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DOMAINS WITH INVERTIBLE-RADICAL FACTORIZATION

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Abstract. We study those integral domains in which every proper ideal can be written as an invertible ideal multiplied by a nonempty product of proper radical ideals.

In [15] Vaughan and Yeagy introduced and studied the notion of *SP-domain*, i.e., an integral domain whose ideals are products of radical (also called semiprime) ideals. They proved that an SP-domain is always almost Dedekind (i.e., every localization at a maximal ideal is a rank one discrete valuation domain (DVR)). They also gave an example of an SP-domain which is not Dedekind. For examples of almost Dedekind domains which are not SP, see [16] and [6, Example 3.4.1]. The study of SP-domains was continued by Olberding (in [12]) who gave several characterizations for SP-domains inside the class of almost Dedekind domains and also gave a method to construct SP-domains starting from Boolean topological spaces.

In a sequence of papers ([10], [11], [13]) Olberding introduced and studied the concept of *ZPUI* (*Zerlegung Prim und Umkehrbaridealen*) *domain*, i.e., a domain for which every proper nonzero ideal can be factored as a product of an invertible ideal times a nonempty product of pairwise comaximal prime ideals (Olberding did his study for commutative rings, but we are interested here only in domain case). He showed that a domain A is ZPUI if and only if every proper nonzero ideal can be factored as a product of a finitely generated ideal times a nonempty finite product of prime ideals if and only if A is a strongly discrete h-local Prüfer domain [13, Theorem 1.1]. Let A be a domain. We recall that A is *h-local* if the factor ring A/I is local (resp. semilocal) for each nonzero prime ideal (resp. nonzero ideal) I of A. Also A is a *Prüfer domain* if its nonzero finitely generated ideals are invertible. A Prüfer domain is *strongly discrete* if it has no idempotent prime ideal except zero.

In this paper we study a new class of domains. Call a domain A an *ISP-domain* (*invertible semiprime domain*) if each proper ideal of A is can be written as an invertible ideal multiplied by a nonempty product of proper radical ideals. So any SP-domain (resp. ZPUI-domain) is an ISP-domain.

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In Section 1 we prove the following results. If A is an ISP-domain, then any factor domain of A and any (flat) overring of A are also ISP-domains (Propositions 2 and 3, see also Proposition 9). Any one-dimensional ISP-domain is almost Dedekind and, consequently, any Noetherian ISP-domain is a Dedekind domain (Corollary 4). In Section 2 we prove that if A is an ISP-domain, then A is a strongly discrete Prüfer domain and every nonzero prime ideal of A is contained in a unique maximal ideal (Theorem 5). Consequently, an ISP-domain such that every ideal has finitely many minimal prime ideals is a ZPUI-domain (Corollary 10). In Section 3 we consider the question whether every one-dimensional ISP-domain is an SP-domain. We provide a positive answer for domains in which every nonzero element is contained in at most finitely many noninvertible maximal ideals (Theorem 13). In particular, a one-dimensional ISP-domain having only finitely many noninvertible maximal ideals is an SP-domain (Corollary 14). In Section 4 we give an example of a two-dimensional ISP-domain A which is not h-local. Hence A is neither an SP-domain nor a ZPUI-domain.

Throughout this paper, our rings are commutative and unitary. For any undefined terminology, we refer the reader to [8] or [9].

1. Basic results

We recall the key definition of our paper.

Definition 1. We say that a domain A is an *ISP-domain* (*invertible semiprime domain*) if every proper nonzero ideal I of A can be written as $JQ_1 \cdots Q_n$ where $n \geq 1$, J is an invertible ideal and each Q_i is a proper radical ideal.

