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LOCALLY SYMMETRIC ALMOST COKÄHLER 5-MANIFOLDS WITH KÄHLERIAN LEAVES

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Abstract. Let M be a compact almost coKähler 5-manifold with Kählerian leaves. In this paper, we prove that M is locally symmetric if and only if it is locally isometric to a Riemannian product of a unit circle S^1 and a locally symmetric compact Kähler 4-manifold.

1. Introduction

The study of topology and geometry of (almost) coKähler manifolds dates back to the 1960s. In early literature, (almost) coKähler manifolds were usually referred to as (almost) cosymplectic manifolds. For example, in 1967, D. E. Blair and S. I. Goldberg in [4] obtained that the Betti numbers of any compact cosymplectic manifold are non-zero. Also, a characterization for a quasi-Sasakian manifold to be a cosymplectic manifold was given by D. E. Blair [2]. In 1969, S. I. Goldberg and K. Yano in [8] obtained a condition for almost cosymplectic structures to be integrable. Later, Z. Olszak in [12] established many interesting curvature properties of almost cosymplectic manifolds. Until recently, H. Li in [10] studied topology construction of coKähler manifolds via Kähler mapping tori, showing that coKähler manifolds are really odd dimensional analog of Kähler manifolds. In some recent papers, many authors started to adopt the new terminology "(almost) coKähler manifolds" instead of "(almost) cosymplectic manifolds". We refer the reader to a recent review paper [5] and many references therein for more details.

An almost coKähler manifold with Kählerian leaves, introduced by Z. Olszak [13], means that any leaf of the contact distribution of the manifold is Kählerian. Such manifolds was later studied by P. Dacko and Z. Olszak [7] who proved that a conforamlly flat almost coKähler manifold of dimension greater than three with Kählerian leaves is locally flat and coKähler. D. Perrone in [15] gave a complete classification of homogeneous almost coKähler 3-manifolds. Motivated by the above results, the present author in [17] studied

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almost coKähler manifolds under condition of local symmetry, generalizing partially Perrone's results to any odd dimensions. Since coKähler manifolds can be regarded as a special case of locally conformal almost coKähler manifolds, then some results of locally symmetric coKähler manifolds can be seen in V. F. Kirichenko and S. V. Kharitonova [9, Theorem 8]. It is worth pointing out that a complete classification theorem of locally symmetric coKähler manifolds with vanishing Bochner curvature tensors was obtained by C. Qu and C. Z. Ouyang [16, Theorem 2.3].

In the present paper, improving the corresponding result shown in [17, Theorem 3.1] we obtain some new classification results regarding locally symmetric five-dimensional almost coKähler manifolds as the following.

Theorem 1.1. *If a five-dimensional almost coKähler manifold M with Kählerian leaves is locally symmetric, then one of the following two cases occur.*

- *M is coKähler and is locally isometric to a Riemannian product of a real line or a unit circle and a four-dimensional locally symmetric Kähler manifold.*
- *M is non-coKähler and is Einstein with negative scalar curvature.*

2. Preliminaries

Let M^{2n+1} be a smooth manifold of dimensional $2n+1$. An almost contact metric structure defined on M^{2n+1} means that there exist a $(1,1)$-type tensor field ϕ, a global vector field ξ, a 1-form η and a Riemannian metric g such that

$$\begin{aligned} &\phi^2 = -\mathrm{id} + \eta\otimes\xi, \quad \eta(\xi)=1, \\ &g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \end{aligned} \tag{2.1}$$

where id denotes the identity endomorphism. We call η an *almost contact* 1-*form* and ξ its dual *Reeb vector field*. According to (2.1) we have $\phi(\xi)=0$, $\eta\circ\phi=0$ and $\mathrm{rank}(\phi)=2n$. On the product manifold $M^{2n+1}\times\mathbb{R}$ we define an almost complex structure J by

$$J\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

where X denotes a vector field tangent to an almost contact metric manifold M^{2n+1}, t is the coordinate of $\mathbb{R}$, f is a smooth function defined on the product.

