

ON NEARLY PARAKÄHLER MANIFOLDS

AYDIN GEZER AND SIBEL TURANLI

ABSTRACT. The purpose of the present paper is to study on nearly paraKähler manifolds. Firstly, to investigate some properties of the Ricci tensor and the Ricci* tensor of nearly paraKähler manifolds. Secondly, to define a special metric connection with torsion on nearly paraKähler manifolds and present its some properties.

1. Introduction

An almost product structure on a $2k$ -dimensional smooth manifold M is a $(1, 1)$ -tensor field P squaring to the identity. In this case, the pair (M, P) is called an almost product manifold. An almost paracomplex manifold is an almost product manifold (M, P) such that the two eigenbundles T^+M and T^-M associated with the two eigenvalues ± 1 of P have the same rank. The Nijenhuis tensor N of an almost paracomplex structure P is given by

$$N_P(X, Y) = [PX, PY] - P[PX, Y] - P[X, PY] + [X, Y].$$

It is well known that an almost paracomplex structure is integrable if and only if the corresponding Nijenhuis tensor N vanishes. An integrable almost paracomplex structure is a paracomplex structure. For a survey on paracomplex geometry we refer to [1].

An almost paraHermitian manifold consists of a smooth manifold M endowed with an almost paracomplex structure P and a pseudo-Riemannian metric g compatible in the sense that

$$(1.1) \quad g(PX, Y) = -g(X, PY) \text{ or equivalently } g(PX, PY) = -g(X, Y).$$

Note that the metric g is neutral, i.e., it has signature (k, k) and the eigenbundles $T^\pm M$ are totally isotropic with respect to g . The condition (1.1) also implies that g is hybrid with respect to P . The 2-covariant skew-symmetric tensor field F defined by $F(X, Y) = g(PX, Y)$ is the fundamental 2-form of the almost paraHermitian manifold (M, g, P) . Recall the defining conditions of some of the classes:

$$-N_P = 0, \text{ paraHermitian manifolds,}$$

Received April 14, 2017; Accepted September 14, 2017.

2010 *Mathematics Subject Classification.* Primary 53C55; Secondary 53C05.

Key words and phrases. metric connection, nearly paraKähler manifold, Ricci tensor.

$-dF = 0$, almost paraKähler manifolds,
 $-\nabla^g P = 0 \Leftrightarrow dF = 0$ and $N_P = 0$, paraKähler manifolds,
 $-(\nabla_X^g P)X = 0$, nearly paraKähler manifolds. Here and in the following, let ∇^g always denote the Levi-Civita connection of the pseudo-Riemannian metric g of an almost paraHermitian manifold.

Nearly paraKähler manifolds have been introduced in [5]. These manifolds naturally appear as one class in the classification of almost paraHermitian manifolds [2, 6], which generalizes type \mathcal{W}_1 in the well-known Gray–Hervella classification [4] of almost Hermitian structures. In the present note, we study some properties concerning nearly paraKähler structures on a differentiable manifold endowed with a naturally associated metric which is necessarily of neutral signature. We give some of local and global results in nearly paraKähler manifolds. Throughout this paper, manifolds, tensor fields and connections under consideration are all assumed to be differentiable and of class C^∞ .

2. Nearly paraKähler manifolds

An almost paraHermitian manifold (M, g, P) is called nearly paraKähler if the almost paraHermitian structure is not paraKähler and satisfies the identity

$$(2.1) \quad (\nabla_X^g F)(Y, Z) + (\nabla_Y^g F)(X, Z) = 0$$

for any vector fields X, Y, Z on M , where F is the fundamental 2–form of the almost paraHermitian manifold (M, g, P) . It is easy to see that the condition (2.1) reduces to

$$(2.2) \quad (\nabla_X^g P)Y + (\nabla_Y^g P)X = 0$$

for the almost paracomplex structure P .

Recall that the Nijenhuis tensor of the almost paracomplex structure P satisfies

$$\begin{aligned} N_P(X, Y) &= [PX, PY] - P[PX, Y] - P[X, PY] + [X, Y] \\ &= -P[(\nabla_{PX}^g P)PY - (\nabla_{PY}^g P)PX] - P[(\nabla_X^g P)Y - (\nabla_Y^g P)X] \end{aligned}$$

for vector fields X, Y, Z on M . Here we use the formula:

$$\nabla_X^g Y - \nabla_Y^g X = [X, Y] \quad \text{and} \quad (\nabla_X^g P)PY = -P(\nabla_X^g P)Y.$$

On the nearly paraKähler manifold (M, g, P) , the Nijenhuis tensor N_P simplifies to

$$N_P(X, Y) = -4P(\nabla_X^g P)Y \quad (\text{see also [5]}).$$

When the Nijenhuis tensor N_P vanishes, we obtain $\nabla^g P = 0$ which gives the following proposition.

