Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b170386
pISSN: 1015-8634 / eISSN: 2234-3016

m-ADIC RESIDUE CODES OVER $F_q[v]/(v^2 - v)$ AND DNA CODES

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Abstract. In this study we determine the structure of m-adic residue codes over the non-chain ring $F_q[v]/(v^2 - v)$ and present some promising examples of such codes that have optimal parameters with respect to Griesmer Bound. Further, we show that the generators of m-adic residue codes serve as a natural and suitable application for generating reversible DNA codes via a special automorphism and sets over $F_{4^{2k}}[v]/(v^2 - v)$.

1. Introduction

Quadratic residue codes constitute an important class of cyclic codes. Due to this fact, many researchers have worked on further generalizations of these families of codes [4,5,9,15]. Especially, m-adic residue codes are a generalization of these codes [6]. In this direction, Pless and Brualdi have defined polyadic codes [4], and later Pless has studied polyadic codes via idempotent generators and some specific ideals [4]. After these works, Job has introduced m-adic residue codes in terms of generator polynomials over fields [6].

Another research direction that has been initiated by Leonard Adleman that serves as a solution to the famous Travelermans Problem (an NP-complete problem) is presented by employing DNA molecules in [2].

DNA sequences consist of four bases (nucleotides) which are (A) Adenine, (G) Guanine, (T) Thymine and (C) Cytosine. DNA strands obey the famous Watson Crick complement (WCC) rule, i.e., “A” pairs with “T” and “G” pairs with “C”. Symbolically, we represent the WCC pairings as $A^c = T, T^c = A, G^c = C$ and $C^c = G$ where the superscript c stands for the complement.

The error correction and detection quality observed naturally in DNA strands mainly is based on WCC property. This property has attracted the attention on studying algebraic codes that enjoys such similar properties. Hence, studies on these directions have been one of the main focuses in algebraic coding theory. Such algebraic codes are referred to as DNA codes. To mention some

Received April 27, 2017; Revised September 27, 2017; Accepted November 22, 2017.
2010 *Mathematics Subject Classification.* Primary 94B05, 94B15.
Key words and phrases. m-adic residue codes, polyadic codes, non-chain rings, cyclic codes, DNA codes.

of these studies but surely not all, DNA codes are studied over F_4 [1], F_{16} [10], $F_{4^{2k}}$ [11], the chain rings $F_2[u]/(u^2-1)$ [13], $F_2[u]/(u^4-1)$ [14] and very recently over non-chain rings $F_4[v]/(v^2-v)$ [3]. Due to the complex and still not well understood structure of DNA, researchers have restricted their studies to specific (local) regions of DNA strands such as protein, binding sites that have important role in the protein production processes. In [7], reversible complement 8-bases (8-mers) are observed intensively in some specific and important regions of DNA. This sequences are believed to play important roles on DNA structures.

In this work, we extend the definition of m-adic residue codes over a non-chain ring $(F_q[v]/(v^2-v))$ in terms of idempotent generators. We study codes over the ring $F_q[v]/(v^2-v)$ that is also a non-chain ring unlike in the previous studies in the literature which are over finite fields. We obtain some optimal codes with respect to Griesmer bound introduced in [12]. After exploring structures of these codes, we introduce a special automorphism and a generator set to construct reversible DNA codes by means of the generators of m-adic residue codes over $F_{4^{2k}}[v]/(v^2-v)$. This generator methods for constructing DNA codes has proven to be less complex than [3]. This result is based on the fact that the structure of the generators of m-adic residue codes over non-chain rings $(F_q[v]/(v^2-v))$ is more suitable for reversible DNA codes.

The rest of the paper is organized as follows: In Section 2, we give some preliminaries and definitions about linear codes, cyclic codes, m-adic residue codes and reversible codes. Section 3 is devoted to definition of m-adic residue codes over the ring $F_q[v]/(v^2-v)$ and special classes of cyclic codes. In addition, we give some properties of the generator polynomials and idempotent generators of m-adic residue codes over the ring $F_q[v]/(v^2-v)$ and present two promising examples of these codes in this section. We present a new DNA code construction by palindromic generator polynomials of m-adic residue codes over the ring $F_{4^{2k}}[v]/(v^2-v)$ and present some examples. In Section 5, we finalize this study by summarizing our findings.

2. Preliminaries

In this section we present some necessary definitions in a concise way [6, 8]. Let q be a prime and F_q be a field with q elements. A subset C of F_q^n is called a code and a subspace of F_q^n is called a linear code. The Hamming weight $w_H(x)$ of a codeword $x=(x_0,x_1,\ldots,x_{n-1})\in C$ is the number of non-zero coordinates of x. Hamming distance between $x=(x_0,x_1,\ldots,x_{n-1})\in C$ and $y=(y_0,y_1,\ldots,y_{n-1})\in C$ is $d_H(x,y)=w_H(x-y)$ and minimum distance of C is $d_H(C)=min\{w_H(x-y):x,y\in C, x\neq y\}$. A linear code of length n, dimension k and minimum distance d over the finite field F_q is referred to as an $[n,k,d]_q$ code. If $(a_{n-1},a_0,\ldots,a_{n-2})\in C$ for all $(a_0,a_1,\ldots,a_{n-1})\in C$, then C is called a cyclic code.

Let C_1 and C_2 be two cyclic codes which are generated by the polynomials $g_1(x)$ and $g_2(x)$, respectively. Also, $e_1(x)$ and $e_2(x)$, be the idempotent generators of these codes respectively. Then, a generator polynomial of C_1+C_2 is $gcd(g_1(x), g_2(x))$ and an idempotent generator of C_1+C_2 is $e_1(x)+e_2(x)-e_1(x)e_2(x)$. Moreover, a generator polynomial of $C_1\cap C_2$ is $lcm(g_1(x), g_2(x))$ and an idempotent generator of $C_1\cap C_2$ is $e_1(x)e_2(x)$. If $C_1\subseteq C_2$, then the complementary code of C_1 relative to C_2 is denoted by $\overline{C_1}$ having property that $C_1+\overline{C_1}=C_2$, and $C_1\cap\overline{C_1}=\overline{0}$. The complementary code of C_1 relatively to $V=F_q^n$ (the whole space) is called the complement of C_1. If $\sum_{i=0}^{n-1} v_i=0$ for $v=(v_0,v_1,\ldots,v_{n-1})\in V$, then v is called even-like,otherwise it is odd-like. If all codewords of a code are even-like, then this code is an even-like code otherwise it is an odd-like code. Let E be the set of all even-like vectors and $h(x)$ be a polynomial in which all coefficients are 1 that corresponds to the vector notation $(1,1,\ldots,1)$. If $(q,n)=1$, then the dimension of E is $n-1$ and this code is a cyclic code with idempotent generator $1-(1/n)h(x)$.