An almost contact structure is said to be *normal* if the above almost complex structure J is integrable, i.e., J is a complex structure. According to Blair [3], the normality of an almost contact structure is expressed by

$$[\phi,\phi] = -2\mathrm{d}\eta\otimes\xi,$$

where $[\phi,\phi]$ denotes the Nijenhuis tensor of ϕ defined by

$$[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X,\phi Y] - \phi[\phi X, Y] - \phi[X,\phi Y]$$

for any vector fields X, Y on M^{2n+1}. The *fundamental* 2-*form* Φ on an almost contact metric manifold M^{2n+1} is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any vector fields X and Y.

Throughout this paper, by an *almost coKähler manifold*, we mean an almost contact metric manifold such that both the 1-form η and 2-form Φ are closed (cf. [5]). In particular, an almost coKähler manifold is said to be a *coKähler manifold* if the associated almost contact structure is normal, which is also equivalent to

$$\nabla\phi = 0, \text{ or equivalently } \nabla\Phi = 0. \tag{2.2}$$

We remark that the above (almost) coKähler manifolds are in fact (almost) cosymplectic manifolds studied in these papers [2–8], Olszak [12–14] and [15–17]. On an almost coKähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, we set $h = \frac{1}{2}\mathcal{L}_\xi\phi$ and $h' = h \circ \phi$. Note that both h and h' are symmetric operators. The following formulas can be found in Olszak [12, 13] and Perrone [15]:

$$h\xi = 0, \ \ h\phi + \phi h = 0, \ \ \mathrm{tr}(h) = \mathrm{tr}(h') = 0, \tag{2.3}$$

$$\nabla_\xi\phi = 0, \ \ \nabla\xi = h', \ \ \mathrm{div}\xi = 0, \tag{2.4}$$

$$\nabla_\xi h = -h^2\phi - \phi l, \tag{2.5}$$

$$\phi l\phi - l = 2h^2, \tag{2.6}$$

where $l := R(\cdot, \xi)\xi$ is the Jacobi operator along the Reeb vector field and the Riemannian curvature tensor R is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

and tr and div denote the trace and divergence operators, respectively.

3. Loclly symmetric almost coKähler 5-manifolds

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost coKähler manifold. We denote by $\mathcal{D}$ the distribution $\ker\eta$. It is easily to check that any leaf of the distribution $\mathcal{D}$ admits an almost Kähler structure. Z. Olszak in [13] obtained that the above almost Kähler structure is Kähler if and only if

$$(\nabla_X\phi)(Y) = g(X, hY)\xi - \eta(Y)hX \tag{3.1}$$

for any vector fields X, Y. According to [13], if (3.1) holds on an almost coKähler manifold, then the manifold is said to be with Kählerian leaves. From (2.2) we see that an almost coKähler manifold is coKähler if and only if it has Kählerian leaves and h vanishes identically.

In earlier literature, there are few results on locally symmetric almost coKähler manifolds. We now recall these results.

An almost contact metric manifold is called a *locally conformal almost coKähler* manifold (cf. [14]) if it admits a 1-form ω satisfying

$$\mathrm{d}\omega = 0, \ \mathrm{d}\eta = \omega \wedge \eta, \ \mathrm{d}\Phi = 2\omega \wedge \Phi.$$

Notice that any almost coKähler manifold is a locally conformal almost coKähler manifold with ω vanishing. Kirichenko and Kharitonova [9, Theorem 8] proved that any locally symmetric normal locally conformal almost coKähler manifold is either a coKähler manifold with locally symmetric Kähler component or a manifold of constant non-positive curvature.

Qu and Ouyang [16, Theorem 2.3] proved that a locally symmetric coKähler manifold with vanishing Bochner curvature tensor is either a coKähler manifold of constant ϕ-holomorphic sectional curvature or a product of a coKähler manifold of constant ϕ-holomorphic sectional curvature c and a Kähler manifold of constant holomorphic sectional curvature $-c$ with $c \neq 0$.

Perrone [15, Proposition 3.1] proved that an almost coKähler 3-manifold is locally symmetric if and only if it is locally isometric to a product $\mathbb{R} \times N^2(c)$, where N^2 denotes a Kähler surface of constant Gauss curvature c.

After presenting some useful lemmas, the present author in [17, Theorem 3.1] proved that a locally symmetric almost coKähler manifold of dimension greater than 3 with Kählerian leaves is either a coKähler manifold with locally symmetric Kählerian leaves or ξ is an eigenvector field of the Ricci operator with negative constant eigenvalue.