Proposition 2.1. *An integrable nearly paraKähler manifold is always a paraKähler manifold.*

2.1. Properties concerning Ricci and Ricci* tensors

Let (M, g, P) be a $2k$ -dimensional nearly paraKähler manifold. Coordinate systems in M are denoted (U, x^i) , where U is the coordinate neighbourhood and $x^i, i = 1, 2, \dots, 2k$ are the coordinate functions. On putting $X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial x^j}$, the conditions (2.1) and (2.2) can be respectively written in local coordinates as

$$\nabla_i^g F_{jm} + \nabla_j^g F_{im} = 0$$

and

$$\nabla_i^g P_j^h + \nabla_j^g P_i^h = 0.$$

Contraction with respect to i and h in the last equation gives $\nabla_i^g P_j^i = 0$.

Proposition 2.2. *In a nearly paraKähler manifold (M, g, P) , the Ricci tensor is hybrid with respect to the almost paracomplex structure P .*

Proof. In a nearly paraKähler manifold (M, g, P) , if we apply the Ricci identity to the tensor P_j^i , we get

$$\nabla_k^g \nabla_j^g P_i^h - \nabla_j^g \nabla_k^g P_i^h = R_{kjm}^g P_i^m - R_{kji}^g P_m^h,$$

where R_{kji}^g are components of the Riemannian curvature tensor R^g . Contracting the above equation with respect to k and h , with help of $\nabla_i^g P_j^i = 0$, we obtain

$$\begin{aligned} \nabla_h^g \nabla_j^g P_i^h &= R_{j m}^g P_i^m - R_{h j i}^g P_m^h = R_{j m}^g P_i^m - R_{h j i l}^g g^{lm} P_m^h \\ (2.3) \quad &= R_{j m}^g P_i^m - R_{h j i l}^g F^{lh} = R_{j m}^g P_i^m - H_{ji}. \end{aligned}$$

Here $R_{j m}^g$ are the components of the Ricci tensor of g , F^{lh} are the contravariant components of the fundamental 2-form F and $H_{ji} = R_{h j i l}^g F^{lh}$. The tensor H_{ji} is an anti-symmetric tensor, i.e., $H_{(ji)} = 0$. In fact, by means of $F^{lh} = -F^{hl}$, $R_{(hj)il}^g = R_{hj(il)} = 0$, we have

$$H_{ji} = \frac{1}{2} \left(R_{h j i l}^g + R_{i j i h}^g \right) F^{lh} = \frac{1}{2} \left(R_{h j i l}^g - R_{i h l j}^g \right) F^{lh}$$

and similarly

$$H_{ij} = \frac{1}{2} \left(R_{h i j l}^g - R_{j h l i}^g \right) F^{lh}$$

from which,

$$H_{ji} + H_{ij} = \left(R_{h j i l}^g - R_{i h l j}^g + R_{h i j l}^g - R_{j h l i}^g \right) F^{lh} = 0.$$

Changing the role j and i in (2.3), we write

$$(2.4) \quad \nabla_h \nabla_i P_j^h = R_{im}^g P_j^m - H_{ij}.$$

Adding (2.3) to (2.4), we have

$$\nabla_h (\nabla_j P_i^h + \nabla_i P_j^h) = R_{jm}^g P_i^m + R_{im}^g P_j^m - 2H_{(ij)},$$

which gives

$$R_{mj}^g P_i^m = -R_{im}^g P_j^m$$

by virtue of $R_{jm}^g = R_{mj}^g$. \square

Theorem 2.3. *In a nearly paraKähler manifold (M, g, P) , in order that $R^g = -R^{g*}$, it is necessary and sufficient that:*

$$\nabla^{gm} \nabla_m^g F_{ji} = 0,$$

where R^g and R^{g*} are respectively the Ricci tensor and the Ricci* tensor of g and F is the fundamental 2-form of (M, g, P) .

