

EXISTENCE AND UNIQUENESS THEOREMS OF SECOND-ORDER EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we consider the second-order nonlinear differential equation with the nonlocal boundary conditions. We first reformulate this boundary value problem as a fixed point problem for a Fredholm integral equation operator, and then present a result on the existence and uniqueness of the solution by using the contraction mapping theorem. Furthermore, we establish a sufficient condition on the functions μ and h_i , $i = 1, 2$ that guarantee a unique solution for this nonlocal problem in a Hilbert space. Also, accurate analytic solutions in series forms for this boundary value problems are obtained by the Adomian decomposition method (ADM).

1. Introduction

The theory of boundary-value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions [8–14, 17–23]. This problem is used in different areas of physics, engineering and mathematics such as plate deflection theory.

Many authors have studied the second-order nonlinear differential equation under various boundary conditions and by different approaches [9–11, 20–23] and the references therein.

Consider the second-order boundary value problem with integral boundary conditions

$$(1) \quad -u''(x) = f(x, u(x)), \quad 0 < x < 1,$$

$$(2) \quad u(0) = \int_0^1 h_1(x)u(x)dx, \quad u(1) = \int_0^1 h_2(x)u(x)dx,$$

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where $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function and $h_i : [0, 1] \rightarrow \mathbb{R}$, $i = 1, 2$ are integrable functions on $[0, 1]$.

In this paper, we first reformulate this boundary value problem as a fixed point problem for a Fredholm integral equation operator, and then present a result on the existence and uniqueness of the solution by using the contraction mapping theorem. Furthermore, we establish a sufficient condition on the functions μ and h_i , $i = 1, 2$ that guarantee a unique solution for Pr. (1)-(2) in a Hilbert space. Also, the existence and accurate analytic solutions in series forms for this boundary value problems are obtained by the Adomian decomposition method (ADM) [1–7, 15, 16].

2. The existence and uniqueness theorem

We will use the following Lemma to reformulate the transformed boundary value problem Pr. (1)-(2) as a fixed point problem for Fredholm integral equation.

Lemma 2.1. *Let $g : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. The unique solution u of the following boundary value problem with nonlocal boundary conditions*

$$(3) \quad \begin{cases} -u''(x) = g(x), \\ u(0) = \int_0^1 h_1(x)u(x)dx \text{ and } u(1) = \int_0^1 h_2(x)u(x)dx \end{cases}$$

is given by

$$(4) \quad u(x) = \int_0^1 G_1(x, y)g(y)dy,$$

where

$$(5) \quad G_1(x, y) = G(x, y) + H_1 + H_2 + (1-x)H_3 + (1-x)H_4,$$

where

$$(6) \quad G(x, y) = \begin{cases} x(1-y), & 0 \leq x \leq y \leq 1, \\ y(1-x), & 0 \leq y \leq x \leq 1, \end{cases}$$

$$(7) \quad H_1 = \frac{[1 - \int_0^1 sh_2(s)ds] \int_0^1 h_1(s)[\int_0^1 G(s, y)g(y)dy]ds}{\kappa},$$

$$(8) \quad H_2 = \frac{\int_0^1 sh_1(s)ds \int_0^1 h_2(s)[\int_0^1 G(s, y)g(y)dy]ds}{\kappa},$$

$$(9) \quad H_3 = \frac{(1 - \int_0^1 (1-s)h_1(s)ds) \int_0^1 h_2(s)[\int_0^1 G(s, y)g(y)dy]ds}{\kappa},$$

$$(10) \quad H_4 = \frac{\int_0^1 sh_1(s)ds \int_0^1 h_2(s)[\int_0^1 G(s, y)g(y)dy]ds}{\kappa}$$

and

(11)

$$\kappa = [1 - \int_0^1 (1-s)h_1(s)ds][1 - \int_0^1 sh_2(s)ds] - \int_0^1 sh_1(s)ds \int_0^1 (1-s)h_2(s)ds \neq 0.$$