Clearly a ZPUI-domain or an SP-domain is an ISP-domain. The well-known Bezout domain $A = \mathbb{Z} + X\mathbb{Q}[X]$ (see [4] for its basic properties) is not an ISP-domain. Indeed, consider the ideal $I = X\mathbb{Z}[1/2] + X^2\mathbb{Q}[X]$. The radical ideals containing I are $X\mathbb{Q}[X]$ and $nA = n\mathbb{Z} + X\mathbb{Q}[X]$ with n a positive square-free integer. So there is no element $f \in A$ such that $I \subseteq fA$ and If^{-1} is a product of radical ideals. Note that every proper nonzero principal ideal gA can be written in the form required by Definition 1. Indeed, if $g \notin X\mathbb{Q}[X]$, then g is a product of principal primes and if $g \in X\mathbb{Q}[X]$, then $g = 2(g/2)A$. Note also that A is strongly discrete.

In this section we prove a few basic properties of ISP-domains.

Proposition 2. *If A is an ISP-domain and P a prime ideal of A, then A/P is an ISP-domain.*

Proof. Let $I \supset P$ be a proper ideal of A. As A is an ISP-domain, we can write $I = JH_1 \cdots H_n$ with J an invertible ideal, $n \geq 1$ and each H_i a proper radical ideal. Since all ideals $I, H_1, \ldots, H_n$ contain P, we get

$$I/P = (J/P)(H_1/P) \cdots (H_n/P)$$

with J/P invertible and each H_i/P a proper radical ideal. $\square$

Proposition 3. *Let A be an ISP-domain and B a flat overring of A. Then B is an ISP-domain.*

Proof. Let H be a proper nonzero ideal of B and $I = H \cap A$. By [2, Theorem 2], $IB = H$. As A is an ISP-domain, we can write $I = JQ_1 \cdots Q_n$ with J an invertible ideal, $n \geq 1$ and all Q_i's proper radical ideals. Then $H = IB = (JB)(Q_1B) \cdots (Q_nB)$, where JB is invertible and each Q_iB is a radical ideal. Indeed, since $A_{M \cap A} = B_M$ for every $M \in \mathrm{Max}(B)$ (cf. [2, Theorem 2]), it is easy to check locally that a radical ideal of A extends to a radical ideal of B. If every Q_iB is equal to B, then $H = JB$ and $WB = B$ where $W = Q_1 \cdots Q_n$. Hence $J \subseteq JB \cap A = H \cap A = I = JW \subseteq J$, so $J = JW$, thus $W = A$ (because J is invertible), a contradiction. □

We give a simple application of Proposition 3.

Corollary 4. *Any one-dimensional ISP-domain is almost Dedekind. Consequently, a Noetherian ISP-domain is a Dedekind domain.*

Proof. Let A be a one-dimensional ISP-domain. By Proposition 3, we may assume that A is local with maximal ideal M. Let $x \in M - \{0\}$. Since the radical ideals of A are 0 and M, we get $xA = yM^k$ for some $y \in A$ and $k \geq 1$, so M is invertible, hence A is a DVR. For the "Consequently" part, assume, by the contrary, that A is a Noetherian ISP-domain which is not Dedekind. By the first part, $\dim(A) \geq 2$, so, using Proposition 3, we may assume that A is a two-dimensional local domain (with maximal ideal M). Let $x \in M - M^2$, P a height one prime ideal containing x and let $y \in M - P$. Since $P \not\subseteq M^2$, M is minimal over (P, y^2) and A is an ISP-domain, we get $(P, y^2) = M$. Modding out by P, we get a contradiction. □

2. ISP domains are Prüfer strongly discrete

The following theorem is the main result of this paper.

Theorem 5. *If A is an ISP-domain, then*

(a) *A is a strongly discrete Prüfer domain, and*
(b) *every nonzero prime ideal of A is contained in a unique maximal ideal.*

In particular, a local domain is an ISP-domain if and only if it is a strongly discrete valuation domain.

We need a string of three lemmas.