Definition ([6])**.** Let p be a prime and b be a primitive element of $Z_p^*=Z_p\backslash\{0\}$. The set of nonzero m-adic residues modulo p is defined as $Q_0=\{a^m : a\in Z_p^*\}$ where $m\geq 2$, $m\in Z$ and $m|(p-1)$. Also, we let $Q_i=b^iQ_0$ and $\mu_a : i\to ai \pmod p$ where $a\in Q_1$ be a coordinate permutation such that μ_a cyclically permutes the sets $Q_0,Q_1,\ldots,Q_{m-1}$.

Example 2.1. Let $p=19$ and 2 be a primitive element of Z_{19}^*. Since $6|19-1$, we can take $m=6$. Then, $Q_0=\{1,7,11\}$, $Q_1=\{2,3,14\}$, $Q_2=\{4,6,9\}$, $Q_3=\{8,12,18\}$, $Q_4=\{5,16,17\}$, $Q_5=\{10,13,15\}$.

Definition ([6])**.** Let p be a prime and q be a prime power such that $gcd(p,q)=1$. Let b be a primitive element of Z_p^* and α be a primitive pth root of unity in some field extension of F_q. Let Q_0 be the set of nonzero m-adic residues modulo p and $Q_i=b^iQ_0$. If q is an m-adic residue modulo p, i.e., $q\in Q_0$, then the codes generated by the polynomials $g_i(x)=\frac{x^p-1}{\prod\limits_{k\in Q_i}(x-\alpha^k)}$ $(i=0,1,\ldots,m-1)$ are called an even-like family of m-adic residue codes of class I with length p over F_q.

The following families of m-adic residue codes defined below are derivations of even-like family of m-adic residue codes of class I.

Definition ([6])**.**
- A family of codes generated by polynomials $\widehat{g_i}(x)=\prod\limits_{k\in Q_i}(x-\alpha^k)$ $(i=0,1,\ldots,m-1)$ is called a family of odd-like class I m-adic residue codes of length p over F_q and the code generated by $\widehat{g_i}(x)$ is the complement of the code generated by $g_i(x)$.
- A family of codes generated by polynomials $h_i(x)=(x-1)\widehat{g_i}(x)$ $(i=0,1,\ldots,m-1)$ is called a family of even-like class II m-adic

residue codes of length p over F_q and the code generated by $h_i(x)$ is the complementary code of the code generated by $g_i(x)$ relative to E.

- A family of codes generated by polynomials $\widehat{h}_i(x) = \frac{g_i(x)}{x-1}$ $(i = 0, 1, \ldots, m-1)$ is called a family of odd-like class II m-adic residue codes of length p over F_q and these codes are the complements of the codes generated by $h_i(x)$.

Example 2.2. Even-like class I 4-adic residue codes of length 17 over $F_4 = \{0, 1, w, 1+w \mid w^2 = w+1\}$ are

$$\begin{aligned} C_0 &= \langle g_0(x)\rangle \\ &= \langle 1+x+w^2x^2+x^4+w^2x^5+wx^6+wx^7+w^2x^8+x^9+w^2x^{11}+x^{12}+x^{13}\rangle, \\ C_1 &= \langle g_1(x)\rangle \\ &= \langle 1+wx+wx^2+w^2x^3+x^4+wx^6+wx^7+x^9+w^2x^{10}+wx^{11}+wx^{12}+x^{13}\rangle, \\ C_2 &= \langle g_2(x)\rangle \\ &= \langle 1+x+wx^2+x^4+wx^5+w^2x^6+w^2x^7+wx^8+x^9+wx^{11}+x^{12}+x^{13}\rangle, \end{aligned}$$

and

$$\begin{aligned} C_3 &= \langle g_3(x)\rangle \\ &= \langle 1+w^2x+w^2x^2+wx^3+x^4+w^2x^6+w^2x^7+x^9+wx^{10}+w^2x^{11}+w^2x^{12}+x^{13}\rangle \end{aligned}$$

and the minimum distance of these codes is 12.

Theorem 2.3 ([6])**.** *Let C be an arbitrary m-adic residue code with generating idempotent e. Then, e is a linear combination of the polynomials $l_0(x)$, $l_1(x)$,…,$l_{m-1}(x)$ and 1 over F_q where $l_i(x) = \sum\limits_{k \in Q_i} x^k$.*

In the following we give a preliminary definition for necessary notions to be introduced later.

Definition. Let C be a code of length n over an finite alphabet. If $c^r = (c_{n-1}, c_{n-2}, \ldots, c_1, c_0) \in C$ for all $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, then C is called a reversible code.

3. m-adic residue codes over $F_q[v]/(v^2 - v)$

Pless introduced the idempotent generators of m-adic residue codes inspired by quadratic residue codes and she has studied properties of these generators [4] . Here, we extend these ideas of idempotent generators to m-adic residue codes over the non-chain ring $F_q[v]/(v^2 - v)$ and identify the idempotent generators for all classes of m-adic residue codes over this ring.

