Bull. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/BKMS.b170374
pISSN: 1015-8634 / eISSN: 2234-3016

EXISTENCE AND UNIQUENESS THEOREMS OF SECOND-ORDER EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

Lazhar Bougoffa and Ammar Khanfer

Abstract. In this paper, we consider the second-order nonlinear differential equation with the nonlocal boundary conditions. We first reformulate this boundary value problem as a fixed point problem for a Fredholm integral equation operator, and then present a result on the existence and uniqueness of the solution by using the contraction mapping theorem. Furthermore, we establish a sufficient condition on the functions μ and h_i, $i = 1, 2$ that guarantee a unique solution for this nonlocal problem in a Hilbert space. Also, accurate analytic solutions in series forms for this boundary value problems are obtained by the Adomian decomposition method (ADM).

1. Introduction

The theory of boundary-value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions [8–14, 17–23]. This problem is used in different areas of physics, engineering and mathematics such as plate deflection theory.

Many authors have studied the second-order nonlinear differential equation under various boundary conditions and by different approaches [9–11, 20–23] and the references therein.

Consider the second-order boundary value problem with integral boundary conditions

$$-u''(x) = f(x, u(x)),\ 0 < x < 1, \tag{1}$$

$$u(0) = \int_0^1 h_1(x)u(x)dx,\ u(1) = \int_0^1 h_2(x)u(x)dx, \tag{2}$$

Received April 25, 2017; Revised August 16, 2017; Accepted December 15, 2017.
2010 *Mathematics Subject Classification.* Primary 34B10.
Key words and phrases. second-order equation, nonlocal boundary conditions, existence and uniqueness solution, Adomian decomposition method.

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given function and $h_i : [0,1] \to \mathbb{R}$, $i = 1, 2$ are integrable functions on $[0,1]$.

In this paper, we first reformulate this boundary value problem as a fixed point problem for a Fredholm integral equation operator, and then present a result on the existence and uniqueness of the solution by using the contraction mapping theorem. Furthermore, we establish a sufficient condition on the functions μ and h_i, $i = 1, 2$ that guarantee a unique solution for Pr. (1)-(2) in a Hilbert space. Also, the existence and accurate analytic solutions in series forms for this boundary value problems are obtained by the Adomian decomposition method (ADM) [1–7, 15, 16].

2. The existence and uniqueness theorem

We will use the following Lemma to reformulate the transformed boundary value problem Pr. (1)-(2) as a fixed point problem for Fredholm integral equation.

Lemma 2.1. *Let $g : [0,1] \to \mathbb{R}$ be a continuous function. The unique solution u of the following boundary value problem with nonlocal boundary conditions*

$$\begin{cases} -u''(x) = g(x), \\ \quad u(0) = \int_0^1 h_1(x)u(x)dx \text{ and } u(1) = \int_0^1 h_2(x)u(x)dx \end{cases} \tag{3}$$

is given by

$$u(x) = \int_0^1 G_1(x,y)g(y)dy, \tag{4}$$

where

$$G_1(x,y) = G(x,y) + H_1 + H_2 + (1-x)H_3 + (1-x)H_4, \tag{5}$$

where

$$G(x,y) = \begin{cases} x(1-y), & 0 \le x \le y \le 1, \\ y(1-x), & 0 \le y \le x \le 1, \end{cases} \tag{6}$$

$$H_1 = \frac{[1 - \int_0^1 sh_2(s)ds] \int_0^1 h_1(s)[\int_0^1 G(s,y)g(y)dy]ds}{\kappa}, \tag{7}$$

$$H_2 = \frac{\int_0^1 sh_1(s)ds \int_0^1 h_2(s)[\int_0^1 G(s,y)g(y)dy]ds}{\kappa}, \tag{8}$$

$$H_3 = \frac{(1 - \int_0^1 (1-s)h_1(s)ds) \int_0^1 h_2(s)[\int_0^1 G(s,y)g(y)dy]ds}{\kappa}, \tag{9}$$

$$H_4 = \frac{\int_0^1 sh_1(s)ds \int_0^1 h_2(s)[\int_0^1 G(s,y)g(y)dy]ds}{\kappa} \tag{10}$$

and

(11)

$$\kappa = [1-\int_0^1 (1-s)h_1(s)ds][1-\int_0^1 sh_2(s)ds]-\int_0^1 sh_1(s)ds \int_0^1 (1-s)h_2(s)ds \neq 0.$$