Proof. Let (M, g, P) be a nearly paraKähler manifold. The tensor R^{g*} which is locally expressed as

$$R_{ji}^{g*} = -H_{jm} P_i^m = -R_{hjml}^g F^{lh} P_i^m$$

is called the Ricci* tensor of M [8]. It is easy to see that $R_{jm}^{g*} P_i^m = -H_{ji}$. In the case, using $\nabla_j^g F_{im} = -\nabla_j^g F_{mi} = \nabla_m^g F_{ji}$, (2.3) becomes

$$\begin{aligned} \nabla_h^g \nabla_j^g P_i^h &= R_{jm}^g P_i^m + R_{jm}^{g*} P_i^m \\ \nabla_h^g \nabla_j^g (g^{mh} F_{im}) &= (R_{jm}^g + R_{jm}^{g*}) P_i^m \\ g^{mh} \nabla_h^g \nabla_j^g F_{im} &= (R_{jm}^g + R_{jm}^{g*}) P_i^m \\ \nabla^{gm} \nabla_m^g F_{ji} &= (R_{jm}^g + R_{jm}^{g*}) P_i^m, \end{aligned}$$

which completes the proof. \square

As a direct result of the Theorem 2.3, we have:

Corollary 2.4. *In an integrable nearly paraKähler manifold (M, g, P) , $R^g = -R^{g*}$.*

Proposition 2.5. *In a nearly paraKähler manifold (M, g, P) , the Ricci* tensor is hybrid with respect to the almost paracomplex structure P .*

Proof. For the Ricci* tensor in a nearly paraKähler manifold (M, g, P) , we have

$$\begin{aligned} R_{ji}^{g*} &= -H_{jm} P_i^m \\ -R_{jn}^{g*} P_m^n &= H_{jm}. \end{aligned}$$

From this, with help of $H_{(ij)} = 0$, we get

$$\begin{aligned} H_{jm} + H_{mj} &= -R_{jn}^{g*} P_m^n - R_{mn}^{g*} P_j^n \\ 0 &= R_{jn}^{g*} P_m^n + R_{mn}^{g*} P_j^n. \end{aligned}$$

Since $R_{mn}^{g*} = R_{nm}^{g*}$, we have

$$R_{jn}^{g*} P_m^n = -R_{nm}^{g*} P_j^n. \quad \square$$

Now, consider an almost paraHermitian manifold (M, g, P) , if the square norm $\|\nabla^g P\|^2$ of $\nabla^g P$, locally expressed by

$$\|\nabla^g P\|^2 = g^{ij} g^{mn} g_{lk} (\nabla^g P)_{im}^l (\nabla^g P)_{jn}^k$$

is zero, then the almost paraHermitian manifold is an isotropic paraKähler manifold. The isotropic paraKähler structure is analogue of the isotropic Kähler structure originally introduced in [3]. In [7], authors studied the isotropy property of antiKähler Codazzi manifolds. The similar problem can be taken into account for the case of nearly paraKähler manifolds.

Theorem 2.6. *In a nearly paraKähler manifold (M, g, P) , in order that $R^g = -R^{g^*}$, it is necessary and sufficient that the nearly paraKähler manifold is an isotropic paraKähler, where R^g and R^{g^*} are respectively the Ricci tensor and the Ricci* tensor of g .*

Proof. In a nearly paraKähler manifold (M, g, P) , transvecting $\nabla_j^g F_{im} = \nabla_m^g F_{ji}$ with F^{ji} , it follows that

$$(\nabla_j^g F_{im}) F^{ji} = 0$$

from which, by taking covariant derivative ∇_k^g

$$\begin{aligned} \nabla_k^g \{ (\nabla_j^g F_{im}) F^{ji} \} &= 0 \\ (\nabla_k^g \nabla_j^g F_{im}) F^{ji} + (\nabla_j^g F_{im}) (\nabla_k^g F^{ji}) &= 0 \\ (\nabla_k^g \nabla_m^g F_{ji}) F^{ji} + (\nabla_m^g F_{ji}) (\nabla_k^g F^{ji}) &= 0. \end{aligned}$$

