z - x$ that

$$2t_1 \leq s_2 - s_1 + t_2 + t_3,$$

and so

$$s_2 \geq s_1 - 1.$$

Then elementary computations lead to

$$f(x) - f(y) \leq s_1 + \frac{1}{2} \leq \frac{5}{2}s_2 \text{ and } f(z) - f(x) \geq s_2 - \frac{1}{2} \geq \frac{1}{2}s_2.$$

Hence

$$f(x) - f(y) \leq 5(f(z) - f(x)),$$

as required, and thus the claim is proved.

To finish the proof of this example, it remains to check the truth of the statement (3) in the example. On the one hand, a direct computation gives

$$|\cdot|_h(n, n+1) = 2\log\frac{1 + \frac{1}{2}}{\sqrt{\frac{1}{4}}} = 2\log 3,$$

and on the other hand, we have

$$|\cdot|_h(f(n), f(n+1)) = |\cdot|_h(n, n+1) = 2\log\frac{1+\max\{\frac{1}{2^n}, \frac{1}{2^{n+1}}\}}{\sqrt{\frac{1}{2^n}\frac{1}{2^{n+1}}}}$$
$$= 2\log(2^n+1) + \log 2.$$

Since $|\cdot|_h(f(n), f(n+1)) \to \infty$ as $n \to \infty$, we see that the homeomorphism f in the example is not (λ, ν)-quasi-isometric for any $\lambda \geq 1$ and $\nu \geq 0$. This implies that the statement (3) holds, and hence the proof of the example is complete. □

Obviously, power quasisymmetry implies quasisymmetry, and quasisymmetry implies weak quasisymmetry. We do not know if [2, Theorem 4.1] is true for quasisymmetric maps.

References

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