Proof. Integrating the first equation of (3) twice, we obtain

$$u(x) = C_1 x + C_2 - \int_0^x \int_1^y g(s)dsdy, \tag{12}$$

where $C_1 = u'(1)$ is a constant to be determined later and $C_2 = u(0)$. Integration by parts of the integral with respect to y in Eq. (12) gives

$$u(x) = C_1 x + C_2 + x\int_x^1 g(y)dy + \int_0^x yg(y)dy. \tag{13}$$

We determine C_1 and C_2 from the nonlocal boundary conditions (2), we obtain

$$C_1 = -\int_0^1 yg(y)dy + \int_0^1 [h_2(y) - h_1(y)]\, u(y)dy \tag{14}$$

and

$$C_2 = \int_0^1 h_1(y)u(y)dy. \tag{15}$$

Substituting Eqs. (14)-(15) into Eq. (13) we obtain

$$\begin{aligned} u(x) = &\left(-\int_0^1 yg(y)dy + \int_0^1 (h_2(y) - h_1(y))u(y)\right) x + \int_0^1 h_1(y)u(y)dy \\ &+ x\int_x^1 g(y)dy + \int_0^x yg(y)dy. \end{aligned} \tag{16}$$

It follows that

$$\begin{aligned} u(x) = &\int_0^x (1-x)yg(y)dy + \int_x^1 x(1-y)g(y)dy \\ &+ \int_0^1 [xh_2(y) + (1-x)h_1(y)]\, u(y)dy, \end{aligned} \tag{17}$$

that is,

$$u(x) = \int_0^1 G(x,y)g(y)dy + \int_0^1 [xh_2(y) + (1-x)h_1(y)]\, u(y)dy. \tag{18}$$

Multiplying both sides of Eq. (18) by h_1 and integrate over $[0,1]$, we obtain

$$\begin{aligned} \int_0^1 h_1(s)u(s)ds = &\int_0^1 h_1(s)\left[\int_0^1 G(s,y)g(y)dy\right] ds \\ &+ \int_0^1 h_1(s)\left[\int_0^1 [sh_2(y) + (1-s)h_1(y)]\, u(y)dy\right] ds. \end{aligned} \tag{19}$$

Thus

$$\left(1-\int_0^1 (1-s)h_1(s)ds\right)\int_0^1 h_1(s)u(s)ds$$
$$=\int_0^1 h_1(s)\left[\int_0^1 G(s,y)g(y)dy\right]ds+\int_0^1 yh_1(y)dy\int_0^1 h_2(s)u(s)ds. \tag{20}$$

Similarly, multiplying both sides of Eq. (18) by h_2 and integrate over $[0,1]$, we obtain

$$\left(1-\int_0^1 sh_2(s)ds\right)\int_0^1 h_2(s)u(s)ds$$
$$=\int_0^1 h_2(s)\left[\int_0^1 G(s,y)g(y)dy\right]ds+\int_0^1 (1-y)h_2(y)dy\int_0^1 h_1(s)u(s)ds. \tag{21}$$

Solving Eq. (20) and Eq. (21) for $\int_0^1 h_1(s)u(s)ds$ and $\int_0^1 h_2(s)u(s)ds$, to get

$$\int_0^1 h_1(s)u(s)ds=\frac{[1-\int_0^1 sh_2(s)ds]\int_0^1 h_1(s)[\int_0^1 G(s,y)g(y)dy]ds}{[1-\int_0^1(1-s)h_1(s)ds][1-\int_0^1 sh_2(s)ds]-\int_0^1 sh_1(s)ds\int_0^1(1-s)h_2(s)ds}$$
$$+\frac{\int_0^1 sh_1(s)ds\int_0^1 h_2(s)[\int_0^1 G(s,y)g(y)dy]ds}{[1-\int_0^1(1-s)h_1(s)ds][1-\int_0^1 sh_2(s)ds]-\int_0^1 sh_1(s)ds\int_0^1(1-s)h_2(s)ds} \tag{22}$$

and

$$\int_0^1 h_2(s)u(s)ds=\frac{(1-\int_0^1(1-s)h_1(s)ds)\int_0^1 h_2(s)[\int_0^1 G(s,y)g(y)dy]ds}{[1-\int_0^1(1-s)h_1(s)ds][1-\int_0^1 sh_2(s)ds]-\int_0^1 sh_1(s)ds\int_0^1(1-s)h_2(s)ds}$$
$$+\frac{\int_0^1(1-s)h_2(s)ds\int_0^1 h_1(s)[\int_0^1 G(s,y)g(y)dy]ds}{[1-\int_0^1(1-s)h_1(s)ds][1-\int_0^1 sh_2(s)ds]-\int_0^1 sh_1(s)ds\int_0^1(1-s)h_2(s)ds}. \tag{23}$$