Proposition 3.1. *Let p be a prime and q be a prime power. Let q be m-adic residues modulo p, i.e., $q \in Q_0$ for a positive integer m such that $m|(p-1)$. If e_i and e_j are idempotents in $F_q[x]/(x^p-1)$, then $ve_i+(1-v)e_j$ is an idempotent in $(F_q[v]/(v^2 - v))[x]/(x^p - 1)$.*

Proof. By the definition of being an idempotent, we have $(ve_i + (1-v)e_j)^2 = ve_i^2 + (1-v)e_j^2 = ve_i + (1-v)e_j$. □

Proposition 3.2. *Let e_i and e_j be idempotent generators of an even-like class I m-adic residue codes of length p over F_q such that $0 \le i, j \le m-1$ where i and j are integers. Let $E_k = ve_i + (1-v)e_j \in (F_q[v]/(v^2-v))[x]/(x^p-1)$ and $h = 1 + x + x^2 + \cdots + x^{p-1}$. Then,*

i. *$\mu_a(E_k)$ is idempotent.*
ii. *If $i \ne j$, then $E_iE_j = 0$.*
iii. *$E_0 + E_1 + \cdots + E_{m-1} = 1 - h$.*

Proof. i. If $E_k = ve_i+(1-v)e_j$, then $\mu_a(ve_i+(1-v)e_j) = v\mu_a(e_i)+(1-v)\mu_a(e_j)$. Since $\mu_a(e_i)$ and $\mu_a(e_j)$ are idempotents,

$$\begin{aligned}(\mu_a(E_k))^2 &= (v\mu_a(e_i) + (1-v)\mu(e_j))^2\\ &= v^2(\mu_a(e_i))^2 + (1-v)^2(\mu_a(e_j))^2\\ &= v\mu_a(e_i) + (1-v)\mu_a(e_j) = \mu_a(E_k).\end{aligned}$$

ii. Let $E_i = ve_a + (1-v)e_b$, $E_j = ve_c + (1-v)e_d$ where $a, b, c, d \in Z^+ \cup \{0\}$ are distinct nonnegative integers.

$$\begin{aligned}E_iE_j &= (ve_a + (1-v)e_b)(ve_c + (1-v)e_d)\\ &= v^2e_ae_c + (1-v)^2e_be_d\\ &= v^2 0 + (1-v)^2 0 = 0.\end{aligned}$$

iii. Let $E_0 = ve_0 + (1-v)e_0'$, $E_1 = ve_1 + (1-v)e_1', \ldots, E_{m-1} = ve_{m-1} + (1-v)e_{m-1}'$.

$$\begin{aligned}&E_0 + E_1 + \cdots + E_{m-1}\\ &= v(e_0 + e_1 + \cdots + e_{m-1}) + (1-v)(e_0' + e_1' + \cdots + e_{m-1}')\\ &= v(1-h) + (1-v)(1-h) = 1-h.\end{aligned}$$

□

Since the idempotent elements $E_k = ve_i + (1-v)e_j$ satisfy the properties above, we naturally consider them as idempotent generators of m-adic residue codes over $F_q[v]/(v^2-v)$.

Definition. A family of codes generated by idempotent elements $E_k = ve_i + (1-v)e_j$ is called a family of even-like class I m-adic residue codes of length p over $F_q[v]/(v^2-v)$ where e_i and e_j are idempotent generators of class I even-like m-adic residue codes over F_q.

Proposition 3.3. *Let E_i be an idempotent generator of a class I even-like m-adic residue codes of length p over $F_q[v]/(v^2-v)$. Assume $E_i' = 1 - E_i$ such that $0 \le i, j \le m-1$ where i and j are integers. Then E_i' satisfies the followings*:

i. *$\mu_a(E_i') = E_j'$ $(i \ne j)$.*
ii. *If $i \ne j$, then $E_i' + E_j' - E_i'E_j' = 1$.*

iii. $E'_0 E'_1 \cdots E'_{m-1} = h$.

Proof. i. $\mu_a(E'_i) = \mu_a(1 - E_i) = 1 - \mu_a(E_i) = 1 - E_j = E'_j$.

ii.

$$\begin{aligned} E'_i + E'_j - E'_i E'_j &= 1 - E_i + 1 - E_j - (1 - E_i)(1 - E_j) \\ &= 2 - E_i - E_j - (1 - E_i - E_j + E_i E_j) \\ &= 1 + E_i E_j = 1 + 0 = 1. \end{aligned}$$

iii.

$$\begin{aligned} E'_0 E'_1 \cdots E'_{m-1} &= (1 - E_0)(1 - E_1) \cdots (1 - E_{m-1}) \\ &= 1 - (E_0 + E_1 + \cdots + E_{m-1}) \\ &\quad + (E_0 E_1 + E_0 E_2 + \cdots + E_{m-2} E_{m-1}) \\ &\quad - (E_0 E_1 E_2 + E_0 E_1 E_3 + \cdots + E_{m-3} E_{m-2} E_{m-1}) \\ &\quad + \cdots + (-1)^m E_0 E_1 \cdots E_{m-1} \\ &= 1 - (E_0 + E_1 + \cdots + E_{m-1}) + 0 \\ &= 1 - (1 - h) = 1 - 1 + h = h. \end{aligned}$$

□

Definition. A family of codes generated by the idempotent element $E'_i = 1 - E_i$ $(0 \leq i \leq m-1\ )$ is called a family of odd-like class I m-adic residue codes of length p over $F_q[v]/(v^2 - v)$.

Proposition 3.4. *Let E_i be an idempotent generator of a class I even-like m-adic residue codes of length p over $F_q[v]/(v^2 - v)$. Assume $F_i = 1 - h - E_i$ such that $0 \leq i, j \leq m-1$ where i and j are integers. Then, F_i satisfies the followings*:

i. $\mu_a(F_i) = F_j\ (i \neq j)$.
ii. $F_i + F_j - F_i F_j = 1 - h$, *where* $i \neq j$.
iii. $F_0 F_1 \cdots F_{m-1} = 0$.