Lemma 6. *If A is an ISP-domain and $P \subset M$ are nonzero prime ideals of A, then $P \subseteq M^2A_M$.*

Proof. By Proposition 3, we may assume that A is local with maximal ideal M. Assume that $P \not\subseteq M^2$ and take $x \in M - P$. Since A is an ISP-domain and $P \not\subseteq M^2$, we get that (P, x^2) is a radical ideal, so $(P, x^2) = (P, x)$ which gives a contradiction after modding out by P. □

Lemma 7. *Let A be an ISP-domain, $P \subset M$ prime ideals and $x \in M - P$ such that M is minimal over (P, x). Then MA_M is a principal ideal.*

Proof. By Proposition 3, we may assume that A is local with maximal ideal M. We show first that M is not idempotent. On contrary assume that $M^2 = M$. Note that $\sqrt{(P, x)} = M$ is the only radical ideal containing (P, x). As A is an ISP-domain and $M = M^2$, we get $(P, x) = yM$ for some $y \in A$. As $P \subseteq yM$, we get $y \notin P$ (otherwise $P = yA \subseteq yM$), hence $P = Py$. From $x \in yM$, we get $x = yz$ for some $z \in M$. Now from $(Py, yz) = yM$, we get $(P, z) = M$, so M/P is a principal idempotent nonzero maximal ideal of A/P, a contradiction. Thus M is not idempotent and let us pick $w \in M - M^2$. By Lemma 6, M is the only prime ideal containing w, so $wA = M$ because A is an ISP-domain. □

Lemma 8. *If A is an ISP-domain and I an invertible radical proper ideal of A, then A/I is zero-dimensional.*

Proof. On contrary assume that $\dim(A/I) \geq 1$. Then there exist two prime ideals $P \subset M$ and $x \in M - P$ such that $I \subseteq P$ and M is minimal over (P, x). By Lemma 7, MA_M is principal. Localizing at M, we may assume that A is local with maximal ideal M. Then $I = yA$ and $M = zA$ for some $y, z \in A$. As $I \subset M$, we get $y = az^2$ for some $a \in A$, so $az \in \sqrt{yA} = yA$, hence $y = az^2 \in yzA$, thus $1 \in zA = M$, a contradiction. □

Proof of Theorem 5. (a) By [13, Lemma 3.2], it suffices to show that PA_P is a principal ideal for every nonzero prime ideal P of A. Set $B = A_P$ and $M = PA_P$. By Proposition 3, B is an ISP-domain. Given $x \in M - \{0\}$, we write $xB = yH_1 \cdots H_n$ with $y \in B$, $n \geq 1$ and H_i a proper radical ideal for $i = 1$ to n. Then each H_i is invertible hence principal, because B is local. By Lemma 8, we have $Spec(B/H_1) = \{M/H_1\}$, hence $H_1 = \sqrt{H_1} = M$.

(b) By Proposition 3, we may assume that A is semilocal. Indeed, if M_1 and M_2 are two distinct maximal ideals containing a nonzero prime ideal, then (b) fails for A_S, where $S = A - (M_1 \cup M_2)$. Now let I be a nonzero radical ideal. Since A is a semilocal Prüfer domain, it follows that I has finitely many minimal primes, say $P_1, \ldots, P_n$. Then $I = P_1 \cap \cdots \cap P_n = P_1 \cdots P_n$ because $P_1, \ldots, P_n$ are incomparable prime ideals in a Prüfer domain, hence pairwise comaximal. Since A is an ISP-domain and every nonzero radical ideal is a product of primes, A is a ZPUI-domain. By [13, Theorem 1.1], A is h-local, so (b) holds. The "in particular" assertion follows from [13, Theorem 1.1]. □

We give two corollaries of Theorem 5.

Corollary 9. *Any overring of an ISP-domain is also an ISP-domain.*

Proof. Let A be an ISP-domain and B an overring of A. By Theorem 5, A is a Prüfer domain, so B is A-flat, cf. [14, page 798]. Apply Proposition 3. □

Corollary 10. *For a domain A, the following are equivalent.*

(a) *A is a ZPUI-domain.*

(b) *A is an h-local strongly discrete Prüfer domain.*
(c) *A is an h-local ISP-domain.*
(d) *A is a generalized Dedekind ISP-domain.*
(e) *A is an ISP-domain such that* $\mathrm{Min}(I)$ *is finite for each ideal* I.

