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TWO POINTS DISTORTION ESTIMATES FOR CONVEX UNIVALENT FUNCTIONS

Mari Okada and Hiroshi Yanagihara

Abstract. We study the class $\mathcal{CV}(\Omega)$ of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfying

$$1 + \frac{zf''(z)}{f'(z)} \in \Omega, \quad z \in \mathbb{D},$$

where Ω is a convex and proper subdomain of $\mathbb{C}$ with $1 \in \Omega$. Let ϕ_Ω be the unique conformal mapping of $\mathbb{D}$ onto Ω with $\phi_\Omega(0) = 1$ and $\phi'_\Omega(0) > 0$ and

$$k_\Omega(z) = \int_0^z \exp\left(\int_0^t \zeta^{-1}(\phi_\Omega(\zeta) - 1)\, d\zeta\right) dt.$$

Let $z_0, z_1 \in \mathbb{D}$ with $z_0 \neq z_1$. As the first result in this paper we show that the region of variability $\{\log f'(z_1) - \log f'(z_0) : f \in \mathcal{CV}(\Omega)\}$ coincides with the set $\{\log k'_\Omega(z_1 z) - \log k'_\Omega(z_0 z) : |z| \leq 1\}$. The second result deals with the case when Ω is the right half plane $\mathbb{H} = \{w \in \mathbb{C} : \mathrm{Re}\, w > 0\}$. In this case $\mathcal{CV}(\Omega)$ is identical with the usual normalized class of convex univalent functions on $\mathbb{D}$. And we derive the sharp upper bound for $|\log f'(z_1) - \log f'(z_0)|$, $f \in \mathcal{CV}(\mathbb{H})$. The third result concerns how far two functions in $\mathcal{CV}(\Omega)$ are from each other. Furthermore we determine all extremal functions explicitly.

1. Introduction

Let $\mathbb{C}$ be the complex plane, $\mathbb{D}(c,r) = \{z \in \mathbb{C} : |z - c| < r\}$ and $\overline{\mathbb{D}}(c,r) = \{z \in \mathbb{C} : |z - c| \leq r\}$ with $c \in \mathbb{C}$ and $r > 0$. In particular we denote the unit disk $\mathbb{D}(0,1)$ by $\mathbb{D}$. Let $\mathcal{A}$ be the linear space of analytic functions in the unit disk $\mathbb{D}$, endowed with the topology of uniform convergence on every compact subset of $\mathbb{D}$. Set $\mathcal{A}_0 = \{f \in \mathcal{A} : f(0) = f'(0) - 1 = 0\}$ and denote by S the subclass of $\mathcal{A}_0$ consisting of all univalent functions as usual. Then S is a compact subset of the metrizable space $\mathcal{A}$. See [1, Chap. 9] for details.

Unless otherwise stated explicitly, throughout the discussion let Ω be a simply connected domain in $\mathbb{C}$ with $1 \in \Omega \neq \mathbb{C}$ and ϕ_Ω the unique conformal

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mapping of $\mathbb{D}$ onto Ω with $\phi_\Omega(0) = 1$ and $\phi'_\Omega(0) > 0$. Ma and Minda [3] considered the classes $S^*(\Omega)$ and $\mathcal{CV}(\Omega)$

$$S^*(\Omega) = \left\{ f \in \mathcal{A}_0 : \frac{zf'(z)}{f(z)} \in \Omega \text{ on } \mathbb{D} \right\},$$
$$\mathcal{CV}(\Omega) = \left\{ f \in \mathcal{A}_0 : 1 + \frac{zf''(z)}{f'(z)} \in \Omega \text{ on } \mathbb{D} \right\}.$$

with some mild conditions, e.g. Ω is starlike with respect to 1 and the symmetry with respect to the real axis $\mathbb{R}$, i.e., $\overline{\Omega} = \Omega$. It is easy to see that for $f \in \mathcal{A}_0$, $f \in \mathcal{CV}(\Omega)$ if and only if $zf' \in S^*(\Omega)$. Note that, with the special choice of $\Omega = \mathbb{H} := \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$, these two classes consist of starlike and convex functions in the standard sense, and are denoted simply by S^* and $\mathcal{CV}$, respectively.