Proof. i. $\mu_a(F_i) = \mu_a(1 - h - E_i) = 1 - \mu_a(h) - \mu_a(E_i) = 1 - h - E_j = F_j$.

ii.

$$\begin{aligned} &F_i + F_j - F_i F_j \\ &= (1 - h - E_i) + (1 - h - E_j) - (1 - h - E_i)(1 - h - E_j) \\ &= 2 - 2h - E_i - E_j - (1 - h - E_j - h + h^2 - hE_j - E_i + hE_i + E_i E_j) \\ &= 1 - h. \end{aligned}$$

iii. Let $a = 1 - h$.

$$\begin{aligned} F_0 F_1 \cdots F_{m-1} &= (1 - h - E_0)(1 - h - E_1) \cdots (1 - h - E_{m-1}) \\ &= (a - E_0)(a - E_1) \cdots (a - E_{m-1}) \\ &= a^m - a^{m-1}(E_0 + E_1 + \cdots + E_{m-1}) \end{aligned}$$

$$+ a^{m-2}\sum_{i\neq j} E_iE_j + \cdots + (-1)^m E_0E_1\cdots E_{m-1}$$
$$= a^m - a^{m-1}(1-h) + 0 = a^m - a^m = 0. \qquad \square$$

Definition. A family of codes generated by the idempotent elements $F_i = 1-h-E_i$ is called a family of even-like class II m-adic residue codes of length p over $F_q[v]/(v^2-v)$ such that $0 \le i \le m-1$ where i is an integer.

Proposition 3.5. *Let E_i and F_i be class I even-like m-adic residue code of length p and class II even-like m-adic residue code, respectively. Let $F_i' = 1 - F_i = h + E_i$ such that $0 \le i, j \le m-1$ where i and j are integers. Then F_i' satisfies the following*:

i. $\mu_a(F_i') = F_j'$ $(i \neq j)$.
ii. *If $i \neq j$, then* $F_i'F_j' = h$.
iii. $F_0' + F_1' + \cdots + F_{m-1}' = 1 - (m-1)h$.

Proof. i. $\mu_a(F_i') = \mu_a(1-F_i) = 1 - \mu_a(F_i) = 1 - F_j = F_j'$.
ii. $F_i'F_j' = (h+E_i)(h+E_j) = h^2 + h(E_iE_j) + E_iE_j = h$.
iii.

$$\begin{aligned} F_0' + F_1' + \cdots + F_{m-1}' &= (h+E_0) + (h+E_1) + \cdots + (h+E_{m-1}) \\ &= mh + E_0 + \cdots + E_{m-1} \\ &= 1 - (m-1)h. \end{aligned} \qquad \square$$

Definition. A family of codes generated by idempotent element $F_i' = 1 - F_i = h + E_i$ is called a family of odd-like class II m-adic residue codes of length p over $F_q[v]/(v^2-v)$ such that $0 \le i \le m-1$ where i is an integer.

Example 3.6. Idempotent generators of even-like class I 4-adic residue codes of length 17 over F_4 are $e_0 = l_0 + wl_1 + l_2 + w^2l_3$, $e_1 = wl_0 + l_1 + w^2l_2 + l_3$, $e_2 = l_0 + w^2l_1 + l_2 + wl_3$ and $e_3 = w^2l_0 + l_1 + wl_2 + l_3$ where $l_0 = x + x^4 + x^{13} + x^{16}$, $l_1 = x^3 + x^5 + x^{12} + x^{14}$, $l_2 = x^2 + x^8 + x^9 + x^{15}$, $l_3 = x^6 + x^7 + x^{10} + x^{11}$. If $E_0 = ve_0 + (1-v)e_2$ is considered, then idempotent generators of even-like class I 4-adic residue codes of length 17 over $F_4[v]/(v^2-v)$ are $E_0 = ve_0 + (1-v)e_2$, $E_1 = \mu_3(E_0) = ve_3 + (1-v)e_1$, $E_2 = \mu_3(E_1) = ve_2 + (1-v)e_0$, $E_3 = \mu_3(E_2) = ve_1 + (1-v)e_3$(also $E_0 = \mu_3(E_3)$).

Let $g_i(x)$ be the generator polynomial of corresponding to idempotent generator E_i, then

$$\begin{aligned} g_0(x) &= 1 + x + (v+w)x^2 + x^4 + (v+w)x^5 + (v+w^2)x^6 + (v+w^2)x^7 \\ &\quad + (v+w)x^8 + x^9 + (v+w)x^{11} + x^{12} + x^{13}, \\ g_1(x) &= 1 + (v+w)x + (v+w)x^2 + (v+w^2)x^3 + x^4 + (v+w)x^6 + (v+w)x^7 \\ &\quad + x^9 + (v+w^2)x^{10} + (v+w)x^{11} + (v+w)x^{12} + x^{13}, \\ g_2(x) &= 1 + x + (v+w^2)x^2 + x^4 + (v+w^2)x^5 + (v+w)x^6 + (v+w)x^7 \end{aligned}$$

$$
\begin{aligned}
&+(v+w^2)x^8+x^9+(v+w^2)x^{11}+x^{12}+x^{13},\\
g_3(x) &= 1+(v+w^2)x+(v+w^2)x^2+(v+w)x^3+x^4+(v+w^2)x^6\\
&+(v+w^2)x^7+x^9+(v+w)x^{10}+(v+w^2)x^{11}+(v+w^2)x^{12}+x^{13}.
\end{aligned}
$$

All these polynomials generate the same parameter $[17, 4, 12]$ codes and these parameters attain the Griesmer bound given in [12]. Thus these codes are optimal.

Let $\widehat{g_i}(x)$ be the generator polynomial of a code generated by the idempotent E_i'. Then,

$$
\begin{aligned}
\widehat{g_0}(x) &= 1+x+(v+w^2)x^2+x^3+x^4,\\
\widehat{g_1}(x) &= 1+(v+w)x+x^2+(v+w)x^3+x^4,\\
\widehat{g_2}(x) &= 1+x+(v+w)x^2+x^3+x^4,\\
\widehat{g_3}(x) &= 1+(v+w^2)x+x^2+(v+w^2)x^3+x^4.
\end{aligned}
$$

Let $h_i(x)$ be the generator polynomial of a code generated by the idempotent F_i. Then,

$$
\begin{aligned}
h_0(x) &= 1+(v+w)x^2+(v+w)x^3+x^5,\\
h_1(x) &= 1+(v+w^2)x+(v+w^2)x^2+(v+w^2)x^3+(v+w^2)x^4+x^5,\\
h_2(x) &= 1+(v+w^2)x^2+(v+w^2)x^3+x^5,\\
h_3(x) &= 1+(v+w)x+(v+w)x^2+(v+w)x^3+(v+w)x^4+x^5.
\end{aligned}
$$