Substituting these into Eq. (18), we obtain

$$u(x)=\int_0^1 G(x,y)g(y)dy$$
$$+\frac{(1-\int_0^1(1-s)h_1(s)ds)\int_0^1 h_2(s)[\int_0^1 G(s,y)g(y)dy]ds}{[1-\int_0^1(1-s)h_1(s)ds][1-\int_0^1 sh_2(s)ds]-\int_0^1 sh_1(s)ds\int_0^1(1-s)h_2(s)ds}$$
$$+\frac{\int_0^1(1-s)h_2(s)ds\int_0^1 h_1(s)[\int_0^1 G(s,y)g(y)dy]ds}{[1-\int_0^1(1-s)h_1(s)ds][1-\int_0^1 sh_2(s)ds]-\int_0^1 sh_1(s)ds\int_0^1(1-s)h_2(s)ds}$$
$$+(1-x)\frac{[1-\int_0^1 sh_2(s)ds]\int_0^1 h_1(s)[\int_0^1 G(s,y)g(y)dy]ds}{[1-\int_0^1(1-s)h_1(s)ds][1-\int_0^1 sh_2(s)ds]-\int_0^1 sh_1(s)ds\int_0^1(1-s)h_2(s)ds}$$
$$+(1-x)\frac{\int_0^1 sh_1(s)ds\int_0^1 h_2(s)[\int_0^1 G(s,y)g(y)dy]ds}{[1-\int_0^1(1-s)h_1(s)ds][1-\int_0^1 sh_2(s)ds]-\int_0^1 sh_1(s)ds\int_0^1(1-s)h_2(s)ds},$$

which is what we had to prove. □

Lemma 2.2. *The nonlocal boundary value problem Pr.* (1)-(2) *can be written as a Fredholm integral equation for* u,

$$u=\int_0^1 G_1(x,y)f(y,u(y))dy. \tag{24}$$

Proof. Replacing $g(x)$ by $f(x,u(x))$ in Lemma 2.1, we obtain Eq. (24). □

Theorem 2.3. *Suppose that $f(x,\ u)$ is continuous on $[0,1]\times\mathbb{R}$ and there is a continuous function $\mu:[0,1]\to\mathbb{R}^+$ such that*

$$| f(x,\ y)-f(x,\ z)| \leq \mu(x) \mid y-z\mid, \forall\ y,z\in\mathbb{R},\ x\in[0,1]. \tag{25}$$

Moreover, we assume

$$\sup_{0\leq x\leq 1}\int_0^1 \mid G_1(x,y)\mid \mu(y)dy<1, \tag{26}$$

where $G_1(x,y)$ is defined by (5). Then the nonlocal boundary value problem Pr. (1)-(2) has a unique solution.

Proof. The integral equation (24) may be written as a fixed point equation $T(u)=u$, where the map T is defined by

$$Tu(x)=\int_0^1 G_1(x,y)f(y,u(y))dy. \tag{27}$$

We show T is contraction, since for any $v,w\in C([0,1])$, we have

$$\begin{aligned}\| Tv-Tw\|_\infty &= \sup_{0\leq x\leq 1}\left|\int_0^1 G_1(x,y)\left[f(y,\ v(y))-f(y,\ w(y))\right]dy\right| \\ &\leq \sup_{0\leq x\leq 1}\int_0^1 \mid G_1(x,y)\mid\mid f(y,\ v(y))-f(y,\ w(y))\mid dy \\ &\leq \| u-v\|_\infty \sup_{0\leq x\leq 1}\int_0^1 \mid G_1(x,y)\mu(y)\mid dy \\ &\leq c\,\| v-w\|_\infty, \end{aligned} \tag{28}$$

where

$$c=\sup_{0\leq x\leq 1}\int_0^1 \mid G_1(x,y)\mid \mu(y)dy<1. \tag{29}$$

Thus T is a contraction and from the contraction mapping theorem, T has a unique fixed point, i.e., there exists a unique $u\in C([0,1])$ such that $Tu=u$. This fixed point u is a unique solution of this problem. □

3. The uniqueness solution in an Hilbert space

The generalization of the result on the uniqueness of the solution of Pr. (1)-(2) can be obtained in the Hilbert space U defined as

$$U=\left\{u:\ u,\sqrt{x(1-x)}\frac{du}{dx},\ \frac{d^2u}{dx^2}\in L_2(0,1)\right\} \tag{30}$$

with respect to the norm

$$\| u\|_U^2=\int_0^1\left[u^2+x(1-x)(\frac{du}{dx})^2+(\frac{d^2u}{dx^2})^2\right]dx<\infty. \tag{31}$$

Suppose there are two solutions u and v such that $u \neq v$. Then from Eq. (1) with (2), we have

$$-w'' = f(x,u) - f(x,v),\ 0 < x < 1 \tag{32}$$

subject to

$$w(0) = \int_0^1 h_1(x)w(x)dx,\ w(1) = \int_0^1 h_2(x)w(x)dx, \tag{33}$$

where $w = u - v$.