Proof. Integrating the first equation of (3) twice, we obtain

$$(12) \quad u(x) = C_1x + C_2 - \int_0^x \int_1^y g(s)dsdy,$$

where $C_1 = u'(1)$ is a constant to be determined later and $C_2 = u(0)$. Integration by parts of the integral with respect to y in Eq. (12) gives

$$(13) \quad u(x) = C_1x + C_2 + x \int_x^1 g(y)dy + \int_0^x yg(y)dy.$$

We determine C_1 and C_2 from the nonlocal boundary conditions (2), we obtain

$$(14) \quad C_1 = - \int_0^1 yg(y)dy + \int_0^1 [h_2(y) - h_1(y)] u(y)dy$$

and

$$(15) \quad C_2 = \int_0^1 h_1(y)u(y)dy.$$

Substituting Eqs. (14)-(15) into Eq. (13) we obtain

$$(16) \quad \begin{aligned} u(x) = & \left(- \int_0^1 yg(y)dy + \int_0^1 (h_2(y) - h_1(y))u(y) \right) x + \int_0^1 h_1(y)u(y)dy \\ & + x \int_x^1 g(y)dy + \int_0^x yg(y)dy. \end{aligned}$$

It follows that

$$(17) \quad \begin{aligned} u(x) = & \int_0^x (1-x)yg(y)dy + \int_x^1 x(1-y)g(y)dy \\ & + \int_0^1 [xh_2(y) + (1-x)h_1(y)] u(y)dy, \end{aligned}$$

that is,

$$(18) \quad u(x) = \int_0^1 G(x,y)g(y)dy + \int_0^1 [xh_2(y) + (1-x)h_1(y)] u(y)dy.$$

Multiplying both sides of Eq. (18) by h_1 and integrate over $[0, 1]$, we obtain

$$(19) \quad \begin{aligned} \int_0^1 h_1(s)u(s)ds = & \int_0^1 h_1(s) \left[\int_0^1 G(s,y)g(y)dy \right] ds \\ & + \int_0^1 h_1(s) \left[\int_0^1 [sh_2(y) + (1-s)h_1(y)] u(y)dy \right] ds. \end{aligned}$$

Thus

$$(20) \quad \begin{aligned} & \left(1 - \int_0^1 (1-s)h_1(s)ds\right) \int_0^1 h_1(s)u(s)ds \\ &= \int_0^1 h_1(s) \left[\int_0^1 G(s,y)g(y)dy \right] ds + \int_0^1 yh_1(y)dy \int_0^1 h_2(s)u(s)ds. \end{aligned}$$

Similarly, multiplying both sides of Eq. (18) by h_2 and integrate over $[0, 1]$, we obtain

$$(21) \quad \begin{aligned} & \left(1 - \int_0^1 sh_2(s)ds\right) \int_0^1 h_2(s)u(s)ds \\ &= \int_0^1 h_2(s) \left[\int_0^1 G(s,y)g(y)dy \right] ds + \int_0^1 (1-y)h_2(y)dy \int_0^1 h_1(s)u(s)ds. \end{aligned}$$

Solving Eq. (20) and Eq. (21) for $\int_0^1 h_1(s)u(s)ds$ and $\int_0^1 h_2(s)u(s)ds$, to get

$$(22) \quad \begin{aligned} \int_0^1 h_1(s)u(s)ds &= \frac{[1 - \int_0^1 sh_2(s)ds] \int_0^1 h_1(s) [\int_0^1 G(s,y)g(y)dy] ds}{[1 - \int_0^1 (1-s)h_1(s)ds][1 - \int_0^1 sh_2(s)ds] - \int_0^1 sh_1(s)ds \int_0^1 (1-s)h_2(s)ds} \\ &+ \frac{\int_0^1 sh_1(s)ds \int_0^1 h_2(s) [\int_0^1 G(s,y)g(y)dy] ds}{[1 - \int_0^1 (1-s)h_1(s)ds][1 - \int_0^1 sh_2(s)ds] - \int_0^1 sh_1(s)ds \int_0^1 (1-s)h_2(s)ds} \end{aligned}$$