Proof. (a) ⇔ (b) is a part of [13, Theorem 1.1]. Implications [(a) and (b)] ⇒ (c) ⇒ (d) ⇒ (e) are well-known. For (e) ⇒ (a), repeat the second half of the proof of Theorem 5 part (b). □

3. Almost Dedekind ISP-domains

In this section, we consider the question whether any one-dimensional ISP-domain is an SP-domain. First, we recall some terminology from [12]. Let A be an almost Dedekind domain. The maximal ideals of A containing a radical invertible ideal are called *non-critical*, while the others are called *critical*. Given I an ideal of A and $n \geq 1$, we set $V_n(I) = \{M \in \mathrm{Max}(A) \mid I \subseteq M^n\}$. Note that $V_{n+1}(I) \subseteq V_n(I)$ and $V_1(I)$ is the usual Zariski closed set $V(I)$. Next, we recall [12, Theorem 2.1] and add a new assertion (g).

Theorem 11 ([12, Theorem 2.1])**.** *For an almost Dedekind domain A, the following assertions are equivalent.*

(a) *A is an SP-domain.*
(b) *A has no critical maximal ideals.*
(c) *The radical of an invertible ideal is invertible.*
(d) *Ever principal ideal is a product of radical ideals.*
(e) *For every nonzero proper (principal) ideal* I *and* $n \geq 1$*, the set* $V_n(I)$ *is (Zariski) closed in Spec(A) and* $V_m(I)$ *is empty for some large* m.
(f) *Every nonzero proper ideal* I *can be factorized (uniquely) as* $I = J_1 J_2 \cdots J_n$ *with radical ideals* $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n$.
(g) *For every nonzero proper ideal* I*, we have* $I = \sqrt{I}H$ *for some ideal* H.

Proof. Since only (g) is new, it suffices to prove the equivalence of (f) and (g). (g) ⇒ (f) We have $I = \sqrt{I}H_1$ and $H_1 = \sqrt{H_1}H_2$ for some ideals H_1 and H_2. Set $J_1 = \sqrt{I}$ and $J_2 = \sqrt{H_1}$, so $I = J_1 J_2 H_2$. From $I \subseteq H_1$, we get $J_1 \subseteq J_2$. Repeating, we get $I = J_1 J_2 \cdots J_n H_n$ with radical ideals $J_1 \subseteq \cdots \subseteq J_n$. If some H_n is A, we are done. If not, let M be a maximal ideal containing all J_i's. Then $I = J_1 J_2 \cdots J_n H_n \subseteq M^n$ for each $n \geq 1$, which is a contradiction because A_M is a DVR. Conversely, from $I = J_1 \cdots J_n$ with $J_1 \subseteq \cdots \subseteq J_n$ radical ideals, we get $\sqrt{I} = J_1$, so we are done. □

In the next lemma, we recall two known facts.

Lemma 12. *If A is an almost Dedekind domain which is not Dedekind, then*

(a) *Every noninvertible nonzero ideal of A is contained in some noninvertible maximal ideal.*
(b) *Every infinite closed subset of* $\mathrm{Max}(A)$ *contains some noninvertible maximal ideal.*

Proof. (a) is a well-known application of Zorn's Lemma (every non finitely generated ideal is contained in a non finitely generated prime ideal).

(b) Let I be a nonzero ideal such that $V(I)$ is infinite. By (a), we may assume that I is invertible, so the assertion follows from [6, Proposition 3.2.2]. We give an alternative proof. For each $P \in V(I)$, we have $IA_P = (PA_P)^{n_P}$ for some (unique) positive integer n_P. Consider the ideal $H = \sum_{P \in V(I)} IP^{-n_P}$. It suffices to show that H is not finitely generated, because $I \subseteq H$ implies $V(H) \subseteq V(I)$, so part (a) applies. Suppose that H is finitely generated. Then there exist distinct ideals $P_1, \dots, P_{k+1} \in V(I)$ such that $IP_{k+1}^{-n_{k+1}} \subseteq \sum_{i=1}^{k} IP_i^{-n_i}$ where $n_j = n_{P_j}$. Since the ideals P_j are mutually comaximal, we have $IP_{k+1}^{-n_{k+1}} \subseteq I(\cap_{i=1}^{k} P_i^{n_i})^{-1}$, cf. [10, Lemma 5.1]. We cancel I and get $\cap_{i=1}^{k} P_i^{n_i} \subseteq P_{k+1}$, which is a contradiction. $\square$