If $0 < \alpha \le 1$ and $\Omega = \{w \in \mathbb{C} : |\operatorname{Arg} w| < 2^{-1}\pi\alpha\}$, then $\phi_\Omega(z) = \{(1 + z)/(1-z)\}^\alpha$, and hence, in this choice $\mathcal{CV}(\Omega)$ reduces to the class of strongly convex functions of order α. Furthermore for $\Omega = \mathbb{H}_\beta := \{w \in \mathbb{C} : \operatorname{Re} w > \beta\}$ with $0 \le \beta < 1$ the class $\mathcal{CV}(\mathbb{H}_\beta)$ coincides with the class of convex functions of order β. Also $\mathcal{CV}(\{\operatorname{Re} w > k|w-1|\})$ with $0 \le k < \infty$ called the class of k-uniformly convex functions, which was introduced in [2]. Various subclasses of $\mathcal{CV}$ can be expressed in this way. For details we refer to [3] and [4]. We notice that it may be possible that $\mathbb{H} \subset \Omega$, and in this case we have $\mathcal{CV} \subset \mathcal{CV}(\Omega)$.

Since Ω is simply connected and $\Omega \ne \mathbb{C}$, $\mathbb{C} \backslash \Omega$ has an unbounded component. Therefore $f \in \mathcal{CV}(\Omega)$ forces that $f'(z) \ne 0$ in $\mathbb{D}$ and the single valued branch $\log f'(z)$ with $\log f'(0) = 0$ exists on $\mathbb{D}$. Let $z_0, z_1 \in \mathbb{D}$ with $z_0 \ne z_1$. One of the aims of the present article is to study the variability regions

$$V_\Omega(z_0, z_1) = \{\log f'(z_1) - \log f'(z_0) : f \in \mathcal{CV}(\Omega)\} \tag{1.1}$$

for various classes $\mathcal{CV}(\Omega)$ in a unified manner. Let

$$k_\Omega(z) = \int_0^z \exp\left(\int_0^t \frac{\phi_\Omega(\zeta) - 1}{\zeta}\, d\zeta \right) dt, \quad z \in \mathbb{D}. \tag{1.2}$$

Then $k_\Omega \in \mathcal{CV}(\Omega)$ and k_Ω plays the role of the extremal function.

Theorem 1.1. *If Ω is convex, then*

$$V_\Omega(z_0, z_1) = \{\log k'_\Omega(z_1 z) - \log k'_\Omega(z_0 z) : z \in \overline{\mathbb{D}}\}. \tag{1.3}$$

Furthermore the set in the right hand side of the equation is a convex closed Jordan domain enclosed by the simple closed curve given by

$$\partial\mathbb{D} \ni \varepsilon \mapsto \log k'_\Omega(z_1\varepsilon) - \log k'_\Omega(z_0\varepsilon),$$

and $\log f'(z_1) - \log f'(z_0) = \log k'_\Omega(z_1\varepsilon) - \log k'_\Omega(z_0\varepsilon)$ holds for some $f \in \mathcal{CV}(\Omega)$ and $\varepsilon \in \partial\mathbb{D}$ if and only if $f(z) = \overline{\varepsilon} k_\Omega(\varepsilon z)$ in $\mathbb{D}$.

When $\Omega = \mathbb{H}_\beta$, the functions $\phi_{\mathbb{H}_\beta}$, $k_{\mathbb{H}_\beta}$ and the set $V_{\mathbb{H}_\beta}(z_0, z_1)$ will be written simply as ϕ_β, k_β and $V_\beta(z_0, z_1)$, respectively. Then we have

$$\phi_\beta(z) = \frac{1 + (1 - 2\beta)z}{1 - z}, \tag{1.4}$$

$$\log k'_\beta(z) = 2(1 - \beta) \log \frac{1}{1 - z}, \tag{1.5}$$

$$k_\beta(z) = \begin{cases} \frac{1}{2\beta-1}\{1 - (1 - z)^{2\beta-1}\}, & \beta \neq \frac{1}{2} \\ \log \frac{1}{1-z}, & \beta = \frac{1}{2}. \end{cases} \tag{1.6}$$

As a simple application of Theorem 1.1 we have the following simple estimate.