Let $\widehat{h_i}(x)$ be the generator polynomial of a code generated by the idempotent F_i'. Then,

$$
\begin{aligned}
\widehat{h_0}(x) &= 1+(v+w)x^2+(v+w)x^3+(v+w^2)x^4+x^5+(v+w)x^6+x^7\\
&+(v+w^2)x^8+(v+w)x^9+(v+w)x^{10}+x^{12},\\
\widehat{h_1}(x) &= 1+(v+w^2)x+x^2+(v+w)x^3+(v+w^2)x^4+(v+w^2)x^5+x^6\\
&+(v+w^2)x^7+(v+w^2)x^8+(v+w)x^9+x^{10}+(v+w^2)x^{11}+x^{12},\\
\widehat{h_2}(x) &= 1+(v+w^2)x^2+(v+w^2)x^3+(v+w)x^4+x^5+(v+w^2)x^6+x^7\\
&+(v+w)x^8+(v+w^2)x^9+(v+w^2)x^{10}+x^{12},\\
\widehat{h_3}(x) &= 1+(v+w)x+x^2+(v+w^2)x^3+(v+w)x^4+(v+w)x^5+x^6\\
&+(v+w)x^7+(v+w)x^8+(v+w^2)x^9+x^{10}+(v+w)x^{11}+x^{12}.
\end{aligned}
$$

A polynomial is called palindromic whenever the order of its coefficients is reversed then it is still equal to itself. Later we are going to use the palindromic property of the generators of these families of codes. For instance, $g(x) = w+x+(1+w)x^2+x^3+wx^4$ is a palindromic polynomial. Here, we give a property of m-adic residue codes that is going to be related to palindromic property in the sequel.

Lemma 3.7. *i. If $(p-1)/m$ is even and $i \in Q_j$, then $-i \in Q_j$.*
ii. $\sum\limits_{i \in Q_j} i = 0.$

Proof. i. If $(p-1)/m$ is even, then $(p-1)/(2m)$ is an integer. Assume that $Z_p^* = \langle b \rangle$. Then, $b^{(p-1)/(2m)} \in Z_p^* \Rightarrow (b^{(p-1)/(2m)})^m = b^{(p-1)/2} = -1 \in Q_0$ (since b is a generator). Further, since Q_i's are multiplicative groups, for all $i \in Q_0$, we have $-i \in Q_0$. Hence, for all $i \in Q_j$, we have $-i \in Q_j$.

ii. Let $Z_p^* = \langle b \rangle$ and $p-1 = mt$. Then, $\langle b^m \rangle = \{1, b^m, b^{2m}, \dots, b^{(t-1)m}\} = Q_0$. $(1 + b^m + b^{2m} + \cdots + b^{(t-1)m})(b^m - 1) = b^{tm} - 1 = 0$. Since $b^m - 1 \neq 0$, $1 + b^m + b^{2m} + \cdots + b^{(t-1)m} = 0$.

Thus, $\sum\limits_{i \in Q_j} i = b^j \sum\limits_{i \in Q_0} i = 0.$ □

Proposition 3.8. *If $q = 2^k$ where $k \geq 1 \in Z$ and $(p-1)/m$ is even, then the generator polynomials of the m-adic residue codes over F_q of length p are palindromic.*

Proof. Let $f(x)$ be a generator polynomial of an m-adic residue code corresponding to Q_0. Then, $f(x) = \prod\limits_{i \in Q_0} (x - \alpha^i)$. We know that if $f'(x) = x^{deg(f)} f(x^{-1})$ is equal to $f(x)$, then $f(x)$ is palindromic. To show this:

$$\begin{aligned} f'(x) &= x^{\deg(f)} f(x^{-1}) \\ &= x^{\deg(f)} \prod_{i \in Q_0} (x^{-1} - \alpha^i) = x^{\deg(f)} \prod_{i \in Q_0} (x^{-1} - \alpha^{-i}) \quad \text{(by Lemma 7(i))} \\ &= x^{\deg(f)} \prod_{i \in Q_0} (\frac{\alpha^i - x}{x\alpha^i}) = \frac{x^{\deg(f)}}{x^{\deg(f)}} \prod_{i \in Q_0} (\frac{\alpha^i - x}{\alpha^i}) \\ &= \frac{1}{\alpha^{\sum_{i \in Q_0} i}} \prod_{i \in Q_0} (\alpha^i - x) = \frac{1}{\alpha^0} \prod_{i \in Q_0} (x - \alpha^i) \quad \text{(by Lemma 7(ii))} \\ &= f(x). \end{aligned}$$

□

Proposition 3.9. *Let p be a prime and q be a power of 2. Assume that $m \in Z^+$ such that $m|(p-1)$ and $a, q \in Q_0$. If e_i and e_j are idempotent generators of m-adic residue codes of length p over F_q, then $v(\sum\limits_{S \subseteq I, i \in S} e_i) + (1-v)(\sum\limits_{P \subseteq I, j \in P} e_j)$'s are idempotents in the ring $(F_q[v]/(v^2 - v))[x]/(x^p - 1)$.*

Proof. Since q is a power of 2, the characteristic of $(F_q[v]/(v^2 - v))[x]/(x^p - 1)$ is 2. Let I be an index set and $S, P \subseteq I$, then

$$\begin{aligned} v(\sum_{i \in S} e_i) + (1-v)(\sum_{j \in P} e_j)^2 &= (v(\sum_{i \in S} e_i))^2 + ((1-v)(\sum_{j \in P} e_j))^2 \\ &= v(\sum_{i \in S} e_i)^2 + (1-v)(\sum_{j \in P} e_j)^2 \end{aligned}$$

$$= v(\sum_{i\in S} e_i) + (1-v)(\sum_{j\in P} e_j).$$
□

Example 3.10. If we choose $E_0 = v(e_0+e_1)+(1-v)(e_1+e_2)$ as in the previous example, then $E_1 = v(e_3+e_0)+(1-v)(e_0+e_1)$, $E_2 = v(e_2+e_3)+(1-v)(e_3+e_0)$ and $E_3 = v(e_1+e_2)+(1-v)(e_2+e_3)$.