and

$$(23) \quad \begin{aligned} \int_0^1 h_2(s)u(s)ds &= \frac{(1 - \int_0^1 (1-s)h_1(s)ds) \int_0^1 h_2(s) [\int_0^1 G(s,y)g(y)dy] ds}{[1 - \int_0^1 (1-s)h_1(s)ds][1 - \int_0^1 sh_2(s)ds] - \int_0^1 sh_1(s)ds \int_0^1 (1-s)h_2(s)ds} \\ &+ \frac{\int_0^1 (1-s)h_2(s)ds \int_0^1 h_1(s) [\int_0^1 G(s,y)g(y)dy] ds}{[1 - \int_0^1 (1-s)h_1(s)ds][1 - \int_0^1 sh_2(s)ds] - \int_0^1 sh_1(s)ds \int_0^1 (1-s)h_2(s)ds}. \end{aligned}$$

Substituting these into Eq. (18), we obtain

$$\begin{aligned} u(x) &= \int_0^1 G(x,y)g(y)dy \\ &+ \frac{(1 - \int_0^1 (1-s)h_1(s)ds) \int_0^1 h_2(s) [\int_0^1 G(s,y)g(y)dy] ds}{[1 - \int_0^1 (1-s)h_1(s)ds][1 - \int_0^1 sh_2(s)ds] - \int_0^1 sh_1(s)ds \int_0^1 (1-s)h_2(s)ds} \\ &+ \frac{\int_0^1 (1-s)h_2(s)ds \int_0^1 h_1(s) [\int_0^1 G(s,y)g(y)dy] ds}{[1 - \int_0^1 (1-s)h_1(s)ds][1 - \int_0^1 sh_2(s)ds] - \int_0^1 sh_1(s)ds \int_0^1 (1-s)h_2(s)ds} \\ &+ (1-x) \frac{[1 - \int_0^1 sh_2(s)ds] \int_0^1 h_1(s) [\int_0^1 G(s,y)g(y)dy] ds}{[1 - \int_0^1 (1-s)h_1(s)ds][1 - \int_0^1 sh_2(s)ds] - \int_0^1 sh_1(s)ds \int_0^1 (1-s)h_2(s)ds} \\ &+ (1-x) \frac{\int_0^1 sh_1(s)ds \int_0^1 h_2(s) [\int_0^1 G(s,y)g(y)dy] ds}{[1 - \int_0^1 (1-s)h_1(s)ds][1 - \int_0^1 sh_2(s)ds] - \int_0^1 sh_1(s)ds \int_0^1 (1-s)h_2(s)ds}, \end{aligned}$$

which is what we had to prove. \square

Lemma 2.2. *The nonlocal boundary value problem Pr. (1)-(2) can be written as a Fredholm integral equation for u ,*

$$(24) \quad u = \int_0^1 G_1(x,y)f(y,u(y))dy.$$

Proof. Replacing $g(x)$ by $f(x, u(x))$ in Lemma 2.1, we obtain Eq. (24). \square

Theorem 2.3. *Suppose that $f(x, u)$ is continuous on $[0, 1] \times \mathbb{R}$ and there is a continuous function $\mu : [0, 1] \rightarrow \mathbb{R}^+$ such that*

$$(25) \quad |f(x, y) - f(x, z)| \leq \mu(x) |y - z|, \forall y, z \in \mathbb{R}, x \in [0, 1].$$

Moreover, we assume

$$(26) \quad \sup_{0 \leq x \leq 1} \int_0^1 |G_1(x, y)| \mu(y) dy < 1,$$

where $G_1(x, y)$ is defined by (5). Then the nonlocal boundary value problem Pr. (1)-(2) has a unique solution.