Proposition 1.2. *Let $f \in \mathcal{CV}(\mathbb{H}_\beta)$ with $0 < \beta \leq 1$. For $z_0, z_1 \in \mathbb{D}$ with $z_0 \neq z_1$ we have*

$$|\log f'(z_1) - \log f'(z_0)| \leq 2(1 - \beta) \frac{|z_1 - z_0|}{1 - \max\{|z_0|, |z_1|\}}. \tag{1.7}$$

The inequality (1.7) is not sharp. Applying Theorem 1.1 more precisely we can determine

$$\max_{f \in \mathcal{CV}(\mathbb{H}_\beta)} |\log f'(z_1) - \log f'(z_0)|, \qquad \max_{f,g \in \mathcal{CV}(\mathbb{H}_\beta)} |\log f'(z_1) - \log g'(z_1)|.$$

Theorem 1.3. *For $z_0, z_1 \in \mathbb{D}$ with $|z_0| \leq |z_1|$ and $z_0 \neq z_1$ let*

$$c = \frac{1 - z_0 \overline{z_1}}{1 - |z_1|^2}, \qquad \rho = \frac{|z_1 - z_0|}{1 - |z_1|^2} \tag{1.8}$$

and $\varphi_0 = \operatorname{Arg} c$. Then the equation

$$\frac{|c| \sin\theta}{\sqrt{\rho^2 - |c|^2 \sin^2\theta}} \log\left(|c| \cos\theta + \sqrt{\rho^2 - |c|^2 \sin^2\theta}\right) - \theta = \varphi_0 \tag{1.9}$$

has the unique solution $\theta_0 \in \left(-\sin^{-1} \frac{\rho}{|c|}, \sin^{-1} \frac{\rho}{|c|}\right)$, and

$$\begin{aligned} &\max_{f \in \mathcal{CV}(\mathbb{H}_\beta)} |\log f'(z_1) - \log f'(z_0)| \\ =&2(1 - \beta) \left|\log\left(|c| \cos\theta_0 + \sqrt{\rho^2 - |c|^2 \sin^2\theta_0}\right) + i(\theta_0 + \varphi_0)\right| \end{aligned}$$

and the maximum is attained if and only if $f(z) = \overline{\varepsilon_0} k_\beta(\varepsilon_0 z)$, where $\varepsilon_0 \in \partial\mathbb{D}$ is given by

$$\left(|c| \cos\theta_0 + \sqrt{\rho^2 - |c|^2 \sin^2\theta_0}\right) e^{i(\theta_0 + \varphi_0)} = \frac{1 - \varepsilon_0 z_0}{1 - \varepsilon_0 z_1}. \tag{1.10}$$

Particularly when $z_0/z_1 \geq 0$ or $z_0/z_1 < 0$, we have $\varphi_0 = \theta_0 = 0$ and the maximum coincides with $2(1 - \beta) \log \frac{1-|z_0|}{1-|z_1|}$ or $2(1 - \beta) \log \frac{1+|z_0|}{1-|z_1|}$, respectively.

The following theorem shows that how far two functions in $\mathcal{CV}(\mathbb{H}_\beta)$ are from each other.

Theorem 1.4. *For* $z_1 \in \mathbb{D}\backslash\{0\}$ *we have*

$$\max_{f,g\in\mathcal{CV}(\mathbb{H}_\beta)} |\log f'(z_1) - \log g'(z_1)| = 2(1-\beta)\log\frac{1+|z_1|}{1-|z_1|}$$

and the maximum is attained if and only if

$$f(z) = -\frac{z_1}{|z_1|}k_\beta\left(-\frac{\overline{z_1}}{|z_1|}\right) \quad \text{and} \quad g(z) = \frac{z_1}{|z_1|}k_\beta\left(\frac{\overline{z_1}}{|z_1|}\right)$$

or permutation of them.

2. Determination of $V_\Omega(z_0, z_1)$

Assume Ω is convex and let $z_0, z_1 \in \mathbb{D}$ with $z_0 \neq z_1$ be fixed. For $f \in \mathcal{CV}(\Omega)$ let

$$p_f(z) = 1 + z\frac{f''(z)}{f'(z)}, \quad z \in \mathbb{D}.$$

Lemma 2.1. *The set* $V_\Omega(z_0, z_1)$ *is a compact and convex subset of* $\mathbb{C}$ *and has* 0 *as an interior point. Particularly* $\partial V_\Omega(z_0, z_1)$ *is a simple closed curve and* $V_\Omega(z_0, z_1)$ *is the closed Jordan domain enclosed by* $\partial V_\Omega(z_0, z_1)$*, i.e.,* $V_\Omega(z_0, z_1)$ *is the union of* $\partial V_\Omega(z_0, z_1)$ *and the domain surrounded by* $\partial V_\Omega(z_0, z_1)$*.*

Proof. It is easy to see that $\mathcal{CV}(\Omega)$ is a compact subset of the metric space $\mathcal{A}$. Since $V_\Omega(z_0, z_1)$ is the image of $\mathcal{CV}(\Omega)$ with respect to the continuous functional $\mathcal{CV}(\Omega) \ni f \mapsto \log f'(z_1) - \log f'(z_0)$, it is a compact subset of $\mathbb{C}$.