Let $g_i(x)$ be the generator polynomials of a code generated by the idempotent E_i. Then

$$\begin{aligned}
g_0(x) &= 1 + w^2x + (v+w)x^2 + (vw+w)x^3 + vwx^4 + vwx^5 + (vw+w)x^6 \\
&\quad + (v+w)x^7 + w^2x^8 + x^9, \\
g_1(x) &= 1 + (v+w^2)x + w^2x^2 + vw^2x^3 + (vw+w)x^4 + (vw+w)x^5 + vw^2x^6 \\
&\quad + w^2x^7 + (v+w^2)x^8 + x^9, \\
g_2(x) &= 1 + wx + (v+w^2)x^2 + (vw^2+w^2)x^3 + vw^2x^4 + vw^2x^5 \\
&\quad + (vw^2+w^2)x^6 + (v+w^2)x^7 + wx^8 + x^9, \text{ and} \\
g_3(x) &= 1 + (v+w)x + wx^2 + vwx^3 + (vw^2+w^2)x^4 + (vw^2+w^2)x^5 \\
&\quad + vwx^6 + wx^7 + (v+w)x^8 + x^9.
\end{aligned}$$

All these polynomials generate the same parameter $[17, 8, 8]$ codes and these parameters attain the Griesmer bound given in [12]. So these codes are all optimal.

Theorem 3.11. *Generator polynomials of m-adic residue codes over the ring $F_{2^k}[v]/(v^2-v)$ of length p are palindromic if $(p-1)/m$ is even where $k \geq 1 \in Z$.*

Proof. Let I be an index set, $S, P \subseteq I$ and $v(\sum_{i\in S} e_i) + (1-v)(\sum_{j\in P} e_j)$ be an idempotent generator of an m-adic residue codes over the ring $F_{2^k}[v]/(v^2-v)$ of length p, $(p-1)/m$ be even and $k \geq 1 \in Z$. Let f_i and f_j be generator polynomials of m-adic residue codes over F_{2^k} corresponding to $\sum_{i\in S} e_i$ and $\sum_{j\in P} e_j$ respectively. From now on, we will denote $\sum_{i\in S} e_i$ and $\sum_{j\in P} e_j$ by $\sum e_i$ and $\sum e_j$ respectively. Then $gcd(\sum e_i, x^n-1) = f_i$ and $gcd(\sum e_j, x^n-1) = f_j$. There exists $a, b, c, d \in F_{2^k}[x]$ such that $a\sum e_i + b(x^n-1) = f_i$ and $c\sum e_i + d(x^n-1) = f_j$.

Let f be the generator polynomial corresponding to codes generated by the idempotents $v(\sum_{i\in S} e_i)+(1-v)(\sum_{j\in P} e_j)$. Then $gcd(v(\sum_{i\in S} e_i)+(1-v)(\sum_{j\in P} e_j), x^n-1) = f$.

We claim that $f = vf_i + (1-v)f_j$.

$$\begin{aligned}
&(va + (1-v)c)(v\sum e_i + (1-v)\sum e_j) + (vb + (1-v)d)(x^n-1) \\
&= (va\sum e_i + (1-v)c\sum e_j) + (vb + (1-v)d(x^n-1))
\end{aligned}$$

$$= v(a\sum e_i + b(x^n-1)) + (1-v)(c\sum e_j + d(x^n-1))$$
$$= vf_i + (1-v)f_j.$$

Since s $f = gcd(v(\sum_{i\in S} e_i) + (1-v)(\sum_{j\in P} e_j), x^n-1)$ and $vf_i + (1-v)f_j$ is a linear combination $v(\sum_{i\in S} e_i)+(1-v)(\sum_{j\in P} e_j)$ and x^n-1, f divides $vf_i+(1-v)f_j$.

On the other hand, since $f_i|\sum e_i$ and $f_j|\sum e_j$, then there exists $t, s \in F_{2^k}[x]$ such that $tf_i = \sum e_i$ and $sf_j = \sum e_j$. Thus,

$$\begin{aligned}(tv + s(1-v))(vf_i + (1-v)f_j) &= vtf_i + (1-v)sf_j \\ &= v\sum e_i + (1-v)\sum e_j.\end{aligned}$$

Hence,

$$(vf_i + (1-v)f_k)|(v\sum e_i + (1-v)\sum e_j). \tag{1}$$

In addition, since $f_i|(x^n-1)$ and $f_i|\sum e_i$, then there exists $k, m \in F_{2^k}[x]$ such that $kf_i = x^n-1$ and $mf_j = x^n-1$. Thus,

$$(kv + m(1-v))(vf_i + (1-v)f_j) = x^n - 1.$$

Then

$$(vf_i + (1-v)f_k)|(x^n-1). \tag{2}$$

$(vf_i+(1-v)f_k)|f$, i.e., $gcd(v(\sum_{i\in S} e_i)+(1-v)(\sum_{j\in P} e_j), x^n-1) = (vf_i+(1-v)f_k)$ is obtained by (1) and (2).

Since f_i and f_j are palindromic, $vf_i + (1-v)f_k$ is also palindromic. □

4. Reversible DNA codes over $F_{4^{2k}}[v]/(v^2-v)$

In this section, we utilize the results obtained above for a special family of chain rings where $q = 4^{2k}$. We present a general theorem for reversible DNA codes over $R_{2k} = F_{4^{2k}}[v]/(v^2-v)$ with palindromic factors and apply these for generators of m-adic residue codes. A general form of a ψ-set with an automorphism has been introduced recently by the authors for solving the reversibility problem for DNA codes over R_{2k}. ψ-set has been introduced in [3] over $F_4[v]/(v^2-v)$.