Proof. The integral equation (24) may be written as a fixed point equation $T(u) = u$, where the map T is defined by

$$(27) \quad Tu(x) = \int_0^1 G_1(x, y) f(y, u(y)) dy.$$

We show T is contraction, since for any $v, w \in C([0, 1])$, we have

$$(28) \quad \begin{aligned} \|Tv - Tw\|_\infty &= \sup_{0 \leq x \leq 1} \left| \int_0^1 G_1(x, y) [f(y, v(y)) - f(y, w(y))] dy \right| \\ &\leq \sup_{0 \leq x \leq 1} \int_0^1 |G_1(x, y)| |f(y, v(y)) - f(y, w(y))| dy \\ &\leq \|u - v\|_\infty \sup_{0 \leq x \leq 1} \int_0^1 |G_1(x, y) \mu(y)| dy \\ &\leq c \|v - w\|_\infty, \end{aligned}$$

where

$$(29) \quad c = \sup_{0 \leq x \leq 1} \int_0^1 |G_1(x, y)| \mu(y) dy < 1.$$

Thus T is a contraction and from the contraction mapping theorem, T has a unique fixed point, i.e., there exists a unique $u \in C([0, 1])$ such that $Tu = u$. This fixed point u is a unique solution of this problem. \square

3. The uniqueness solution in an Hilbert space

The generalization of the result on the uniqueness of the solution of Pr. (1)-(2) can be obtained in the Hilbert space U defined as

$$(30) \quad U = \left\{ u : u, \sqrt{x(1-x)} \frac{du}{dx}, \frac{d^2u}{dx^2} \in L_2(0, 1) \right\}$$

with respect to the norm

$$(31) \quad \|u\|_U^2 = \int_0^1 \left[u^2 + x(1-x) \left(\frac{du}{dx} \right)^2 + \left(\frac{d^2u}{dx^2} \right)^2 \right] dx < \infty.$$

Suppose there are two solutions u and v such that $u \neq v$. Then from Eq. (1) with (2), we have

$$(32) \quad -w'' = f(x, u) - f(x, v), \quad 0 < x < 1$$

subject to

$$(33) \quad w(0) = \int_0^1 h_1(x)w(x)dx, \quad w(1) = \int_0^1 h_2(x)w(x)dx,$$

where $w = u - v$.

Multiplying both sides of Eq. (32) by $\varphi(x)w$, where $\varphi(x) = x(x-1)$. Employing integration by parts, we obtain

$$(34) \quad \int_0^1 \varphi''(x)w^2(x)dx - 2 \int_0^1 \varphi'(x)w'^2(x)dx \\ = \varphi'(1)w^2(1) - \varphi'(0)w^2(0) - 2 \int_0^1 g(x, u, v)\varphi(x)w(x)dx,$$

where $g(x, u, v) = f(x, u) - f(x, v)$ and $|g(x, u, v)| \leq \mu(x)|w|$.

Replacing $\varphi(x) = x(x-1)$, $\varphi'(x) = 2x-1$ and $\varphi''(x) = 2$ into Eq. (34) and taking into account the nonlocal boundary conditions (2), we get

$$(35) \quad 2 \int_0^1 w^2(x)dx + 2 \int_0^1 x(1-x)w'^2(x)dx \\ = \left(\int_0^1 h_1(x)w(x)dx \right)^2 + \left(\int_0^1 h_2(x)w(x)dx \right)^2 + 2 \int_0^1 g(x, u, v)x(1-x)w(x)dx.$$

From the Holder inequality, we have

$$(36) \quad \left(\int_0^1 h_i(x)w(x)dx \right)^2 \leq \int_0^1 h_i^2(x)dx \int_0^1 w^2(x)dx, \quad i = 1, 2.$$

Since $\max_{0 \leq x \leq 1} x(1-x) = \frac{1}{4}$, Eq. (35) becomes

$$(37) \quad 2 \int_0^1 w^2(x)dx + 2 \int_0^1 x(1-x)w'^2(x)dx \leq (\|h_1\|^2 + \|h_2\|^2) \int_0^1 w^2(x)dx \\ + \frac{1}{4} \int_0^1 \mu(x)w^2(x)dx,$$

where $\|h_i\|^2 = \int_0^1 h_i^2(x)dx$, $i = 1, 2$.