In this section, we improve the above result and show that the second case of the result in fact means Einstein for dimension five. Firstly, we need

Lemma 3.1 ([15]). *On any locally symmetric almost coKähler manifold we have* $\nabla_\xi h = 0$.

Applying the above lemma, the present author in [17] obtained the following two propositions.

Proposition 3.1 ([17]). *Let M^{2n+1} be a locally symmetric almost coKähler manifold, then the multiplicity of the eigenvalue zero of h is at least three and we have*

$$g((\nabla_X h')h'Y + (\nabla_Y h^2)X, Z) + g((\nabla_Z h')h'Y, X) = 2g((\nabla_{h'Y} h')X, Z) \tag{3.2}$$

for any vector fields X, Y, Z.

Proposition 3.2 ([17]). *Let M^{2n+1} be a locally symmetric almost coKähler manifold of dimension greater than three with Kählerian leaves. Then, either M^{2n+1} is coKähler or it is non-coKähler such that ξ is an eigenvector field of the Ricci operator with negative constant eigenvalue.*

Applying the above two propositions we now present the proof of Theorem 1.1, improving the later case of Proposition 3.2.

Proof of Theorem 1.1. Let M^5 be an almost coKähler manifold of dimension five. If $h = 0$, from (3.1) we see that ϕ is parallel along the Levi-Civita connection and hence by (2.2) it is seen that the manifold is coKähler. Also, it is well-known that any coKähler manifold is locally isometric to a Riemannian product of a real line or a unit circle S^1 and a Kähler manifold (see [3]). Then the proof for the first case follows.

From Proposition 3.1 we next consider the case $h \neq 0$ on certain open subset, i.e., the spectrum of h is given by $\{0, \lambda, \lambda\}$ such that the multiplicity of the eigenvalue zero of h is three. Applying Lemma 3.1, we obtain from (2.5) that $h^2\phi + \phi l = 0$ and hence using this in (2.6) we get

$$l = -h^2. \tag{3.3}$$

In view of (3.3), in what follows we denote by e and v two unit eigenvector fields of h orthogonal to ξ with corresponding eigenvalues $\lambda > 0$ and 0, respectively. Therefore, ϕe and ϕv are also two unit eigenvector fields of h with corresponding eigenvalues $-\lambda$ and 0, respectively. According to Proposition 3.2 we have $Q\xi = -\mathrm{tr}(h^2)\xi$, where the eigenvalue of ξ of the Ricci operator is a constant (see [17, p. 746]). It follows directly that λ is a positive constant.

Applying Lemma 3.1, (2.3) and (3.1) we write

$$\nabla_\xi e = a_0\phi e,\ \nabla_\xi \phi e = -a_0 e,\ \nabla_\xi v = a_1\phi v,\ \nabla_\xi \phi v = -a_1 v,$$

where the first two terms follow from $\nabla_\xi e = \frac{1}{\lambda}h\nabla_\xi e$ and $\nabla_\xi \phi e = -\frac{1}{\lambda}h\nabla_\xi\phi e$; the last two terms follow from $h\nabla_\xi v = 0$ and $h\nabla_\xi \phi v = 0$ and a_0 and a_1 are assumed to be smooth functions. Moreover, from (2.4) we have $\nabla_e \xi = -\lambda\phi e$ and $\nabla_{\phi e}\xi = -\lambda e$. Hence by a direct calculation we have $R(e, \xi)\xi = -\lambda(\lambda + 2a_0)e$. On the other hand, from (3.3) we have $R(e, \xi)\xi = -\lambda^2 e$. Comparing this two relations we have $a_0 = 0$ because of $\lambda > 0$ and the following

$$\nabla_\xi e = 0,\ \nabla_\xi \phi e = 0,\ \nabla_\xi v = a_1\phi v,\ \nabla_\xi \phi v = -a_1 v. \tag{3.4}$$

Substituting Y with v in (3.2) gives $\nabla_v h^2 = 0$. From this, (2.4) and (3.1) we obtain

$$\nabla_v \xi = 0,\ \nabla_v e = a_3\phi e,\ \nabla_v \phi e = -a_3 e,\ \nabla_v v = a_2\phi v,\ \nabla_v \phi v = -a_2 v, \tag{3.5}$$

where the second and third two terms follow from $\nabla_v e = \frac{1}{\lambda^2}h^2\nabla_v e$ and $\nabla_v \phi e = \frac{1}{\lambda^2}h^2\nabla_v \phi e$; the last two terms follows from $h^2\nabla_v v = 0$ and $h^2\nabla_v \phi v = 0$ and a_2, a_3 are smooth functions.