Multiplying both sides of Eq. (32) by $\varphi(x)w$, where $\varphi(x) = x(x-1)$. Employing integration by parts, we obtain

$$\begin{aligned} &\int_0^1 \varphi''(x)w^2(x)dx - 2\int_0^1 \varphi(x)w'^2(x)dx \\ &= \varphi'(1)w^2(1) - \varphi'(0)w^2(0) - 2\int_0^1 g(x,u,v)\varphi(x)w(x)dx, \end{aligned} \tag{34}$$

where $g(x,u,v) = f(x,u) - f(x,v)$ and $|g(x,u,v)| \leq \mu(x)|w|$.

Replacing $\varphi(x) = x(x-1)$, $\varphi'(x) = 2x-1$ and $\varphi''(x) = 2$ into Eq. (34) and taking into account the nonlocal boundary conditions (2), we get

$$\begin{aligned} &2\int_0^1 w^2(x)dx + 2\int_0^1 x(1-x)w'^2(x)dx \\ &= (\int_0^1 h_1(x)w(x)dx)^2 + (\int_0^1 h_2(x)w(x)dx)^2 + 2\int_0^1 g(x,u,v)x(1-x)w(x)dx. \end{aligned} \tag{35}$$

From the Holder inequality, we have

$$(\int_0^1 h_i(x)w(x)dx)^2 \leq \int_0^1 h_i^2(x)dx \int_0^1 w^2(x)dx,\ i = 1,2. \tag{36}$$

Since $\max_{0\leq x\leq 1} x(1-x) = \frac{1}{4}$, Eq. (35) becomes

$$\begin{aligned} 2\int_0^1 w^2(x)dx + 2\int_0^1 x(1-x)w'^2(x)dx &\leq (\| h_1 \|^2 + \| h_2 \|^2)\int_0^1 w^2(x)dx \\ &+ \frac{1}{4}\int_0^1 \mu(x)w^2(x)dx, \end{aligned} \tag{37}$$

where $\| h_i \|^2 = \int_0^1 h_i^2(x)dx,\ i = 1,2$.

If we assume that $\mu(x) \leq \mu_0, \forall\ x \in [0,1]$. Then,

$$\left(2 - \frac{\mu_0}{4} - (\| h_1 \|^2 + \| h_2 \|^2)\right)\int_0^1 w^2(x)dx + 2\int_0^1 x(1-x)w'^2(x)dx \leq 0. \tag{38}$$

Multiplying now both sides of Eq. (32) by w'', we obtain

$$\int_0^1 w''^2(x)dx = -\int_0^1 g(x,\ u,\ v)w''(x)dx. \tag{39}$$

The term $\int_0^1 g(x,\ u,\ v)w''(x)dx$ can be estimated by means of the Cauchy-Schwarz-Bunyakovski inequality and the ϵ-inequality

$$2 \mid uv \mid \leq \epsilon u^2 + \frac{1}{\epsilon}v^2,\ \epsilon > 0. \tag{40}$$

Thus

$$\int_0^1 w''^2(x)dx \leq \frac{1}{2\epsilon_1}\int_0^1 \mid g(x,u,v) \mid^2 dx + \frac{\epsilon_1}{2}\int_0^1 w''^2(x)dx,\ \epsilon_1 > 0. \tag{41}$$

Therefore

$$(1-\frac{\epsilon_1}{2})\int_0^1 w''^2(x)dx \leq \frac{1}{2\epsilon_1}\int_0^1 \mu^2(x)w^2(x)dx,\ \epsilon_1 > 0. \tag{42}$$

If we choose $\epsilon_1 = 1$, then

$$\int_0^1 w''^2(x)dx \leq \mu_0^2 \int_0^1 w^2(x)dx. \tag{43}$$

Adding side to side Eq. (38) and Eq. (43), we obtain

$$\begin{aligned}&\Big(2-\frac{\mu_0}{4}-\mu_0^2-(\parallel h_1 \parallel^2 + \parallel h_2 \parallel^2)\Big)\int_0^1 w^2(x)dx\\ &+2\int_0^1 x(1-x)w'^2(x) + \int_0^1 w''^2(x)dx \leq 0.\end{aligned} \tag{44}$$

Choosing

$$\frac{\mu_0}{4}+\mu_0^2 + \parallel h_1 \parallel^2 + \parallel h_2 \parallel^2 < 2. \tag{45}$$

Then

$$\int_0^1 \left[w^2 + x(1-x)\left(\frac{dw}{dx}\right)^2 + \left(\frac{d^2w}{dx^2}\right)^2\right] dx \leq 0, \tag{46}$$

that is

$$||w||_U \leq 0. \tag{47}$$

This is a contradiction.