If we assume that $\mu(x) \leq \mu_0, \forall x \in [0, 1]$. Then,

$$(38) \quad \left(2 - \frac{\mu_0}{4} - (\|h_1\|^2 + \|h_2\|^2) \right) \int_0^1 w^2(x)dx + 2 \int_0^1 x(1-x)w'^2(x)dx \leq 0.$$

Multiplying now both sides of Eq. (32) by w'' , we obtain

$$(39) \quad \int_0^1 w''^2(x)dx = - \int_0^1 g(x, u, v)w''(x)dx.$$

The term $\int_0^1 g(x, u, v)w''(x)dx$ can be estimated by means of the Cauchy-Schwarz-Bunyakovski inequality and the ϵ -inequality

$$(40) \quad 2 | uv | \leq \epsilon u^2 + \frac{1}{\epsilon} v^2, \quad \epsilon > 0.$$

Thus

$$(41) \quad \int_0^1 w''^2(x)dx \leq \frac{1}{2\epsilon_1} \int_0^1 |g(x, u, v)|^2 dx + \frac{\epsilon_1}{2} \int_0^1 w''^2(x)dx, \quad \epsilon_1 > 0.$$

Therefore

$$(42) \quad \left(1 - \frac{\epsilon_1}{2}\right) \int_0^1 w''^2(x)dx \leq \frac{1}{2\epsilon_1} \int_0^1 \mu^2(x)w^2(x)dx, \quad \epsilon_1 > 0.$$

If we choose $\epsilon_1 = 1$, then

$$(43) \quad \int_0^1 w''^2(x)dx \leq \mu_0^2 \int_0^1 w^2(x)dx.$$

Adding side to side Eq. (38) and Eq. (43), we obtain

$$(44) \quad \left(2 - \frac{\mu_0}{4} - \mu_0^2 - (\|h_1\|^2 + \|h_2\|^2)\right) \int_0^1 w^2(x)dx + 2 \int_0^1 x(1-x)w'^2(x) + \int_0^1 w''^2(x)dx \leq 0.$$

Choosing

$$(45) \quad \frac{\mu_0}{4} + \mu_0^2 + \|h_1\|^2 + \|h_2\|^2 < 2.$$

Then

$$(46) \quad \int_0^1 \left[w^2 + x(1-x) \left(\frac{dw}{dx} \right)^2 + \left(\frac{d^2w}{dx^2} \right)^2 \right] dx \leq 0,$$

that is

$$(47) \quad \|w\|_U \leq 0.$$

This is a contradiction.

We have proved the following statement.

Theorem 3.1. *Under the hypotheses of Theorem 2.3, and if we assume $\mu(x) \leq \mu_0, \forall x \in [0, 1]$ with (45). Then Pr. (1)-(2) has a unique solution in the Hilbert space U .*

4. Approximate analytic solution by the Adomian decomposition method (ADM)

We propose here to solve this nonlocal boundary value problem by the Adomian decomposition method (ADM) [1–7, 15, 16].

Let us consider the case $f(x, u(x)) = \mu(x)f(u(x))$ and assume $f(0) \neq 0$. We rewrite the equivalent Eq. (24) in Adomian's operator-theoretic notation as

$$(48) \quad Lu = Ru + Nu,$$

where

$$(49) \quad Lu = u, \quad Ru = 0 \quad \text{and} \quad Nu = \int_0^1 G_1(x, y)\mu(x)f(u(y))dy.$$