Multiplying the above last equation with g^{km} , we find

$$\begin{aligned} g^{km} (\nabla_k^g \nabla_m^g F_{ji}) F^{ji} + g^{km} (\nabla_m^g F_{ji}) (\nabla_k^g F^{ji}) &= 0 \\ (\nabla^{gm} \nabla_m^g F_{ji}) P_n^i g^{nj} + g^{km} (\nabla_m^g P_j^l g_{li}) (\nabla_k^g P_n^i g^{nj}) &= 0 \\ (\nabla^{gm} \nabla_m^g F_{ji}) P_n^i g^{nj} + g^{km} g_{li} g^{nj} (\nabla^g P)_{mj}^l (\nabla^g P)_{kn}^i &= 0 \\ (R_{jm}^g + R_{jm}^{g^*}) P_i^m P_n^i g^{nj} + \|\nabla^g P\|^2 &= 0 \\ (R_{jm}^g + R_{jm}^{g^*}) g^{mj} + \|\nabla^g P\|^2 &= 0. \end{aligned}$$

Hence the proof is complete. \square

2.2. A special metric connection

In this section, on a nearly paraKähler manifold (M, g, P) we shall consider a special linear connection $\nabla_X Y = \nabla_X^g Y + S(X, Y)$ satisfying some special conditions, where S is a $(1, 2)$ -tensor field.

Applying the covariant derivative ∇_X to the fundamental 2-form F , we obtain

$$\begin{aligned} (\nabla_X F)(Y, Z) &= X(F(Y, Z)) - F(\nabla_X Y, Z) - F(Y, \nabla_X Z) \\ &= X(F(Y, Z)) - F(\nabla_X^g Y + S(X, Y), Z) \\ &\quad - F(Y, \nabla_X^g Z + S(X, Z)) \end{aligned}$$

$$\begin{aligned}
&= X(F(Y, Z)) - F(\nabla_X^g Y, Z) \\
&\quad - F(S(X, Y), Z) - F(Y, \nabla_X^g Z) - F(Y, S(X, Z)) \\
&= (\nabla_X^g F)(Y, Z) - g(PS(X, Y), Z) - g(PY, S(X, Z)) \\
&= (\nabla_X^g F)(Y, Z) + g(S(X, Y), PZ) - g(S(X, Z), PY) \\
&= (\nabla_X^g F)(Y, Z) + S_P(X, Y, Z) - S_P(X, Z, Y)
\end{aligned}$$

for any vector fields X, Y, Z on M , where $S_P(X, Y, Z) = g(S(X, Y), PZ)$. From on now, we consider a special linear connection ∇ which satisfies the following conditions: *i*) $\nabla F = 0$ and *ii*) $S_P(X, Y, Z) + S_P(Z, Y, X) = 0$. Hence, we can write the followings:

$$\begin{aligned}
&(\nabla_X^g F)(Y, Z) + S_P(X, Y, Z) - S_P(X, Z, Y) = 0 \\
&(\nabla_Y^g F)(Z, X) + S_P(Y, Z, X) - S_P(Y, X, Z) = 0 \\
&(\nabla_Z^g F)(X, Y) + S_P(Z, X, Y) - S_P(Z, Y, X) = 0
\end{aligned}$$

from which

$$\begin{aligned}
2S_P(X, Y, Z) &= (\nabla_X^g F)(Y, Z) - (\nabla_Y^g F)(Z, X) + (\nabla_Z^g F)(X, Y) \\
2S_P(X, Y, Z) &= (\nabla_X^g F)(Y, Z) \\
-2g(PS(X, Y), Z) &= g((\nabla_X^g P)Y, Z) \\
S(X, Y) &= -\frac{1}{2}P(\nabla_X^g P)Y.
\end{aligned}$$

Thus, we get the following theorem.

Theorem 2.7. *On a nearly paraKähler manifold (M, g, P) , a special linear connection ∇ is the linear connection having the covariant derivative given by*

$$\nabla_X Y = \nabla_X^g Y - \frac{1}{2}P(\nabla_X^g P)Y$$

for any vector fields X, Y on M .

The following result follows immediately from Proposition 2.1 and Theorem 2.7.