For $f_0, f_1 \in \mathcal{CV}(\Omega)$ and $t \in (0,1)$ let

$$p_t(z) = (1-t)p_{f_1}(z) + tp_{f_0}(z), \quad f_t(z) = \int_0^z \exp\left(\int_0^\zeta \frac{p_t(\xi)-1}{\xi}\,d\xi\right) d\zeta.$$

Then $f_t \in \mathcal{CV}(\Omega)$ and

$$\log f_t'(z_1) - \log f_t'(z_0) = (1-t)\{\log f_1'(z_1) - \log f_1'(z_0)\} + t\{\log f_0'(z_1) - \log f_0'(z_0)\}.$$

From this it easily follows that $V_\Omega(z_0, z_1)$ is convex.

For $\varepsilon \in \overline{\mathbb{D}}$ and $z \in \mathbb{D}$ let

$$F_\varepsilon(z) = \begin{cases} \frac{1}{\varepsilon}k_\Omega(\varepsilon z), & \varepsilon \neq 0 \\ z, & \varepsilon = 0. \end{cases}$$

Then $p_{F_\varepsilon}(z) = \phi_\Omega(\varepsilon z)$ and hence $F_\varepsilon \in \mathcal{CV}(\Omega)$ for all $\varepsilon \in \overline{\mathbb{D}}$. Let

$$q(\varepsilon) = \log F_\varepsilon'(z_1) - \log F_\varepsilon'(z_0) = \log k_\Omega'({}_\varepsilon z_1) - \log k_\Omega'(\varepsilon z_0).$$

Then $q(\varepsilon) \in V_\Omega(z_0, z_1)$ and we have

$$q'(0) = \frac{k_\Omega''(0)}{k_\Omega'(0)}(z_1 - z_0) = \phi_\Omega'(0)(z_1 - z_0) \neq 0.$$

Therefore q is nonconstant analytic in $\mathbb{D}$ and $0(= q(0))$ is an interior point of $q(\mathbb{D})$. Since $q(\mathbb{D}) \subset V_\Omega(z_0, z_1)$, 0 is an interior point of $V_\Omega(z_0, z_1)$.

Since the latter statement of the lemma is a simple consequence of the former one, proof is left to the reader. □

Proof of Theorem 1.1. For $r \in (0, 1)$, ϕ_Ω maps $\mathbb{D}(0, r)$ conformally onto the convex domain $\phi_\Omega(\mathbb{D}(0, r))$. Also the boundary $\partial\phi_\Omega(\mathbb{D}(0, r))$ is the image of the convex closed curve given by $(-\pi, \pi] \ni \theta \mapsto \phi_\Omega(re^{i\theta})$. By the Schwarz lemma we have $|\phi_\Omega^{-1}(p_f(z)))| \leq |z|$. This implies $p_f(\zeta) \in \overline{\phi_\Omega(\mathbb{D}(0, r))} = \phi_\Omega(\overline{\mathbb{D}}(0, r))$ for $\zeta \in \overline{\mathbb{D}}(0, r)$. Thus for $\zeta \in \overline{\mathbb{D}}(0, r)$, $p_f(\zeta)$ belongs to the left half plane of the tangential line at $\phi_\Omega(re^{i\theta})$ with the tangential vector $ire^{i\theta}\phi'_\Omega(re^{i\theta})$. Hence

$$\mathrm{Re}\left\{\frac{\phi_\Omega(re^{i\theta}) - p_f(\zeta)}{re^{i\theta}\phi'_\Omega(re^{i\theta})}\right\} \geq 0.$$

Let $\varepsilon \in \partial\mathbb{D}(0, r)$. Applying the above inequality to $\phi_\Omega(\varepsilon\cdot)$ instead of ϕ_Ω and letting $\zeta = re^{i\theta} = z$ we have

$$\mathrm{Re}\left\{\frac{\phi_\Omega(\varepsilon z) - p_f(z)}{\varepsilon z\phi'_\Omega(\varepsilon z)}\right\} \geq 0, \quad z \in \mathbb{D} \tag{2.1}$$

with equality at some $z_0 \in \mathbb{D}$ if and only if $p_f(z) \equiv \phi_\Omega(\varepsilon z)$.