We have proved the following statement.

Theorem 3.1. *Under the hypotheses of Theorem 2.3, and if we assume $\mu(x) \leq \mu_0, \forall\ x \in [0,1]$ with (45). Then Pr. (1)-(2) has a unique solution in the Hilbert space U.*

4. Approximate analytic solution by the Adomian decomposition method (ADM)

We propose here to solve this nonlocal boundary value problem by the Adomian decomposition method (ADM) [1–7, 15, 16].

Let us consider the case $f(x,u(x)) = \mu(x)f(u(x))$ and assume $f(0) \neq 0$. We rewrite the equivalent Eq. (24) in Adomian's operator-theoretic notation as

$$Lu = Ru + Nu, \tag{48}$$

where

$$Lu = u,\ Ru = 0 \text{ and } Nu = \int_0^1 G_1(x,y)\mu(x)f(u(y))dy. \tag{49}$$

Define the solutions $u(x)$ by its respective infinite series of components in the form

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{50}$$

and the infinite series

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n) \tag{51}$$

for the nonlinear term $N(u)$, where the A_n are the Adomian polynomials [1–3, 15, 16], which can be obtained from the definitional formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0},\ n = 0, 1, 2, \dots. \tag{52}$$

Consequently, the components u_n can be elegantly determined by setting the following recursion scheme

$$\begin{cases} u_0 = 0, \\ u_{n+1} = \displaystyle\int_0^1 G_1(x,y)\mu(x)A_n(y)dy,\ n \geq 0. \end{cases} \tag{53}$$

Let $\Phi_n(x) = \sum_{i=0}^{n-1} u_i(x)$ be the n th-stage approximation functions of $u(x)$ by the ADM for the nonlinear Eq. (24).

By substitution the recursion scheme (53) into this sum, we obtain

$$\begin{cases} \Phi_0(x) = 0, \\ \Phi_n(x) = \displaystyle\int_0^1 G_1(x,y)\mu(x)f(\Phi_{n-1}(y))dy,\ n \geq 1. \end{cases} \tag{54}$$

Let us prove the following results on the convergence of the Adomian decomposition method.

Theorem 4.1. *Let* (Φ_n) *be a sequence defined by* (54). *Then* $\lim_{n\to\infty}\Phi_n(x) = u(x)$ *and*

$$\| \Phi_n - u \|_\infty \le c^n \frac{1}{1-c} \| \Phi_1 - \Phi_0 \|_\infty, \tag{55}$$

where $u(x)$ *satisfies the Fredholm integral equation*

$$u = \int_0^1 G_2(x,y) f(u(y))dy,\ G_2(x,y) = \mu(x)G_1(x,y). \tag{56}$$

Proof. We show that (Φ_n) is a contractive sequence.

$$\begin{aligned}
\| \Phi_{n+2} - \Phi_{n+1} \|_\infty &= \sup_{0\le x\le 1} \left| \int_0^1 G_2(x,y)\,[f(\Phi_{n+1}(y)) - f(\Phi_n(y))]\,dy \right| \\
&\le \sup_{0\le x\le 1} \int_0^1 | G_2(x,y) || f(\Phi_{n+1}(y)) - f(\Phi_n(y)) | \,dy \\
&\le \| \Phi_{n+1} - \Phi_n \|_\infty \sup_{0\le x\le 1} \int_0^1 | G_2(x,y)\mu(y) | \,dy \\
&\le c \| \Phi_{n+1} - \Phi_n \|_\infty . \qquad (57)
\end{aligned}$$

If $m > n$, say $m = n + p$, $p = 1, 2, \ldots$, then

$$\begin{aligned}
\| \Phi_{n+p} - \Phi_n \|_\infty \le\ & \| \Phi_n - \Phi_{n-1} \|_\infty + \| \Phi_{n-1} - \Phi_{n-2} \|_\infty \\
& + \cdots + \| \Phi_{n+1} - \Phi_n \|_\infty \\
& \le c^n(1 + c + c^2 + \cdots + c^{p-1} + c^p + \cdots) \| \Phi_1 - \Phi_0 \|_\infty . \qquad (58)
\end{aligned}$$

Thus

$$\| \Phi_{n+p} - \Phi_n \|_\infty \le c^n \frac{1}{1-c} \| \Phi_1 - \Phi_0 \|_\infty . \tag{59}$$

As $n, m = n + p \to \infty$, we see that $\| \Phi_{n+p} - \Phi_n \|_\infty \to 0$, that is (Φ_n) is a Cauchy sequence in the Banach space $(C([0,1]), \| \cdot \|_\infty)$. Hence, it must be convergent, say $\lim_{n\to\infty}\Phi_n(x) = u(x)$.