Define the solutions $u(x)$ by its respective infinite series of components in the form

$$(50) \quad u(x) = \sum_{n=0}^{\infty} u_n(x)$$

and the infinite series

$$(51) \quad N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n)$$

for the nonlinear term $N(u)$, where the A_n are the Adomian polynomials [1–3, 15, 16], which can be obtained from the definitional formula

$$(52) \quad A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[f \left(\sum_{i=0}^n \lambda^i u_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots$$

Consequently, the components u_n can be elegantly determined by setting the following recursion scheme

$$(53) \quad \begin{cases} u_0 = 0, \\ u_{n+1} = \int_0^1 G_1(x, y)\mu(x)A_n(y)dy, \quad n \geq 0. \end{cases}$$

Let $\Phi_n(x) = \sum_{i=0}^{n-1} u_i(x)$ be the n th-stage approximation functions of $u(x)$ by the ADM for the nonlinear Eq. (24).

By substitution the recursion scheme (53) into this sum, we obtain

$$(54) \quad \begin{cases} \Phi_0(x) = 0, \\ \Phi_n(x) = \int_0^1 G_1(x, y)\mu(x)f(\Phi_{n-1}(y))dy, \quad n \geq 1. \end{cases}$$

Let us prove the following results on the convergence of the Adomian decomposition method.

Theorem 4.1. *Let (Φ_n) be a sequence defined by (54). Then $\lim_{n \rightarrow \infty} \Phi_n(x) = u(x)$ and*

$$(55) \quad \|\Phi_n - u\|_\infty \leq c^n \frac{1}{1-c} \|\Phi_1 - \Phi_0\|_\infty,$$

where $u(x)$ satisfies the Fredholm integral equation

$$(56) \quad u = \int_0^1 G_2(x, y) f(u(y)) dy, \quad G_2(x, y) = \mu(x) G_1(x, y).$$

Proof. We show that (Φ_n) is a contractive sequence.

$$(57) \quad \begin{aligned} \|\Phi_{n+2} - \Phi_{n+1}\|_\infty &= \sup_{0 \leq x \leq 1} \left| \int_0^1 G_2(x, y) [f(\Phi_{n+1}(y)) - f(\Phi_n(y))] dy \right| \\ &\leq \sup_{0 \leq x \leq 1} \int_0^1 |G_2(x, y)| |f(\Phi_{n+1}(y)) - f(\Phi_n(y))| dy \\ &\leq \|\Phi_{n+1} - \Phi_n\|_\infty \sup_{0 \leq x \leq 1} \int_0^1 |G_2(x, y) \mu(y)| dy \\ &\leq c \|\Phi_{n+1} - \Phi_n\|_\infty. \end{aligned}$$

If $m > n$, say $m = n + p$, $p = 1, 2, \dots$, then

$$(58) \quad \begin{aligned} \|\Phi_{n+p} - \Phi_n\|_\infty &\leq \|\Phi_n - \Phi_{n-1}\|_\infty + \|\Phi_{n-1} - \Phi_{n-2}\|_\infty \\ &\quad + \dots + \|\Phi_{n+1} - \Phi_n\|_\infty \\ &\leq c^n (1 + c + c^2 + \dots + c^{p-1} + c^p + \dots) \|\Phi_1 - \Phi_0\|_\infty. \end{aligned}$$

Thus

$$(59) \quad \|\Phi_{n+p} - \Phi_n\|_\infty \leq c^n \frac{1}{1-c} \|\Phi_1 - \Phi_0\|_\infty.$$

As $n, m = n + p \rightarrow \infty$, we see that $\|\Phi_{n+p} - \Phi_n\|_\infty \rightarrow 0$, that is (Φ_n) is a Cauchy sequence in the Banach space $(C([0, 1]), \|\cdot\|_\infty)$. Hence, it must be convergent, say $\lim_{n \rightarrow \infty} \Phi_n(x) = u(x)$.

By taking the limit in (59) as $p \rightarrow \infty$, we obtain the desired inequality (55).