Since Ω is convex, the line segment connecting $\phi_\Omega(\varepsilon z_0)$ and $\phi_\Omega(\varepsilon z_1)$ entirely lies in Ω. Let Γ be the path defined by

$$z(t) = \overline{\varepsilon}\phi_\Omega^{-1}((1-t)\phi_\Omega(\varepsilon z_0) + t\phi_\Omega(\varepsilon z_1)), \quad 0 \leq t \leq 1.$$

Then Γ is a C^1-path in $\mathbb{D}$ joining z_0 and z_1 and satisfying $\phi_\Omega(\varepsilon z(t)) = (1 - t)\phi_\Omega(\varepsilon z_0) + t\phi_\Omega(\varepsilon z_1)$. By differentiation we have

$$\varepsilon\phi'_\Omega(\varepsilon z(t))z'(t) = \phi_\Omega(\varepsilon z_1) - \phi_\Omega(\varepsilon z_0). \tag{2.2}$$

By (2.1) and (2.2) we have successively

$$\begin{aligned}
0 &\leq \int_0^1 \mathrm{Re}\left\{\frac{\phi_\Omega(\varepsilon z(t)) - p_f(z(t))}{\varepsilon z(t)\phi'_\Omega(\varepsilon z(t))}\right\} dt \\
&= \mathrm{Re}\left\{\int_0^1 \frac{\frac{\phi_\Omega(\varepsilon z(t)) - p_f(z(t))}{z(t)} z'(t)}{\varepsilon\phi'_\Omega(\varepsilon z(t))z'(t)} dt\right\} \\
&= \mathrm{Re}\left\{\int_0^1 \frac{\frac{\phi_\Omega(\varepsilon z(t)) - p_f(z(t))}{z(t)} z'(t)}{\phi_\Omega(\varepsilon z_1) - \phi_\Omega(\varepsilon z_0)} dt\right\} \\
&= \mathrm{Re}\left\{\frac{\int_\Gamma \frac{\phi_\Omega(\varepsilon z) - p_f(z)}{z} dz}{\phi_\Omega(\varepsilon z_1) - \phi_\Omega(\varepsilon z_0)}\right\} \\
&= \mathrm{Re}\left\{\frac{\int_\Gamma \frac{\phi_\Omega(\varepsilon z) - 1}{z} dz - \int_\Gamma \frac{p_f(z) - 1}{z} dz}{\phi_\Omega(\varepsilon z_1) - \phi_\Omega(\varepsilon z_0)}\right\}
\end{aligned}$$

$$= \mathrm{Re}\left\{\frac{\log k'_\Omega(\varepsilon z_1) - \log k'_\Omega(\varepsilon z_0) - (\log f'(z_1) - \log f'(z_0))}{\phi_\Omega(\varepsilon z_1) - \phi_\Omega(\varepsilon z_0)}\right\}.$$

Letting $w_0 = \log k'_\Omega(\varepsilon z_1) - \log k'_\Omega(\varepsilon z_0)$ and $c = \phi_\Omega(\varepsilon z_1) - \phi_\Omega(\varepsilon z_0)$ it easily follows that $\log f'(z_1) - \log f'(z_0)$ always belongs to the half plane $\mathcal{H} = \{w \in \mathbb{C} : \mathrm{Re}\,\{(w_0 - w)/c)\} \geq 0\}$. Thus we have $V_\Omega(z_0, z_1) \subset \mathcal{H}$. From this $w_0 = \log k'_\Omega(\varepsilon z_1) - \log k'_\Omega(\varepsilon z_0) \in V_\Omega(z_0, z_1) \cap \partial\mathcal{H}$. Therefore we obtain $\log k'_\Omega(\varepsilon z_1) - \log k'_\Omega(\varepsilon z_0) \in \partial V_\Omega(z_0, z_1)$ for any $\varepsilon \in \partial\mathbb{D}$.