By taking the limit in (59) as $p \to \infty$, we obtain the desired inequality (55).

It remains to verify that u is a solution

$$\begin{aligned}
& \| u - \int_0^1 G_2(x,y) f(u(y))dy \|_\infty \\
\le\ & \| u - \Phi_n \|_\infty + \| \Phi_n - \int_0^1 G_2(x,y) f(\Phi_{n-1}(y))dy \|_\infty \\
& + \| \int_0^1 G_2(x,y)[f(u(y)) - f(\Phi_{n-1}(y))]dy \|_\infty . \qquad (60)
\end{aligned}$$

Since $\lim_{n\to\infty}\Phi_n(x) = u(x)$, we conclude that u satisfies Eq. (56). $\square$

5. Application

Consider the second-order nonlinear differential equation with the nonlocal boundary conditions

$$-u''(x) = 2(1 + x(1 - x))\frac{1}{1+\mid u\mid},\ 0 < x < 1, \tag{61}$$

$$u(0) = 0,\ u(1) = \int_0^1 (2x - 1)u(x)dx. \tag{62}$$

Here $h_1(x) = 0,\ h_2(x) = (2x - 1),\ \mu(x) = 2(1 + x(1 - x)$ and $f(x, u) = 2(1 + x(1 - x)\frac{1}{1+|u|}$. This function satisfies the condition (25), with

$$\mid f(x,\ u) - f(x,\ v) \mid \leq \mu(x) \mid u - v \mid, \forall\ u, v \in \mathbb{R},\ x \in [0, 1]. \tag{63}$$

Since

$$\mu(x) \mid \tfrac{1}{1+|u|} - \tfrac{1}{1+|v|} \mid = \mu(x)\tfrac{1}{(1+|u|)(1+|v|)} \mid u - v \mid \leq \mu(x) \mid u - v \mid . \tag{64}$$

A simple calculation leads to

$$\sup_{0\leq x\leq 1} \int_0^1 G(x, y)dy = \frac{1}{8},\ \forall\ (x, y) \in [0, 1] \times [0, 1], \tag{65}$$

(66)

$$\parallel h_1 \parallel_{L_1} = 0,\ \parallel h_2 \parallel_{L_1} = \int_0^1 \mid h_2(x) \mid dx = \frac{1}{3},\ \mu_0 = \frac{5}{2},\ \mid 1 - \int_0^1 h_2(s)ds \mid = \frac{2}{3}$$

and

$$\begin{aligned} c &= \sup_{0\leq x\leq 1} \int_0^1 \mid G_1(x, y) \mid \mu(y)dy \\ &< \sup_{0\leq x\leq 1} \int_0^1 \mid G(x, y) \mid dy \parallel \mu \parallel_\infty \left(1 + \frac{\parallel h_2 \parallel_{L_1}}{\mid 1 - \int_0^1 h_2(s)ds \mid}\right) \\ &< 1. \end{aligned} \tag{67}$$

Theorem 3.1 implies that there is a unique solution to this problem, which is given as $u(x) = x(1 - x)$.

The recursion scheme (53) produces a rapidly convergent series as

$$\begin{cases} u_0 = 0, \\ u_1 = \int_0^x (1 - x)y\mu(y)A_0(y)dy + \int_x^1 x(1 - y)\mu(y)A_0(y)dy + \int_0^1 (2y - 1)u_0(y)dy, \\ u_2 = \int_0^x (1 - x)y\mu(y)A_1(y)dy + \int_x^1 x(1 - y)\mu(y)A_1(y)dy + \int_0^1 (2y - 1)u_1(y)dy, \\ u_3 = \int_0^x (1 - x)y\mu(y)A_2(y)dy + \int_x^1 x(1 - y)\mu(y)A_2(y)dy + \int_0^1 (2y - 1)u_2(y)dy, \\ \quad \ldots \end{cases}$$

where the Adomian polynomials A_k, for $k \geq 0$, for the nonlinear term $N(u) = \frac{1}{1+u}$ are given as

$$
\begin{aligned}
&A_0 = f(u_0),\\
&A_1 = u_1 f'(u_0),\\
&A_2 = u_2 f'(u_0) + \frac{1}{2} u_1^2 f''(u_0),\\
&A_3 = u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{1}{3!} u_1^3 f'''(u_0),\\
&A_4 = u_4 f'(u_0) + [\frac{1}{2} u_2^2 + u_1 u_3] f''(u_0) + \frac{1}{2} u_1^2 u_2 f'''(u_0) + \frac{1}{4!} u_1^4 f''''(u_0),\\
&\quad \vdots
\end{aligned}
$$