Recall that a domain A *has weak factorization*, if every nonzero nondivisorial ideal I can be factored as the product of its divisorial closure I_ν and a finite product of maximal ideals; i.e., $I = I_\nu M_1 M_2 \cdots M_n$ where $M_1, M_2, \dots, M_n$ are maximal ideals, cf. [5]. By [6, Proposition 4.2.14], an almost Dedekind domain A has weak factorization if and only if every nonzero element of A is contained in at most finitely many noninvertible maximal ideals.

Now let A be an almost Dedekind domain A which has weak factorization. Denote by Z the set of noninvertible maximal ideals of A. We introduce an ad-hoc concept: call an ideal H of A a *clean ideal*, if H is invertible, $V(H) \cap Z = \{M\}$ and $H \not\subseteq M^2$. Let $M \in Z$ and $f \in M - \{0\}$. By our hypothesis $V(f) \cap Z$ is finite, say equal to $\{M, M_1, \dots, M_n\}$. By Prime Avoidance Lemma (e.g. [8, Proposition 4.9]), we can pick an element $g \in M - (M^2 \cup M_1 \cup \cdots \cup M_n)$, so (f, g) is clean. Hence every $M \in Z$ contains a clean ideal. With terminology and notation above, we have:

Theorem 13. *For an almost Dedekind domain A which has weak factorization, the following assertions are equivalent.*

(a) *A is an SP-domain.*
(b) *A is an ISP-domain.*
(c) *For every clean ideal H, the set $V_2(H)$ is finite.*
(d) *Every $M \in Z$ contains a clean ideal H such that $V_2(H)$ is finite.*

Proof. We may assume that A is not a Dedekind domain. Set $F = \mathrm{Max}(A) - Z$. (a) $\Rightarrow$ (b) is obvious. (b) $\Rightarrow$ (c) Assume, to the contrary, that H is a clean ideal and $V_2(H)$ contains an infinite set $\{P_n \mid n \geq 1\} \subseteq F$. Set $V(H) \cap Z = \{M\}$. Let I be the (integral) ideal $\sum_{n \geq 0} HP_{2n+1}^{-1}$. Since $H \subseteq I$ and $V(H) \cap Z = \{M\}$, we get $V(I) \cap Z = \{M\}$, because $M \supseteq H = P_{2n+1}HP_{2n+1}^{-1}$ implies $M \supseteq HP_{2n+1}^{-1}$. As A is an ISP-domain, we can write $I = JQ$ with J an invertible ideal and $Q \neq A$ a product of radical ideals. Since $M \in V(I) - V_2(I)$, we have one of the two cases below.

Case 1: $M \supseteq J$ and $M \not\supseteq Q$. Then $V(Q) \cap Z$ is empty, so Q is invertible, cf. Lemma 12. So $I = JQ$ is invertible, hence finitely generated. Then $HP_{2n+1}^{-1} \subseteq HP_1^{-1} + \cdots + HP_{2n-1}^{-1}$ for some $n \geq 1$. Since H can be cancelled and the other ideals involved are invertible and comaximal, we get $P_{2n+1}^{-1} \subseteq (P_1 \cap \cdots \cap P_{2n-1})^{-1}$ (cf. [10, Lemma 5.1]), hence $P_{2n+1} \supseteq P_1 \cap \cdots \cap P_{2n-1}$, which is a contradiction.

Case 2: $M \not\supseteq J$ and $M \supseteq Q$. Since $H \subseteq Q$ and $H \not\subseteq M^2$, we have that $V_2(Q) \cap Z = \emptyset$. As Q is a product of radical ideals, [1, Lemma 1.10] shows that $V_2(Q)$ is closed, so $V_2(Q)$ is finite, cf. Lemma 12. Note that $P_{2n} \in V_2(I)$ for every $n \geq 1$. Consequently, there exists some $m \geq 1$ such that $P_{2n} \in V(J)$ for each $n \geq m$. By Lemma 12 and the fact that $H \subseteq J$, we get $V(J) \cap Z = \{M\}$, which is a contradiction.