Similarly, substituting Y with ϕv in (3.2) gives $\nabla_{\phi v} h^2 = 0$. From this, (2.4) and (3.1) we obtain

$$\nabla_{\phi v} \xi = 0,\ \nabla_{\phi v} e = a_5\phi e,\ \nabla_{\phi v} \phi e = -a_5 e,\ \nabla_{\phi v} v = a_4\phi v,\ \nabla_{\phi v} \phi v = -a_4 v, \tag{3.6}$$

where the second and third two terms follow from $\nabla_{\phi v} e = \frac{1}{\lambda^2}h^2\nabla_{\phi v} e$, $\nabla_{\phi v} \phi e = \frac{1}{\lambda^2}h^2\nabla_{\phi v} \phi e$, and the last two terms follow from $h^2\nabla_{\phi v} v = 0$ and $h^2\nabla_{\phi v} \phi v = 0$ and a_4, a_5 are smooth functions.

Putting $X = Y = Z = e$ into (3.2) and using (3.1) give

$$g(\nabla_{\phi e} e, \phi e) = g(\nabla_{\phi e} \phi e, e) = 0. \tag{3.7}$$

Similarly, putting $X = Y = Z = \phi e$ into (3.2) and using (3.1) give

$$g(\nabla_e \phi e, e) = g(\nabla_e e, \phi e) = 0. \tag{3.8}$$

Putting $X = Y = e$ and $Z = v$ into (3.2) and using (3.1) and (3.5) give

$$g(\nabla_e e, v) = g(\nabla_{\phi e}\phi e, v) = -g(\nabla_{\phi e} e, \phi v). \tag{3.9}$$

Similarly, putting $X = Y = e$ and $Z = \phi v$ into (3.2) and using (3.1) and (3.6) give

$$g(\nabla_e e, \phi v) = g(\nabla_{\phi e}\phi e, \phi v) = g(\nabla_{\phi e} e, v). \tag{3.10}$$

In view of the second term of (2.4) we have $\nabla_e \xi = -\lambda\phi e$ and $g(\nabla_e e, \xi) = 0$. Using this and (3.8) we may write

$$\nabla_e e = a_6 v + a_7 \phi v \tag{3.11}$$

and hence from (3.1) we have

$$\nabla_e \phi e = \lambda\xi + a_6\phi v - a_7 v, \tag{3.12}$$

where a_6 and a_7 are smooth functions. Similarly, from the second term of (2.4) we have $\nabla_{\phi e}\xi = -\lambda e$ and $g(\nabla_{\phi e}\phi e, \xi) = 0$. Using this, (3.7), (3.9) and (3.10) we may write

$$\nabla_{\phi e}\phi e = a_6 v + a_7\phi v \tag{3.13}$$

and hence from (3.1) we have

$$\nabla_{\phi e} e = \lambda\xi + a_7 v - a_6\phi v. \tag{3.14}$$

On the other hand, it is known that local symmetry implies Ricci symmetry, i.e., $\nabla Q = 0$. Then, taking the covariant derivative of $Q\xi = -\mathrm{tr}(h^2)\xi = -2\lambda^2\xi$ and using the second term of (2.4) we have

$$Qh'X = -2\lambda^2 h'X$$

for any vector field X. Putting $X = e$ and $X = \phi e$ respectively in this relation gives

$$Qe = -2\lambda^2 e, \; Q\phi e = -2\lambda^2\phi e. \tag{3.15}$$

Taking the covariant derivative of $Qe = -2\lambda^2 e$ gives $Q\nabla_X e = -2\lambda^2\nabla_X e$ for any vector field X. Putting $X = e$ in this relation and using (3.11) give