It remains to verify that u is a solution

$$(60) \quad \begin{aligned} &\|u - \int_0^1 G_2(x, y) f(u(y)) dy\|_\infty \\ &\leq \|u - \Phi_n\|_\infty + \|\Phi_n - \int_0^1 G_2(x, y) f(\Phi_{n-1}(y)) dy\|_\infty \\ &\quad + \|\int_0^1 G_2(x, y) [f(u(y)) - f(\Phi_{n-1}(y))] dy\|_\infty. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \Phi_n(x) = u(x)$, we conclude that u satisfies Eq. (56). \square

5. Application

Consider the second-order nonlinear differential equation with the nonlocal boundary conditions

$$(61) \quad -u''(x) = 2(1+x(1-x))\frac{1}{1+|u|}, \quad 0 < x < 1,$$

$$(62) \quad u(0) = 0, \quad u(1) = \int_0^1 (2x-1)u(x)dx.$$

Here $h_1(x) = 0$, $h_2(x) = (2x-1)$, $\mu(x) = 2(1+x(1-x))$ and $f(x, u) = 2(1+x(1-x))\frac{1}{1+|u|}$. This function satisfies the condition (25), with

$$(63) \quad |f(x, u) - f(x, v)| \leq \mu(x) |u - v|, \quad \forall u, v \in \mathbb{R}, \quad x \in [0, 1].$$

Since

$$(64) \quad \mu(x) \left| \frac{1}{1+|u|} - \frac{1}{1+|v|} \right| = \mu(x) \frac{1}{(1+|u|)(1+|v|)} |u - v| \leq \mu(x) |u - v|.$$

A simple calculation leads to

$$(65) \quad \sup_{0 \leq x \leq 1} \int_0^1 G(x, y)dy = \frac{1}{8}, \quad \forall (x, y) \in [0, 1] \times [0, 1],$$

$$(66) \quad \|h_1\|_{L_1} = 0, \quad \|h_2\|_{L_1} = \int_0^1 |h_2(x)| dx = \frac{1}{3}, \quad \mu_0 = \frac{5}{2}, \quad \left| 1 - \int_0^1 h_2(s)ds \right| = \frac{2}{3}$$

and

$$(67) \quad \begin{aligned} c &= \sup_{0 \leq x \leq 1} \int_0^1 |G_1(x, y)| \mu(y)dy \\ &< \sup_{0 \leq x \leq 1} \int_0^1 |G(x, y)| dy \|\mu\|_\infty \left(1 + \frac{\|h_2\|_{L_1}}{\left| 1 - \int_0^1 h_2(s)ds \right|} \right) \\ &< 1. \end{aligned}$$

Theorem 3.1 implies that there is a unique solution to this problem, which is given as $u(x) = x(1-x)$.

The recursion scheme (53) produces a rapidly convergent series as

$$\begin{cases} u_0 = 0, \\ u_1 = \int_0^x (1-x)y\mu(y)A_0(y)dy + \int_x^1 x(1-y)\mu(y)A_0(y)dy + \int_0^1 (2y-1)u_0(y)dy, \\ u_2 = \int_0^x (1-x)y\mu(y)A_1(y)dy + \int_x^1 x(1-y)\mu(y)A_1(y)dy + \int_0^1 (2y-1)u_1(y)dy, \\ u_3 = \int_0^x (1-x)y\mu(y)A_2(y)dy + \int_x^1 x(1-y)\mu(y)A_2(y)dy + \int_0^1 (2y-1)u_2(y)dy, \\ \dots \end{cases}$$

where the Adomian polynomials A_k , for $k \geq 0$, for the nonlinear term $N(u) = \frac{1}{1+u}$ are given as

$$\begin{aligned} A_0 &= f(u_0), \\ A_1 &= u_1 f'(u_0), \\ A_2 &= u_2 f'(u_0) + \frac{1}{2} u_1^2 f''(u_0), \\ A_3 &= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{1}{3!} u_1^3 f'''(u_0), \\ A_4 &= u_4 f'(u_0) + \left[\frac{1}{2} u_2^2 + u_1 u_3 \right] f''(u_0) + \frac{1}{2} u_1^2 u_2 f'''(u_0) + \frac{1}{4!} u_1^4 f''''(u_0), \\ &\vdots \end{aligned}$$