(c) $\Rightarrow$ (d) is clear.

(d) $\Rightarrow$ (a) By [12, Theorem 2.1], it suffices to show that each $M \in Z$ contains an invertible radical ideal. By (d), M contains a clean ideal H such that $V_2(H)$ is finite, say equal to $\{P_1, \ldots, P_n\}$. For each i between 1 and n, we have $HA_{P_i} = P_i^{k_i} A_{P_i}$ for some $k_i \geq 2$. Then $HP_1^{-k_1} \cdots P_n^{-k_n}$ is an invertible radical ideal contained in M. □

The SP-domain A constructed in [12, Example 4.3] has nonzero Jacobson radical and no $M \in \text{Max}(A)$ finitely generated. Thus A does not have weak factorization.

Corollary 14. *Let A be almost Dedekind domain having only finitely many noninvertible maximal ideals. Then A is an ISP-domain if and only if A is an SP-domain.*

Corollary 15. *Let A be an ISP-domain which has weak factorization and B a one-dimensional overring of A. Then B is an SP-domain.*

Proof. By Theorem 5, A is a strongly discrete Prüfer domain, so B has weak factorization, cf. [6, Corollary 4.3.3]. Now apply Corollary 9 and Theorem 13. □

The following question remains.

Question 16. Is every one-dimensional ISP-domain an SP-domain?

4. An example

In this final section we give an example of a two-dimensional ISP-domain A which is not h-local. Hence A is neither an SP-domain nor a ZPUI-domain.

Proposition 17. *Let C be an SP-domain but not Dedekind, $M = qC$ a maximal principal ideal of C and D a DVR with quotient field C/M. Assume there exists a unit p of C such that $\pi(p)$ generates the maximal ideal of D, where $\pi : C \to C/M$ is the canonical map. Then the pull-back domain $A = \pi^{-1}(D)$ is a two-dimensional ISP-domain which is not h-local.*

Proof. As $\pi(Mp^{-1}) = 0$, it follows that $M \subseteq pA$, so A/pA is the residue field of D, because $A/M = D$ and $\pi(p)$ generates the maximal ideal of D. Also, the only prime ideal of A strictly containing M is the maximal ideal pA. By standard pull-back arguments (see for instance [7, Lemma 1.1.4]), the map $P \mapsto P \cap A$ is a bijection from $Spec(C) - V(M)$ to $Spec(A) - V(M)$ and $A_{P\cap A} = C_P$. By [7, Corollary 1.1.9], A is a two-dimensional Prüfer domain. Also, by [7, Lemma 1.1.6], we have $A[p^{-1}] = C[p^{-1}] = C$. Roughly speaking, $Spec(A)$ is obtained from $Spec(C)$ by adding the maximal ideal $pA \supseteq M$. Since C is an almost Dedekind domain which is not Dedekind, there exists a nonzero element $z \in A$ belonging to infinitely many maximal ideals of A, so A is not h-local. By [7, Proposition 5.3.3], $B = A_{pA}$ is a two-dimensional strongly discrete valuation domain. It follows that $\cap_{t\geq 1} p^t A = M$.

Let I be an ideal of A. We observe that $I = IB \cap IC$. Indeed, if $N \in \text{Max}(A) - \{pA\}$, then $I \subseteq IC_{A-N} = IA_N$, so $IB \cap IC \subseteq \cap_{Q\in\text{Max}(A)} IA_Q = I$. In particular, we have $A = B \cap C$. Since C is almost Dedekind and $M = qC$, we can write $IC = M^i J$ where J is an ideal of C with $M + J = C$ and $i \geq 0$, so $IC = M^i \cap J$. We also see that $H := J \cap A \not\subseteq M$. As $\cap_{t\geq 1} p^t A = M$, we can write $H = p^j L = p^j A \cap L$ where L is an ideal of A with $L \not\subseteq pA$ and $j \geq 0$. Consequently we get

$$IC \cap A = M^i \cap J \cap A = M^i \cap H = M^i \cap p^j A \cap L$$

which equals either $M^i \cap L$ if $i \geq 1$ or $p^j A \cap L$ if $i = 0$. Using basic facts on valuation domains (see [8, Section 17]), it suffices to consider the following three cases. Each time we use the equality $I = (IB \cap A) \cap (IC \cap A)$.