This in turn gives the following solution components

$$
\begin{cases}
u_0 = 0,\\
u_1 = \frac{7}{6}\,x - x^2 - \frac{1}{3}\,x^3 + \frac{1}{6}\,x^4,\\
u_2 = -\frac{121}{504}\,x + \frac{7}{18}\,x^3 + \frac{1}{36}\,x^4 - \frac{1}{4}\,x^5 + \frac{1}{18}\,x^6 + \frac{1}{42}\,x^7 - \frac{1}{168}\,x^8,\\
u_3 = \frac{2921}{24948}\,x - \frac{121}{1512}\,x^3 - \frac{269}{1008}\,x^4 + \frac{269}{1680}\,x^5 + \frac{7}{27}\,x^6 - \frac{38}{189}\,x^7\\
\qquad - \frac{5}{168}\,x^8 + \frac{433}{9072}\,x^9 - \frac{89}{22680}\,x^{10} - \frac{17}{5544}\,x^{11} + \frac{17}{33264}\,x^{12},\\
u_4 = -\frac{26787731}{363242880}\,x + \frac{2921}{74844}\,x^3 + \frac{33793}{299376}\,x^4 + \frac{73391}{498960}\,x^5 - \frac{14969}{45360}\,x^6 - \frac{3169}{19845}\,x^7\\
\qquad + \frac{2453}{5880}\,x^8 - \frac{1901}{25920}\,x^9 - \frac{947}{6804}\,x^{10} + \frac{26947}{498960}\,x^{11} + \frac{6151}{427680}\,x^{12} - \frac{118441}{12972960}\,x^{13}\\
\qquad + \frac{5303}{45405360}\,x^{14} + \frac{79}{166320}\,x^{15} - \frac{79}{1330560}\,x^{16},\\
\qquad \cdots
\end{cases}
$$

The fifth-stage approximate solution is therefore given as

$$
\begin{aligned}
\phi(x) &= \sum_{i=0}^{4} u_i\\
&= \tfrac{x(1-x)}{1089728640}[64701\,x^{14} - 452907\,x^{13} - 580179\,x^{12} + 9368865\,x^{11}\\
&\quad - 6860803\,x^{10} - 62371531\,x^9 + 93576261\,x^8 + 121486143\,x^7\\
&\quad - 294204561\,x^6 + 72964527\,x^5 + 89517063\,x^4 + 27176967\,x^3\\
&\quad - 16910713\,x^2 - 32773833\,x + 1056954807].
\end{aligned}
$$

In order to verify the accuracy of our proposed approach, we compare the approximate solution obtained by the ADM with the exact solution $u(x) = x(1-x)$. The approximate analytic solution obtained by the ADM for the 5th-stage approximation of $u(x)$ has been plotted in Figure 1. The comparison indicates a good agreement between the approximate solution obtained by this new variation of the Adomian decomposition method and the exact solution.

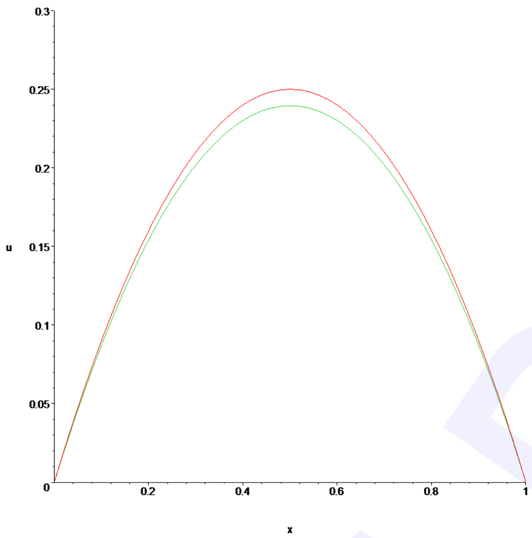

FIGURE 1. Red color is the exact solution and Green color is the approximate solution $\phi(x)$.

References

[1] G. Adomian, *Modification of the decomposition approach to the heat equation*, J. Math. Anal. Appl. **124** (1987), no. 1, 290–291.