$$a_6 Qv + a_7 Q\phi v = -2\lambda^2 a_6 v - 2\lambda^2 a_7\phi v. \tag{3.16}$$

Similarly, putting $X = \phi e$ in previous relation and using (3.14) give

$$a_7 Qv - a_6 Q\phi v = -2\lambda^2 a_7 v + 2\lambda^2 a_6\phi v. \tag{3.17}$$

It follows from relations (3.16) and (3.17) that either $a_6^2 + a_7^2 = 0$ identically or when $a_6^2 + a_7^2 \neq 0$ holds on some open subset we obtain directly

$$Qv = -2\lambda^2 v, \; Q\phi v = -2\lambda^2\phi v. \tag{3.18}$$

The second case means that M^5 is Einstein with Ricci operator $Q = -2\lambda^2\mathrm{id}$, where λ is a positive constant. Next, we show that the first case can not occur. Let us now consider the first case $a_6 = a_7 = 0$. Applying this in relations (3.11)-(3.14) we observe that the distribution $[e] \oplus [\phi e] \oplus [\xi]$ is integrable with totally geodesic leaves, where we denote by $[e]$, $[\phi e]$ and $[\xi]$ the eigendistributions of h

with corresponding eigenvector fields e, ϕe and ξ, respectively. Moreover, from (3.5) and (3.6) it is easily seen that the distribution $[v]\oplus[\phi v]$ is also integrable with totally geodesic leaves, where $[v]$ and $[\phi v]$ denote the eigendistributions of h with corresponding eigenvector fields v and ϕv, respectively. Therefore, now we deduce that M^5 is locally isometric to a Riemannian product $M^3\times M^2$, where M^3 and M^2 are integral manifolds of the distributions $[e]\oplus[\phi e]\oplus[\xi]$ and $[v]\oplus[\phi v]$, respectively.

Since M^5 is locally symmetric, here we remark that both M^2 and M^3 are locally symmetric. Moreover, it follows directly from (3.4) and (3.11)-(3.14) that

$$R(\phi e,e)e=\lambda^2\phi e,\ R(\xi,e)e=-\lambda^2\xi. \tag{3.19}$$

Taking the covariant derivative of the first term of (3.19) along e and using the second term of (3.19) we obtain

$$(\nabla_e R)(\phi e,e)e=2\lambda^3\xi, \tag{3.20}$$

where we have used relations and (3.11) and (3.12).

Taking into account the local symmetry condition of totally geodesic submanifold M^3 we arrive at a contradiction, $\lambda=0$. In fact, on the integral manifold of the distribution $[e]\oplus[\phi e]\oplus[\xi]$ there exists a three-dimensional almost coKähler structure which is locally symmetric. Therefore, applying Perrone [15, Proposition 3.1], that is, any locally symmetric three-dimensional almost coKähler manifold must be coKähler, we still obtain $\lambda=0$ and arrive at again a contradiction. This complete the proof. □

It is well-known that a homogeneous Einstein manifold can not be compact if its the scalar curvature is negative (see Besse [1, Theorem 7.56]). Thus, from Theorem 1.1 we give our main result as the following.

Theorem 3.1. *A five-dimensional compact, simple connected almost coKähler manifold with Kählerian leaves is locally symmetric if and only if it is coKähler and is locally isometric to a Riemannian product of a unit circle S^1 and a four-dimensional locally symmetric compact Kähler manifold.*

Next we show some examples of five-dimensional locally symmetric or Einstein almost coKähler manifolds.

Example 3.1. From Oguro and Sekigawaa [11] we know that the product $\mathbb{H}^3\times\mathbb{R}$ admits a strictly almost Kähler structure. Therefore, we state that the product $\mathbb{R}\times(\mathbb{H}^3\times\mathbb{R})$ admits a locally symmetric non-coKähler almost coKähler structure. However, the structure has no Kählerian leaves and is not Einstein.