In order to explain the reversibility problem, we give a concrete example first. Let (a_1, a_2, a_3) be a codeword where a_i's are elements of R_4 corresponding to a DNA string ATGGCTGATGAG (a 12-string) where the matching is given by $a_1 \to$ATGG, $a_2 \to$CTGA, $\alpha_3 \to$TGAG. The reverse of (a_1, a_2, a_3) is clearly equal to (a_3, a_2, a_1). (a_3, a_2, a_1) corresponds to TGAGCTGAATGG. However,TGAGCTGAATGG is not the reverse of ATGGCTGATGAG. Indeed, the reverse of ATGGCTGATGAG is GAGTAGTCGGTA. Whenever we have a ring element matched with non-single letters this problem occurs. Hence, for a code being reversible over ring elements does not necessarily mean that is going to produce reversibility over DNA letters which is crucial for a DNA code.

TABLE 1. The θ mapping between DNA pairs and F_{16} ([10]).

F_{16}(multiplicative)	F_{16}(additive)	Double DNA pair
0	0	AA
α^0	1	TT
α^1	α	AT
α^2	α^2	GC
α^3	α^3	AG
α^4	$1+\alpha$	TA
α^5	$\alpha+\alpha^2$	CC
α^6	$\alpha^2+\alpha^3$	AC
α^7	$1+\alpha+\alpha^3$	GT
α^8	$1+\alpha^2$	CG
α^9	$\alpha+\alpha^3$	CA
α^{10}	$1+\alpha+\alpha^2$	GG
α^{11}	$\alpha+\alpha^2+\alpha^3$	CT
α^{12}	$1+\alpha+\alpha^2+\alpha^3$	GA
α^{13}	$1+\alpha^2+\alpha^3$	TG
α^{14}	$1+\alpha^3$	TC

This section is focused on defining codes over R_{2k} that enjoy this reversibility property whenever they are mapped to DNA strings. Let C be a subset of $\{A,C,T,G\}^n$. If both elements and their reverses are in C, then C is called a reversible DNA code. In order to obtain a reversible DNA code from codes over rings, the main tools are to be able to discover specific generators and a Gray map that transforms codes over rings to DNA strings with reversible property. I the sequel, we study these two problems over R_{2k}.

R_{2k} is a commutative non-chain ring with $v^2=v$. By Chinese Remainder Theorem we can then decompose R_{2k} as follows: $R_{2k}=vF_{4^{2k}}\oplus(1-v)F_{4^{2k}}$. We define a Gray map;

$$\begin{aligned}\phi : R_{2k} &\to F_{4^{2k}}^2\\ a+vb &\to (a+b,a).\end{aligned}\tag{3}$$

θ is used to transform elements of $F_{4^{2k}}$ to DNA strings of lengths $2k$ as in Tables given in [10,11]. Especially, the DNA table for F_{16} originally introduced in 1 is also presented here table 4. θ_1 is used to convert the elements of the R_{2k} to DNA strings of lengths $4k$. Let $a+vb$ be an element in R_{2k}. $\theta_1(a+vb)=(\theta(a+b),\theta(a))$. Θ is used to convert a codeword to a DNA string. Let $c=(c_0,c_1,\ldots,c_{n-1})$ and $\Theta(c)=(\theta_1(c_0),\theta_1(c_1),\ldots,\theta_1(c_{n-1}))$.

Example 4.1. Let $\beta=\alpha^3+\alpha^6v\in R_2$ and $\phi(\beta)=(\alpha^2,\alpha^3)$. Then, $\theta_1(\beta)=(\theta(\alpha^2),\theta(\alpha^3))=GCAG$.

Example 4.2. Let $c = (\alpha^3+\alpha^6 v, \alpha^3+\alpha^9 v)$ be a codeword of a code. Then, $\phi(\alpha^3+\alpha^6 v) = (\alpha^2, \alpha^3)$ and $\phi(\alpha^3+\alpha^9 v) = (\alpha, \alpha^3)$. $\Theta(c) = (\theta_1(\alpha^3+\alpha^6 v), \theta_1(\alpha^3+\alpha^9 v)) = (\theta(\alpha^2), \theta(\alpha^3), \theta(\alpha), \theta(\alpha^3)) = GCAGATAG$.

We introduce a new automorphism over R_{2k} such that it helps to obtain the DNA reverse of an element in R_{2k};

$$\begin{aligned} &\psi : R_{2k} \to R_{2k} \\ &a+vb \to a^{4^k} + (1+v)b^{4^k} = (a+b)^{4^k} + vb^{4^k}. \end{aligned} \tag{4}$$

Example 4.3. Let $\beta = \alpha^3+\alpha^6 v \in R_2$ and $\theta_1(\beta) = GCAG$ then $\psi(\beta) = \alpha^{12} + \alpha^9(v-1) = \alpha^8 + v\alpha^9$ and $\theta_1(\psi(\beta)) = \theta_1(\alpha^8+v\alpha^9) = (\theta(\alpha^{12}), \theta(\alpha^8)) = GACG$.

Definition. Let $g(x)$ be a polynomial of degree $\deg g(x) = t$ over R. Let C be a linear code over R, generated by a set Λ_g called the ψ set defined by

$$\Lambda_g = \{\Lambda_0, \Lambda_1, \dots, \Lambda_{t-1}\}$$

where

$$\Lambda_i = \begin{cases} x^i g(x), & \text{if } i \text{ is even,} \\ x^i \psi(g(x)), & \text{if } i \text{ is odd.} \end{cases}$$

Theorem 4.4. *Let $f(x)$ be a palindromic factor of x^n-1 over $F_{4^{2k}}[v]/(v^2-v)$ where $n-\deg(f(x))$ is even. If ψ-set generates a linear code C then $\Theta(C)$ is a reversible DNA codes.*

Proof. Proof is similar in [3] and Theorem 4.5. □

The following theorem can be presented as a consequence of properties of the idempotent generators of m-adic residue codes.