[2] ______, *Nonlinear Stochastic Systems Theory and Applications to Physics*, Mathematics and its Applications, **46**, Kluwer Academic Publishers Group, Dordrecht, 1989.

[3] ______, *Solving Frontier Problems of Physics: The Decomposition Method*, Fundamental Theories of Physics, **60**, Kluwer Academic Publishers Group, Dordrecht, 1994.

[4] L. Bougoffa and R. C. Rach, *Solving nonlocal initial-boundary value problems for linear and nonlinear parabolic and hyperbolic partial differential equations by the Adomian decomposition method*, Appl. Math. Comput. **225** (2013), 50–61.

[5] L. Bougoffa, R. C. Rach, and A. Mennouni, *An approximate method for solving a class of weakly-singular Volterra integro-differential equations*, Appl. Math. Comput. **217** (2011), no. 22, 8907–8913.

[6] ______, *A convenient technique for solving linear and nonlinear Abel integral equations by the Adomian decomposition method*, Appl. Math. Comput. **218** (2011), no. 5, 1785–1793.

[7] L. Bougoffa, R. C. Rach, and A.-M. Wazwaz, *Solving nonlocal initial-boundary value problems for the Lotka-von Foerster model*, Appl. Math. Comput. **225** (2013), 7–15.

[8] S. A. Brykalov, *A second order nonlinear problem with two-point and integral boundary conditions*, Georgian Math. J. **225** (1994), no. 3, 243–249.

[9] Y. Cui, *Solvability of second-order boundary-value problems at resonance involving integral conditions*, Electron. J. Differential Equations **2012** (2012), No. 45, 9 pp.

[10] Y. Cui and Y. Zou, *Existence of solutions for second-order integral boundary value problems*, Nonlinear Anal. Model. Control **21** (2016), no. 6, 828–838.

[11] Z. Du and J. Yin, *A second order differential equation with generalized Sturm-Liouville integral boundary conditions at resonance*, Filomat **28** (2014), no. 7, 1437–1444.

[12] M. Feng, X. Li, and C. Xue, *Multiple positive solutions for impulsive singular boundary value problems with integral boundary conditions*, Int. J. Open Probl. Comput. Sci. Math. **2** (2009), no. 4, 546–561.
[13] G. Infante, *Nonlocal boundary value problems with two nonlinear boundary conditions*, Commun. Appl. Anal. **12** (2008), no. 3, 279–288.
[14] A. Lomtatidze and L. Malaguti, *On a nonlocal boundary value problem for second order nonlinear singular differential equations*, Georgian Math. J. **7** (2000), no. 1, 133–154.
[15] A.-M. Wazwaz, *Partial Differential Equations and Solitary Waves Theory*, Nonlinear Physical Science, Higher Education Press, Beijing, 2009.
[16] ———, *Linear and Nonlinear Integral Equations*, Higher Education Press, Beijing, 2011.
[17] J. R. L. Webb, *A unified approach to nonlocal boundary value problems*, in Dynamic systems and applications. Vol. 5, 510–515, Dynamic, Atlanta, GA, 2006.
[18] ———, *Positive solutions of some higher order nonlocal boundary value problems*, Electron. J. Qual. Theory Differ. Equ. **2009** (2009), Special Edition I, no. 29, 15 pp.
[19] J. R. L. Webb and G. Infante, *Positive solutions of nonlocal boundary value problems: a unified approach*, J. London Math. Soc. (2) **74** (2006), no. 3, 673–693.
[20] Z. Yang, *Existence and nonexistence results for positive solutions of an integral boundary value problem*, Nonlinear Anal. **65** (2006), no. 8, 1489–1511.
[21] ———, *Positive solutions of a second-order integral boundary value problem*, J. Math. Anal. Appl. **321** (2006), no. 2, 751–765.
[22] ———, *Existence and uniqueness of positive solutions for an integral boundary value problem*, Nonlinear Anal. **69** (2008), no. 11, 3910–3918.
[23] X. Zhang, M. Feng, and W. Ge, *Existence result of second-order differential equations with integral boundary conditions at resonance*, J. Math. Anal. Appl. **353** (2009), no. 1, 311–319.

Lazhar Bougoffa
Department of Mathematics
Al Imam Mohammad Ibn Saud Islamic University
Riyadh 11623, Riyadh
Email address: lbbougoffa@imamu.edu.sa, bougoffa@hotmail.com

Ammar Khanfer
Department of Mathematics
Al Imam Mohammad Ibn Saud Islamic University
Riyadh 11623, Riyadh
Email address: ahkhanfer@imamu.edu.sa