We deal with uniqueness. Suppose that $\log f'(z_1) - \log f'(z_0) = \log k'_\Omega(\varepsilon z_1) - \log k'_\Omega(\varepsilon z_0)$ holds for some $f \in \mathcal{CV}(\Omega)$ and $\varepsilon \in \partial\mathbb{D}$. Then from the uniqueness part of (2.1) it follows that $\phi_\Omega(\varepsilon z) = p_f(z)$ on the image of Γ. By the identity theorem for analytic functions we obtain that $\phi_\Omega(\varepsilon z) = p_f(z)$ in $\mathbb{D}$. Therefore, $\overline{\varepsilon} k_\Omega(\varepsilon z) = f(z)$ in $\mathbb{D}$ by normalization.

Now we show that the closed curve given by $\partial\mathbb{D} \ni \varepsilon \mapsto \log k'_\Omega(\varepsilon z_1) - \log k'_\Omega(\varepsilon z_0)$ is simple. Assume that $\log k'_\Omega(\varepsilon_1 z_1) - \log k'_\Omega(\varepsilon_1 z_0) = \log k'_\Omega(\varepsilon_0 z_1) - \log k'_\Omega(\varepsilon_1 z_0)$. Then from the uniqueness part of the theorem which is shown above we have $\overline{\varepsilon_1} k_\Omega(\varepsilon_1 z) = \overline{\varepsilon_0} k_\Omega(\varepsilon_0 z)$ in $\mathbb{D}$. Since $k_\Omega(z) = z + 2^{-1}k''_\Omega(0)z^2 + \cdots$ with $k''_\Omega(0) = \phi'_\Omega(0) > 0$, this implies $\varepsilon_1 = \varepsilon_0$.

We have shown that the closed curve given by $\partial\mathbb{D} \ni \varepsilon \mapsto \log k'_\Omega(\varepsilon z_1) - \log k'_\Omega(\varepsilon z_0)$ is simple and its image is contained in $\partial V_\Omega(z_0, z_1)$. By Lemma 2.1 $\partial V_\Omega(z_0, z_1)$ is also a image of simple closed curve. Note that a simple closed curve cannot contain any simple closed curve other than itself, the mapping $\partial\mathbb{D} \ni \varepsilon \mapsto \log k'_\Omega(\varepsilon z_1) - \log k'_\Omega(\varepsilon z_0)$ is a parametrization of the boundary curve $\partial V_\Omega(z_0, z_1)$. □

3. The case that $\Omega = \mathbb{H}_\beta$

Proof of Proposition 1.2. When $\Omega = \mathbb{H}_\beta$, by Theorem 1.1 and (1.5), for $f \in \mathcal{CV}(\mathbb{H}_\beta)$ there exists $z \in \overline{\mathbb{D}}$ with $\log f'(z_1) - \log f'(z_0) = 2(1-\beta)\log\frac{1-z_0 z}{1-z_1 z}$. Since $\log\frac{1}{1-w} = \sum_{k=1}^\infty \frac{w^k}{k}$, we have

$$\begin{aligned}
\left|\frac{\log f'(z_1) - \log f'(z_0)}{z_1 - z_0}\right| &= 2(1-\beta)\left|\sum_{k=1}^\infty \frac{z_1^{k-1} + z_1^{k-2}z_0 + \cdots + z_0^{k-1}}{k} z^{k-1}\right| \\
&\leq 2(1-\beta)\sum_{k=1}^\infty (\max\{|z_1|, |z_0|\})^{k-1}\,|z|^{k-1} \\
&= \frac{2(1-\beta)}{1 - \max\{|z_1|, |z_0|\}|z|} \\
&\leq \frac{2(1-\beta)}{1 - \max\{|z_1|, |z_0|\}}.
\end{aligned}$$

□

Proof of Theorem 1.3. Similarly we have

$$V_\beta(z_0, z_1) = \{\log f'(z_1) - \log f'(z_0) : f \in \mathcal{CV}(\mathbb{H}_\beta)\}$$

$$= \left\{ 2(1-\beta)\log\frac{1-z_0z}{1-z_1z} : z \in \overline{\mathbb{D}} \right\}.$$