Theorem 4.5. *Let $g(x)$ be a generator polynomial of an m-adic residue code over $F_{4^{2k}}[v]/(v^2-v)$ where $\deg(g(x))$ is odd. If a ψ-set generates the linear code C, then $\Theta(C)$ is a reversible DNA code.*

Proof. Let $g(x)$ be a generator polynomial of an m-adic residue code then it is a palindromic polynomial by Theorem 3.11. Let Λ_g be a ψ set of $g(x)$. Reverses of DNA codewords $\psi(c)$, for all $c \in C$, are obtained by the following equation:

$$\Theta\left(\sum_{i=0}^{k-1} \beta_i \Lambda_i\right)^r = \Theta\left(\sum_{i=0}^{k-1} \psi(\beta_i)\Lambda_{k-1-i}\right), \tag{5}$$

where $k = n - deg(g(x))$ and $\beta_i \in R_{2k}$. Note that, the polynomial is considered as a codeword. Since $\sum_i \psi(\beta_i)\Lambda_{k-1-i} \in C$, then $\Theta(C)$ is a reversible DNA code. □

Example 4.6. $g(x) = x^{13} + x^{12} + (v+(\alpha^2+\alpha))x^{11} + x^9 + (v+(\alpha^2+\alpha))x^8 + (v+(\alpha^2+\alpha+1))x^7 + (v+(\alpha^2+\alpha+1))x^6 + (v+(\alpha^2+\alpha))x^5 + x^4 + (v+(\alpha^2+\alpha))x^2 + x + 1$ is a generator polynomial of an m-adic residue code length of 17 over $F_{16}[v]/(v^2-v)$. The ψ set of $g(x)$ is as follows:

$\{\Lambda_0,\Lambda_1,\Lambda_2,\Lambda_3\} = \{\{1,1,v+\alpha^2+\alpha,0,1,v+\alpha^2+\alpha,v+\alpha^2+\alpha+1,v+\alpha^2+\alpha+1,v+\alpha^2+\alpha,1,0,v+\alpha^2+\alpha,1,1,0,0,0\}$, $\{0,1,1,v+\alpha^2+\alpha+1,0,1,v+\alpha^2+\alpha+1,v+\alpha^2+\alpha,v+\alpha^2+\alpha,v+\alpha^2+\alpha+1,1,0,v+\alpha^2+\alpha+1,1,1,0,0\},\{0,0,1,1,v+\alpha^2+\alpha,0,1,v+\alpha^2+\alpha,v+\alpha^2+\alpha+1,v+\alpha^2+\alpha+1,v+\alpha^2+\alpha,1,0,v+\alpha^2+\alpha,1,1,0\},\{0,0,0,1,1,v+\alpha^2+\alpha+1,0,1,v+\alpha^2+\alpha+1,v+\alpha^2+\alpha,v+\alpha^2+\alpha,v+\alpha^2+\alpha+1,1,0,v+\alpha^2+\alpha+1,1,1\}\}$.

The parameters of C are $[17,4,12]$. Let us choose a codeword and investigate its corresponding DNA form.

Let $c_1 = \alpha\Lambda_0 = \{\alpha,\alpha,\alpha v+\alpha^3+\alpha^2,0,\alpha,\alpha v+\alpha^3+\alpha^2,\alpha v+\alpha^3+\alpha^2+\alpha,\alpha v+\alpha^3+\alpha^2+\alpha,\alpha v+\alpha^3+\alpha^2,\alpha,0,\alpha+\alpha^3+\alpha^2,\alpha,\alpha,0,0,0\}$ be a codeword where $\alpha v+\alpha^3+\alpha^2 = \alpha^6+\alpha v \to (\alpha^{11},\alpha^6) \to CTAC$, $\alpha v+\alpha^3+\alpha^2+\alpha \to (\alpha^6,\alpha^{11}) \to ACCT$, $\alpha \to (\alpha,\alpha) \to TTTT$.

Then DNA corresponding of c_1 is

$$\begin{aligned}\Theta(c_1) = \{&ATAT\ ATAT\ CTAC\ AAAA\ ATAT\ CTAC\ ACCT\ ACCT\ CTAC\\ &ATAT\ AAAA\ CTAC\ ATAT\ ATAT\ AAAA\ AAAA\ AAAA\}.\end{aligned}$$

The reversible DNA corresponding of c_1 $(\Theta(c_1)^r)$ is obtained according to Equation 5 as follows:

$$\begin{aligned}\Theta(c_1)^r &= \Theta(\psi(\alpha)\Lambda_3) = \Theta(\alpha^4\Lambda_3)\\ &= \Theta(\alpha^4 \times \{0,0,0,1,1,v+\alpha^2+\alpha+1,0,1,v+\alpha^2+\alpha+1,v+\alpha^2+\alpha,\\ &\quad v+\alpha^2+\alpha,v+\alpha^2+\alpha+1,1,0,v+\alpha^2+\alpha+1,1,1\})\\ &= \Theta(0,0,0,\alpha^4,\alpha^4,\alpha^{14}+v\alpha^4,0,\alpha^4,\alpha^{14}+v\alpha^4,\alpha^9+v\alpha^4,\alpha^9+v\alpha^4,\\ &\quad \alpha^{14}+v\alpha^4,\alpha^4,0,\alpha^{14}+v\alpha^4,\alpha^4,\alpha^4)\\ &= \{AAAA\ AAAA\ AAAA\ TATA\ TATA\ CATC\ AAAA\\ &\quad TATA\ CATC\ TCCA\ TCCA\ CATC\ TATA\ AAAA\\ &\quad CATC\ TATA\ TATA\}.\end{aligned}$$

5. Conclusion

Here we define and construct m-adic residue codes over the non-chain ring $F_q[v]/(v^2-v)$ unlike previous papers which work over finite fields and also have more restrictions on their parameters. We obtain some optimal codes with respect to Griesmer bound [12] which also makes this study even more interesting. We also apply these results for constructing DNA codes for which palindromic polynomials together with a specific automorphism map ψ over non-chain rings are shown to be more suitable. An obstacle which is an open problem is to find palindromic polynomials over these rings. This problem and applications to different non-chain rings are future studies on this direction. Another and the most challenging problem is to find a good match between DNA codes obtained from algebraic structures and real DNA data.

Acknowledgment. The authors wish to express their thanks to the reviewers and their enlightening remarks that led to an improved version of our paper.

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