Case 1: $IB = p^n B$ for some $n \geq 0$. We have $IB \cap A = p^n A$. If $i \geq 1$, we get $I = p^n A \cap M^i \cap L = M^i L$. If $i = 0$, we get $I = p^n A \cap p^j A \cap L = p^k L$ with $k = \max(n, j)$.

Case 2: $IB = M^n$ for some $n \geq 1$. If $i \geq 1$, we get $I = M^n \cap M^i \cap L = M^k L$ with $k = \max(n, i)$. If $i = 0$, we get $I = M^n \cap p^j A \cap L = M^n L$.

Case 3: $IB = p^n q^m A$ for some $m \geq 1$ and $n \in \mathbb{Z}$. We have $IB \cap A = p^n q^m A$, because pA is the only maximal ideal containing q. If $i > m \geq 1$, we get $I = p^n q^m A \cap M^i \cap L = M^i L$. If $m \geq i \geq 1$, we get $I = p^n q^m A \cap M^i \cap L = p^n q^m L$. If $i = 0$, we get $I = p^n q^m A \cap p^j A \cap L = p^n q^m L$.

Consequently, to complete our proof, it suffices to show that L is a product of radical ideals. Since C is an SP-domain, we can write $LC = H_1 \cdots H_n$ with each H_i a radical ideal of C. Then each $J_i = H_i \cap A$ is a radical ideal of A. Note that none of ideals J_i is contained in pA, since $L \not\subseteq pA$. Set $R = J_1 \cdots J_n$. Then $R + pA = A$ and $L + pA = A$, so $R : p = R$ and $L : p = L$. Since $RC = H_1 \cdots H_n = LC$, we get $L = LC \cap A = RC \cap A = R$. □

Finally, we construct a specific domain satisfying the hypothesis of Proposition 17. We modify appropriately [6, Example 3.4.1]. If A is a domain and $P_1, \ldots, P_n$ are prime ideals of A, we denote by $A_{P_1\cup\cdots\cup P_n}$ the fraction ring of A with denominators in $A - (P_1 \cup \cdots \cup P_n)$. Let y and $(x_n)_{n\geq 1}$ be indeterminates over the rational field $\mathbb{Q}$. Consider the domain

$$C = \bigcup_{n\geq 1} \mathbb{Q}[x_1, \ldots, x_n, y/(x_1 \cdots x_n)]_{(x_1)\cup\cdots\cup(x_n)\cup(y/(x_1\cdots x_n))}.$$

As C is a union of an ascending chain of (semi-local) PID's, it is a one-dimensional Bezout domain. Adapting the proof of [6, Example 3.4.1], we see that the maximal ideals of C are $N = \sum_{n\geq 1}(y/(x_1 \cdots x_n))C$ and the principal ideals $(x_nC)_{n\geq 1}$. As $yC_M = MC_M$ for each $M \in \mathrm{Max}(C)$, it follows that yC is a radical ideal, hence N is non-critical. By [12, Corollary 2.2], C is an SP-domain. The residue field C/x_1C is isomorphic to $K(y/x_1)$ where $K = \mathbb{Q}(x_n; n \geq 2)$. Then $D = K[y/x_1]_{(y/x_1)}$ is a DVR with quotient field C/x_1C. Note that $x_1 + y/x_1$ is a unit of $\mathbb{Q}[x_1, y/x_1]_{(x_1)\cup(y/x_1)}$, hence a unit of C. Moreover, the canonical map $C \to C/x_1C$ sends $x_1 + y/x_1$ to y/x_1 which is a generator of the maximal ideal of D. Thus C satisfies the hypothesis of Proposition 17.

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