This in turn gives the following solution components

$$\left\{ \begin{aligned} u_0 &= 0, \\ u_1 &= \frac{7}{6} x - x^2 - \frac{1}{3} x^3 + \frac{1}{6} x^4, \\ u_2 &= -\frac{121}{504} x + \frac{7}{18} x^3 + \frac{1}{36} x^4 - \frac{1}{4} x^5 + \frac{1}{18} x^6 + \frac{1}{42} x^7 - \frac{1}{168} x^8, \\ u_3 &= \frac{2921}{24948} x - \frac{121}{1512} x^3 - \frac{269}{1008} x^4 + \frac{269}{1680} x^5 + \frac{7}{27} x^6 - \frac{38}{189} x^7 \\ &\quad - \frac{5}{168} x^8 + \frac{433}{9072} x^9 - \frac{89}{22680} x^{10} - \frac{17}{5544} x^{11} + \frac{17}{33264} x^{12}, \\ u_4 &= -\frac{26787731}{363242880} x + \frac{2921}{74844} x^3 + \frac{33793}{299376} x^4 + \frac{73391}{498960} x^5 - \frac{14969}{45360} x^6 - \frac{3169}{19845} x^7 \\ &\quad + \frac{2453}{5880} x^8 - \frac{1901}{25920} x^9 - \frac{947}{6804} x^{10} + \frac{26947}{498960} x^{11} + \frac{6151}{427680} x^{12} - \frac{118441}{12972960} x^{13} \\ &\quad + \frac{5303}{45405360} x^{14} + \frac{79}{166320} x^{15} - \frac{79}{1330560} x^{16}, \\ &\dots \end{aligned} \right.$$

The fifth-stage approximate solution is therefore given as

$$\begin{aligned} \phi(x) &= \sum_{i=0}^4 u_i \\ &= \frac{x(1-x)}{1089728640} [64701 x^{14} - 452907 x^{13} - 580179 x^{12} + 9368865 x^{11} \\ &\quad - 6860803 x^{10} - 62371531 x^9 + 93576261 x^8 + 121486143 x^7 \\ &\quad - 294204561 x^6 + 72964527 x^5 + 89517063 x^4 + 27176967 x^3 \\ &\quad - 16910713 x^2 - 32773833 x + 1056954807]. \end{aligned}$$

In order to verify the accuracy of our proposed approach, we compare the approximate solution obtained by the ADM with the exact solution $u(x) = x(1-x)$. The approximate analytic solution obtained by the ADM for the 5th-stage approximation of $u(x)$ has been plotted in Figure 1. The comparison indicates a good agreement between the approximate solution obtained by this new variation of the Adomian decomposition method and the exact solution.

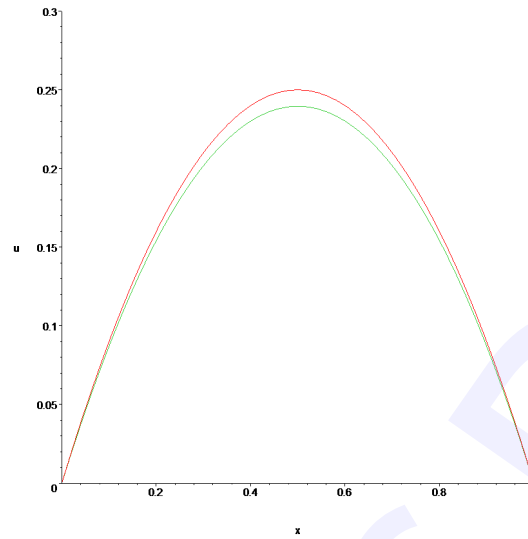


FIGURE 1. Red color is the exact solution and Green color is the approximate solution $\phi(x)$.

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