The image of $\overline{\mathbb{D}}$ under the linear fractional transformation $z \mapsto \frac{1-z_0z}{1-z_1z}$ coincides with $\overline{\mathbb{D}}(c,\rho)$, where c and ρ are defined by (1.8). Notice that $|c| < \rho$. Let $\varphi_0 = \operatorname{Arg} c \in (-\pi,\pi]$. Then for $re^{i(\theta+\varphi_0)} \in \partial\mathbb{D}(c,\rho)$, by the law of cosines we have $r = |c|\cos\theta \pm \sqrt{\rho^2 - |c|^2\sin^2\theta}$, $|\theta| \le \sin^{-1}\frac{\rho}{|c|}$. Then the boundary $\partial V_\beta(z_0,z_1) = 2(1-\beta)\log\partial\mathbb{D}(c,\rho)$ consists of two simple arcs J_1 and J_2 which have parametric representations

$$J_\ell : u + iv = u_\ell(\theta) + i\Theta(\theta), \quad \ell = 1,2, \quad |\theta| \le \sin^{-1}\frac{\rho}{|c|},$$

where

$$\begin{aligned} u_1(\theta) =& 2(1-\beta)\log\left\{|c|\cos\theta - \sqrt{\rho^2 - |c|^2\sin^2\theta}\right\}, \\ u_2(\theta) =& 2(1-\beta)\log\left\{|c|\cos\theta + \sqrt{\rho^2 - |c|^2\sin^2\theta}\right\}, \\ \Theta(\theta) =& 2(1-\beta)(\theta+\varphi_0), \quad \varphi_0 = \operatorname{Arg} c. \end{aligned}$$

Since $\frac{1}{2}\{u_1(\theta) + u_2(\theta)\} = (1-\beta)\log\{|c|^2 - \rho^2\}$, $\partial V_\beta(z_0,z_1)$ is symmetric with respect to the vertical line $L : u = (1-\beta)\log\{|c|^2 - \rho^2\}$ and the horizontal line $v = 2(1-\beta)\varphi_0$. By

$$|c-1| = \frac{|\overline{z_1}(z_1-z_0)|}{1-|z_1|^2} < \frac{|z_1-z_0|}{1-|z_1|^2} = \rho,$$

we also note that the origin is an interior point of $V_\beta(z_0,z_1)$.

Since $V_\beta(z_0,z_1)$ is compact, there exists $w_0 \in \partial V_\beta(z_0,z_1)$ with $|w_0| = \max_{f\in\mathcal{CV}(\mathbb{H}_\beta)} |\log f'(z_1) - \log f'(z_0)|$. From $|z_1| \ge |z_0|$ it follows that $|c|^2 - \rho^2 \ge 1$ and hence the origin lies in the left hand side of the symmetric axis L. Therefore there exists θ_0 with $|\theta_0| \le \sin^{-1}\frac{\rho}{|C|}$ such that $w_0 = u_2(\theta_0) + i\Theta(\theta_0)$ and that the normal line at w_0 passes through the origin.

Claim. There exists uniquely the normal line to the arc J_2, which passes through the origin.

We temporarily assume the claim. Then the unique normal line can be expressed as

$$\begin{cases} u = u_2(\theta_0) - \dfrac{d\Theta}{d\theta}(\theta_0)t = 2(1-\beta)\left\{\log\left(|c|\cos\theta_0 + \sqrt{\rho^2 - |c|^2\sin^2\theta_0}\right) - t\right\}, \\ v = \Theta(\theta_0) + \dfrac{du_2}{d\theta}(\theta_0)t = 2(1-\beta)\left\{\theta_0 + \varphi_0 - \dfrac{|c|\sin\theta_0}{\sqrt{\rho^2 - |c|^2\sin^2\theta_0}}t\right\} \end{cases}$$

$t \in \mathbb{R}$. Since the line passes through the origin, we obtain

$$\theta_0 + \varphi_0 - \frac{|c| \sin \theta_0}{\sqrt{\rho^2 - |c|^2 \sin^2 \theta_0}} \log \left(|c| \cos \theta_0 + \sqrt{\rho^2 - |c|^2 \sin^2 \theta_0} \right) = 0,$$

which is equivalent to (1.9).