Corollary 2.8. *On an integrable nearly paraKähler manifold (M, g, P) , the special linear connection ∇ is equal to ∇^g .*

Theorem 2.9. *Let ∇ be the special linear connection on a nearly paraKähler manifold (M, g, P) . Then*

- i) $(\nabla_X P)Y = 2(\nabla_X^g P)Y$,
- ii) $T_\nabla(X, Y) = -P(\nabla_X^g P)Y$, where T_∇ is the torsion tensor of ∇ . Moreover, the $(0, 3)$ torsion tensor field defined by

$$T_\nabla(X, Y, Z) = g(T_\nabla(X, Y), Z)$$

satisfies:

$$\begin{aligned}
T(PX, Y, Z) &= T(X, PY, Z) = -T(X, Y, PZ) \text{ and} \\
T(PX, PY, Z) &= T(PX, Y, PZ) = -T(X, PY, PZ),
\end{aligned}$$

- iii) $(\nabla_X g)(Y, Z) = 0$, i.e., ∇ is a metric connection with respect to g ,
 iv) $R_{\nabla}(X, Y)Z = R^g(X, Y)Z + \frac{1}{2}(\nabla_X T)(Y, Z) - \frac{1}{2}(\nabla_Y T)(X, Z)$
 $+ \frac{1}{4}T(X, T(Y, Z)) - \frac{1}{4}T(Y, T(X, Z))$

for any vector fields X, Y, Z on M .

$$\begin{aligned} \text{Proof. i) } (\nabla_X P)Y &= \nabla_X(PY) - P(\nabla_X Y) \\ &= \nabla_X^g(PY) - \frac{1}{2}P(\nabla_X^g P)PY - P\nabla_X^g Y + \frac{1}{2}P^2(\nabla_X^g P)Y \\ &= \nabla_X^g(PY) - P\nabla_X^g Y + (\nabla_X^g P)Y \\ &= 2(\nabla_X^g P)Y. \end{aligned}$$

Here we used $(\nabla_X^g P)PY = -P(\nabla_X^g P)Y$.

ii)

$$\begin{aligned} (2.5) \quad T_{\nabla}(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y] \\ &= \nabla_X^g Y - \frac{1}{2}P(\nabla_X^g P)Y - \nabla_Y^g X \\ &\quad + \frac{1}{2}P(\nabla_Y^g P)X - [X, Y] \\ &= -\frac{1}{2}P[(\nabla_X^g P)Y - (\nabla_Y^g P)X] \\ &= -P(\nabla_X^g P)Y. \end{aligned}$$

Using the above equation, it follows that

$$\begin{aligned} T(PX, Y, Z) &= g(T(PX, Y), Z) = g(PT(X, Y), Z) \\ &= -g(T(X, Y), PZ) = -T(X, Y, PZ), \\ T(PX, Y, Z) &= g(T(PX, Y), Z) = g(T(X, PY), Z) = T(X, PY, Z) \end{aligned}$$

and

$$\begin{aligned} T(PX, PY, Z) &= g(T(PX, PY), Z) = g(-T(X, Y), Z) = -T(X, Y, Z), \\ T(PX, Y, PZ) &= g(T(PX, Y), PZ) = g(PT(X, Y), PZ) = -T(X, Y, Z), \\ T(X, PY, PZ) &= g(T(X, PY), PZ) = g(-PT(X, Y), PZ) = T(X, Y, Z). \end{aligned}$$

iii) With help of $g((\nabla_X^g P)Y, Z) = -g(Y, (\nabla_X^g P)Z)$, we find

$$\begin{aligned} (\nabla_X g)(Y, Z) &= X(g(Y, Z)) - g(\nabla_X^g Y, Z) - g(Y, \nabla_X^g Z) \\ &= X(g(Y, Z)) - g(\nabla_X^g Y - \frac{1}{2}P(\nabla_X^g P)Y, Z) \\ &\quad - g(Y, \nabla_X^g Z - \frac{1}{2}P(\nabla_X^g P)Z) \\ &= X(g(Y, Z)) - g(\nabla_X^g Y, Z) + \frac{1}{2}g(P(\nabla_X^g P)Y, Z) \\ &\quad + \frac{1}{2}g(Y, P(\nabla_X^g P)Z) - g(Y, \nabla_X^g Z) \end{aligned}$$

$$\begin{aligned}
&= (\nabla_X^g g)(Y, Z) + \frac{1}{2}g(P(\nabla_X^g P)Y, Z) + \frac{1}{2}g(Y, P(\nabla_X^g P)Z) \\
&= -\frac{1}{2}g((\nabla_X^g P)PY, Z) - \frac{1}{2}g(PY, (\nabla_X^g P)Z) \\
&= \frac{1}{2}g(PY, (\nabla_X^g P)Z) - \frac{1}{2}g(PY, (\nabla_X^g P)Z) \\
&= 0
\end{aligned}$$