Example 3.2. Let G be a connected, simply connected Lie group with Lie algebra $\mathfrak{g}=\{e_1,e_2,e_3,e_4,e_5\}$ whose structure equations are given by

$$\begin{aligned}&\mathrm{d}e^1=\frac{\sqrt{3}}{2}e^2\wedge e^5+\frac{1}{2}e^1\wedge e^4,\ \mathrm{d}e^2=\frac{\sqrt{3}}{2}e^1\wedge e^5+\frac{1}{2}e^2\wedge e^4,\\&\mathrm{d}e^3=e^1\wedge e^2+e^3\wedge e^4,\ \mathrm{d}e^4=0,\ \mathrm{d}e^5=0,\end{aligned}$$

where $\{e^1, e^2, e^3, e^4, e^5\}$ is the dual basis for $\mathfrak{g}^*$. Let g be the left invariant metric on G given as the following

$$g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 + (e^5)^2.$$

Conti and Fernández [6] proved that G admits an Einstein non-coKähler almost coKähler structure $(\eta := e^5, \Phi := -e^1\wedge e^2 - e^3\wedge e^4, g)$. Here we remark that the manifold has Kählerian leaves and but is not locally symmetric. This assertion follows from the following computations.

From the structure equation we obtain the Lie bracket of the Lie group.

$$\begin{aligned}
&[e_1, e_2] = -e_3,\ [e_1, e_3] = 0,\ [e_1, e_4] = -\frac{1}{2}e_1,\\
&[e_1, e_5] = -\frac{\sqrt{3}}{2}e_2,\ [e_2, e_3] = 0,\ [e_2, e_4] = -\frac{1}{2}e_2,\\
&[e_2, e_5] = -\frac{\sqrt{3}}{2}e_1,\ [e_3, e_4] = -e_3,\ [e_3, e_5] = 0,\ [e_4, e_5] = 0.
\end{aligned}$$

By a direct calculation and using the well-known Koszual formula we have

$$\begin{aligned}
&\nabla_{e_1}e_2 = -\frac{1}{2}e_3 + \frac{\sqrt{3}}{2}e_5,\ \nabla_{e_1}e_3 = \frac{1}{2}e_2,\ \nabla_{e_1}e_4 = -\frac{1}{2}e_1,\\
&\nabla_{e_1}e_5 = -\frac{\sqrt{3}}{2}e_2,\ \nabla_{e_2}e_3 = -\frac{1}{2}e_1,\ \nabla_{e_2}e_4 = -\frac{1}{2}e_2,\\
&\nabla_{e_2}e_5 = -\frac{\sqrt{3}}{2}e_1,\ \nabla_{e_3}e_4 = -\frac{1}{2}e_3,\ \nabla_{e_3}e_5 = 0,\ \nabla_{e_4}e_5 = 0,\\
&\nabla_{e_1}e_1 = \frac{1}{2}e_4,\ \nabla_{e_2}e_2 = \frac{1}{2}e_4,\ \nabla_{e_3}e_3 = \frac{1}{2}e_4,\ \nabla_{e_4}e_4 = 0,\ \nabla_{e_5}e_5 = 0.
\end{aligned}$$

It follows directly from the above relations that

$$\begin{aligned}
&R(e_4, e_2)e_3 = \frac{1}{2}e_1,\ R(e_1, e_3, e_3) = 0,\\
&R(e_1, e_5)e_3 = -\frac{\sqrt{3}}{4}e_1,\ R(e_1, e_2)e_3 = 0, R(e_1, e_2)e_2 = -\frac{1}{4}e_1.
\end{aligned}$$

Applying the above relations we have $(\nabla_{e_1}R)(e_1, e_2)e_3 = \frac{1}{4}e_1$ and this means that the manifold is not locally symmetric. Moreover, from the above relations we obtain

$$h'e_2 = -\frac{\sqrt{3}}{2}e_1,\ h'e_1 = -\frac{\sqrt{3}}{2}e_2,\ h'e_3 = 0,\ h'e_4 = 0,$$

where $\{e_1, e_2 = \phi e_1, e_3, e_4 = \phi e_3, \xi = e_5\}$ is a local orthonormal ϕ-basis of the tangent space at each point of the manifold. One can check that relation (3.1) holds for any vector fields X, Y and this means that the structure has Kählerian leaves.

For more examples of almost coKähler manifolds with Kählerian leaves we refer the reader to Olszak [13, Section 3].

Before closing this paper, we now propose a nature question deserving further exploration.

Question. Is there a locally symmetric, Einstein, non-coKähler almost coKähler 5-manifold with Kählerian leaves?

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