By the uniqueness part of Theorem 1.1 the extremal function which attains $\max_{f \in \mathcal{CV}(\mathbb{H}_\beta)} |\log f'(z_1) - \log f'(z_0)|$ is given by $f(z) = \overline{\varepsilon_0} k_\beta(\varepsilon_0 z)$, where ε_0 satisfies

$$w_0 = u_2(\theta_0) + i\Theta(\theta_0) = 2(1-\beta) \log \frac{1 - \varepsilon_0 z_0}{1 - \varepsilon_0 z_1},$$

which is equivalent to (1.10). □

Proof of Claim. Let $h(\theta)$ be the v-coordinate of the intersection of the normal line at $(u_2(\theta), \Theta(\theta))$ and the symmetric axis L. Then

$$\frac{h(\theta)}{2(1-\beta)} = \theta + \varphi_0 - \frac{|c| \sin \theta}{2\sqrt{\rho^2 - |c|^2 \sin^2 \theta}} \log \left(\frac{|c| \cos \theta + \sqrt{\rho^2 - |c|^2 \sin^2 \theta}}{|c| \cos \theta - \sqrt{\rho^2 - |c|^2 \sin^2 \theta}} \right).$$

By an elementary calculation

$$\frac{h'(\theta)}{2(1-\beta)} = \frac{\rho^2}{\rho^2 - |c|^2 \sin^2 \theta} - \frac{|c| \rho^2 \cos \theta}{2(\rho^2 - |c|^2 \sin^2 \theta)^{3/2}} \log \left(\frac{1 + \frac{\sqrt{\rho^2 - |c|^2 \sin^2 \theta}}{|c| \cos \theta}}{1 - \frac{\sqrt{\rho^2 - |c|^2 \sin^2 \theta}}{|c| \cos \theta}} \right).$$

Notice that $\frac{1}{2} \log \frac{1+x}{1-x} = x + \sum_{k=1}^{\infty} \frac{1}{2k+1} x^{2k+1} > x$ for $0 < x < 1$. It is easy to see that $h'(\theta) < 0$ for $|\theta| < \sin^{-1} \frac{\rho}{|c|}$. Thus $h(\theta)$ is strictly decreasing in θ. From a geometric consideration we infer that any two normal lines to the curve J_2 intersect in the right hand side of the symmetric axis L. Therefore a normal line passing through 0 is unique. □

Proof of Theorem 1.4. The maximum in question is obviously the diameter of the variability region $V_\beta(0, z_1) = \{2(1-\beta) \log \frac{1}{1 - z_1 z} : |z| \leq 1\}$, i.e.,

$$\max_{f, g \in \mathcal{CV}(\mathbb{H}_\beta)} |\log f'(z_1) - \log g'(z_1)| = \max_{w, \tilde{w} \in V_\beta(0, z_1)} |w - \tilde{w}|.$$

We may assume that $z_1 = r \in (0, 1)$. Let $a = \frac{1 - \sqrt{1 - r^2}}{r}$. Then we have $0 < a < 1$, $1 - ar = \sqrt{1 - r^2}$ and $r - a = a\sqrt{1 - r^2}$. Consider the function

$$F(z) = \log \frac{1}{1 - r \frac{z + a}{1 + az}} - \frac{1}{2} \log \frac{1}{1 - r^2}, \quad z \in \overline{\mathbb{D}}.$$

Then $V_\beta(0, r) = \{2(1-\beta)(F(z) + \frac{1}{2} \log \frac{1}{1 - r^2}) : z \in \overline{\mathbb{D}}\}$ and we have by an elementary calculation

$$F(z) = \log \frac{1 + az}{1 - az} = 2 \sum_{n=0}^{\infty} \frac{a^{2n+1}}{2n+1} z^{2n+1}.$$

Since $F(z)$ is an odd function of z and has a Taylor expansion of non-negative coefficients, we have

$$|F(z)| \le F(|z|), \quad z \in \overline{\mathbb{D}}$$

with equality if and only if $z \in \overline{\mathbb{D}} \cap \mathbb{R}$. In particular, the diameter of $F(\overline{\mathbb{D}})$ is given only by $F(1) - F(-1) = 2F(1) = \log \frac{1+r}{1-r}$ as is expected. It is easy to determine the extremal functions explicitly. We omit details.

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Mari Okada
Department of Applied Science
Faculty of Engineering
Yamaguchi University
Tokiwadai, Ube 755-8611, Japan
Email address: okada@yamaguchi-u.ac.jp

Hiroshi Yanagihara
Department of Applied Science
Faculty of Engineering
Yamaguchi University
Tokiwadai, Ube 755-8611, Japan
Email address: hiroshi@yamaguchi-u.ac.jp