iv) By virtue of (2.5), for the special metric connection ∇ , we have

$$\nabla_X Y = \nabla_X^g Y + \frac{1}{2}T(X, Y)$$

for any vector fields X, Y on M . With help of this, we calculate the curvature tensor field:

$$\begin{aligned}
R_{\nabla}(X, Y), Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \\
&= \nabla_X (\nabla_Y^g Z + \frac{1}{2}T(Y, Z)) - \nabla_Y (\nabla_X^g Z + \frac{1}{2}T(X, Z)) \\
&\quad - (\nabla_{[X, Y]}^g Z + \frac{1}{2}T([X, Y], Z)) \\
&= \nabla_X^g (\nabla_Y^g Z) + \frac{1}{2}T(X, \nabla_Y^g Z) + \frac{1}{2}\nabla_X^g (T(Y, Z)) \\
&\quad + \frac{1}{4}T(X, T(Y, Z)) - \nabla_Y^g (\nabla_X^g Z) - \frac{1}{2}T(Y, \nabla_X^g Z) \\
&\quad - \frac{1}{2}\nabla_Y^g (T(X, Z)) - \frac{1}{4}T(Y, T(X, Z)) - \nabla_{[X, Y]}^g Z \\
&\quad - \frac{1}{2}T([X, Y], Z) \\
&= R^g(X, Y, Z) + \frac{1}{2}T(X, \nabla_Y^g Z) + \frac{1}{2}\nabla_X^g (T(Y, Z)) \\
&\quad + \frac{1}{4}T(X, T(Y, Z)) - \frac{1}{2}T(Y, \nabla_X^g Z) - \frac{1}{2}\nabla_Y^g (T(X, Z)) \\
&\quad - \frac{1}{4}T(Y, T(X, Z)) - \frac{1}{2}T([X, Y], Z)
\end{aligned}$$

from which, using $[X, Y] = \nabla_X^g Y - \nabla_Y^g X$

$$\begin{aligned}
R_{\nabla}(X, Y), Z &= R^g(X, Y)Z + \frac{1}{2}(\nabla_X T)(Y, Z) - \frac{1}{2}(\nabla_Y T)(X, Z) \\
&\quad + \frac{1}{4}T(X, T(Y, Z)) - \frac{1}{4}T(Y, T(X, Z)). \quad \square
\end{aligned}$$

References

- [1] V. Cruceanu, P. Fortuny, and P. M. Gadea, *A survey on paracomplex geometry*, Rocky Mountain J. Math. **26** (1996), no. 1, 83–115.
- [2] P. M. Gadea and J. M. Masque, *Classification of almost para-Hermitian manifolds*, Rend. Mat. Appl. (7) **11** (1991), no. 2, 377–396.

- [3] E. Garcia-Rio and Y. Matsushita, *Isotropic Kähler structures on Engel 4-manifolds*, J. Geom. Phys. **33** (2000), no. 3-4, 288–294.
- [4] A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. (4) **123** (1980), 35–58.
- [5] S. Ivanov and S. Zamkovoy, *Parahermitian and paraquaternionic manifolds*, Differential Geom. Appl. **23** (2005), no. 2, 205–234.
- [6] V. F. Kirichenko, *Generalized Gray–Hervella classes and holomorphically projective transformations of generalized almost Hermitian structures*, Izv. Math. **69** (2005), no. 5, 963–987; translated from Izv. Ross. Akad. Nauk Ser. Mat. **69** (2005), no. 5, 107–132.
- [7] A. Salimov, K. Akbulut, and S. Turanli, *On an isotropic property of anti-Kähler-Codazzi manifolds*, C. R. Math. Acad. Sci. Paris **351** (2013), no. 21-22, 837–839.
- [8] K. Yano, *Differential geometry on complex and almost complex spaces*, International Series of Monographs in Pure and Applied Mathematics, **49**, A Pergamon Press Book. The Macmillan Co., New York, 1965.

AYDIN GEZER
ATATURK UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
25240, ERZURUM-TURKEY
Email address: agezer@atauni.edu.tr

SIBEL TURANLI
ERZURUM TECHNICAL UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
ERZURUM-TURKEY
Email address: sibel.turanli@erzurum.edu.tr